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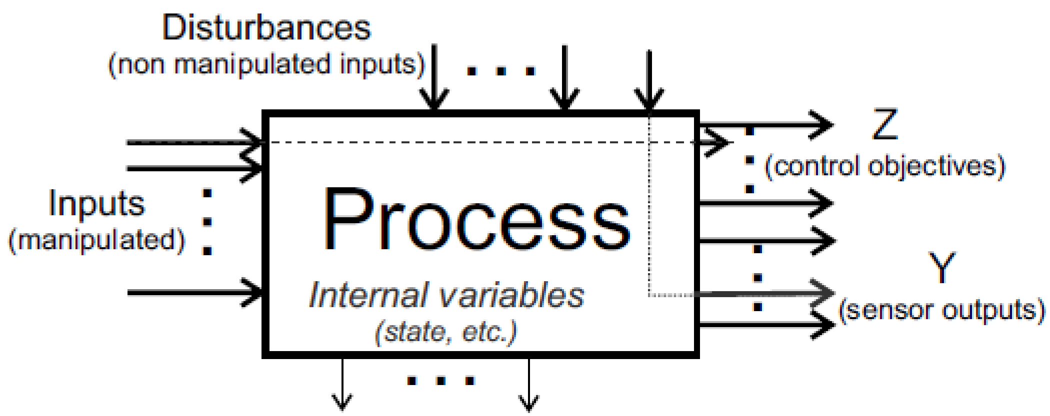
نظم متعددة المتغيرات
Multivariable Systems



الروبوت والأنظمة الذكية

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Control Systems Design in State Space

STATE OBSERVERS

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.

Estimation of unmeasurable state variables is commonly called *observation*. A device (or a computer program) that estimates or observes the state variables is called a *state observer*, or simply an *observer*. If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a *full-order state observer*.

STATE OBSERVERS

There are times when this will not be necessary, when we will need observation of only the unmeasurable state variables, but not of those that are directly measurable as well. For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only ***n-m*** state variables, where ***n*** is the dimension of the state vector and ***m*** is the dimension of the output vector.

An observer that estimates fewer than ***n*** state variables, where ***n*** is the dimension of the state vector, is called a *reduced-order state observer* or, simply, a *reduced-order observer*.

STATE OBSERVERS

If the order of the reduced-order state observer is the minimum possible, the observer is called a *minimum-order state observer* or *minimum-order observer*. In this section, we shall discuss both the full-order state observer and the minimum-order state observer.

State Observer

A state observer estimates the state variables based on the measurements of the output and control variables. Here the concept of observability plays an important role. As we shall see later, state observers can be designed if and only if the observability condition is satisfied.

STATE OBSERVERS

In the following discussions of state observers, we shall use the notation $\tilde{\mathbf{X}}$ to designate the observed state vector. In many practical cases, the observed state vector $\tilde{\mathbf{X}}$ is used in the state feedback to generate the desired control vector.

Consider the plant defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices **A** and **B** and the lack of the initial error.

STATE OBSERVERS

The estimation error or observation error is the difference between the measured output and the estimated output. The initial error is the difference between the initial state and the initial estimated state. Thus, we define the mathematical model of the observer to be

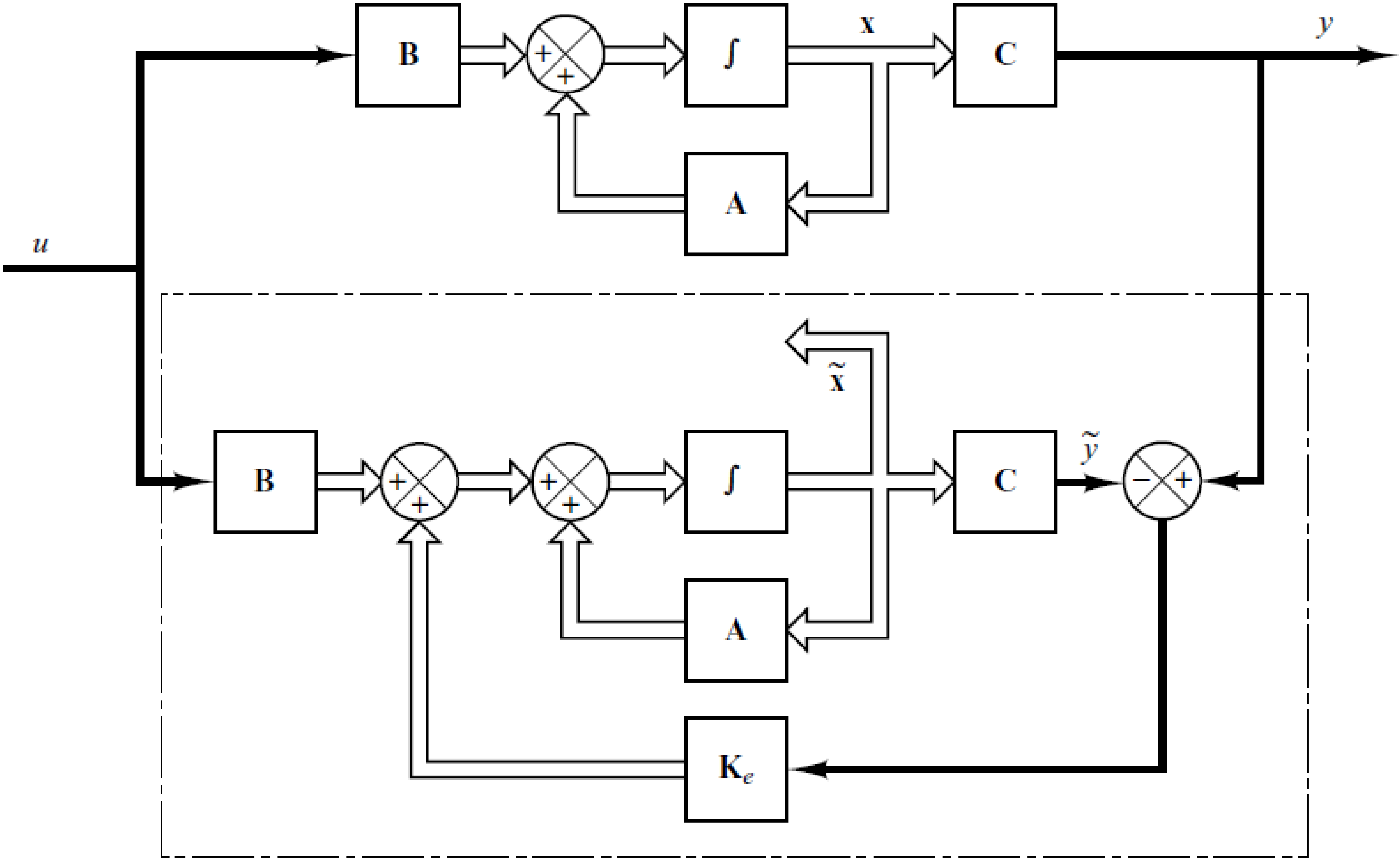
$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y\end{aligned}$$

where $\tilde{\mathbf{X}}$ is the estimated state and $C\tilde{\mathbf{X}}$ is the estimated output. The inputs to the observer are the output ***y*** and the control input ***u***.

STATE OBSERVERS

Matrix $\mathbf{K}_e$, which is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output $\boldsymbol{y}$ and the estimated output $\mathrm{C}\widetilde{\mathbf{X}}$. This term continuously corrects the model output and improves the performance of the observer. Figure shows the block diagram of the system and the full-order state observer.

STATE OBSERVERS



Full-order state observer

Full-Order State Observer

The order of the state observer that will be discussed here is the same as that of the plant. The observer error equation:

$$\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K}_e(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}})$$

Define the difference between $\mathbf{X}$ and $\tilde{\mathbf{X}}$ as the error vector $\mathbf{e}$, or

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$$

Then

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

Full-Order State Observer

We see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K_e}\mathbf{C}$. If matrix $\mathbf{A} - \mathbf{K_e}\mathbf{C}$ is a stable matrix, the error vector will converge to zero for any initial error vector $\boldsymbol{e}(0)$. That is, will converge to $\mathbf{X}(t)$ regardless of the values of $\mathbf{X}(0)$ and If the eigenvalues of matrix $\mathbf{A} - \mathbf{K_e}\mathbf{C}$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.

If the plant is completely observable, then it can be proved that it is possible to choose matrix $\mathbf{K_e}$ such that $\mathbf{A} - \mathbf{K_e}\mathbf{C}$ has arbitrarily desired eigenvalues.

Full-Order State Observer

That is, the observer gain matrix $\mathbf{K_e}$ can be determined to yield the desired matrix $\mathbf{A} - \mathbf{K_e}\mathbf{C}$. We shall discuss this matter in what follows.

Dual Problem

The problem of designing a full-order observer becomes that of determining the observer gain matrix $\mathbf{K_e}$ such that the error dynamics are asymptotically stable with sufficient speed of response.

(The asymptotic stability and the speed of response of the error dynamics are determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K_e}\mathbf{C}$.)

Full-Order State Observer

Hence, the design of the full-order observer becomes that of determining an appropriate $\mathbf{K_e}$ such that $\mathbf{A} - \mathbf{K_e C}$ has desired eigenvalues. Thus, the problem here becomes the same as the pole-placement problem. In fact, the two problems are mathematically the same. This property is called duality.

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system.

Full-Order State Observer

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

assuming the control signal $\boldsymbol{\nu}$ to be $v = -\mathbf{K}\mathbf{z}$

If the dual system is completely state controllable, then the state feedback gain matrix $\mathbf{K}$ can be determined such that matrix $\mathbf{A}^* - \mathbf{C}^*\mathbf{K_e}$ will yield a set of the desired eigenvalues.

If $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_n$ are the desired eigenvalues of the state observer matrix, then by taking the same $\boldsymbol{\mu}_i$'s as the desired eigenvalues of the state-feedback gain matrix of the dual system, we obtain

Full-Order State Observer

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

Noting that the eigenvalues of $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ and those of $\mathbf{A} - \mathbf{K}^*\mathbf{C}$ are the same, we have

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = |s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$$

Comparing the characteristic polynomial and the characteristic polynomial for the observer system, we find that $\mathbf{K_e}$ and $\mathbf{K}^*$ are related by

$$\mathbf{K}_e = \mathbf{K}^*$$

Thus, using the matrix $\mathbf{K}$ determined by the pole-placement approach in the dual system, the observer gain matrix $\mathbf{K_e}$ for the original system can be determined by using the relationship $\mathbf{K_e} = \mathbf{K}^*$.

Necessary and Sufficient Condition for State Observation

As discussed, a necessary and sufficient condition for the determination of the observer gain matrix $\mathbf{K_e}$ for the desired eigenvalues of $\mathbf{A} - \mathbf{K_e}\mathbf{C}$ is that the dual of the original system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

be completely state controllable. The complete state controllability condition for this dual system is that the rank of

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be ***n***. This is the condition for complete observability of the original system. This means that a necessary and sufficient condition for the observation of the state of the system is that the system be completely observable.

Necessary and Sufficient Condition for State Observation

Once we select the desired eigenvalues (or desired characteristic equation), the full order state observer can be designed, provided the plant is completely observable. The desired eigenvalues of the characteristic equation should be chosen so that the state observer responds at least two to five times faster than the closed-loop system considered. As stated earlier, the equation for the full-order state observer is

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

It is noted that thus far we have assumed the matrices **A**, **B**, and **C** in the observer to be exactly the same as those of the physical plant.

Necessary and Sufficient Condition for State Observation

If there are discrepancies in **A**, **B**, and **C** in the observer and in the physical plant, the dynamics of the observer error are no longer governed by last Equation. This means that the error may not approach zero as expected. Therefore, we need to choose $\mathbf{K_e}$ so that the observer is stable and the error remains acceptably small in the presence of small modeling errors.

Transformation Approach to Obtain State Observer Gain Matrix K*e*

By following the same approach as we used in deriving the equation for the state feedback gain matrix **K**, we can obtain the following equation:

$$\mathbf{K}_e = \mathbf{Q}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 - a_1 \end{bmatrix} = (\mathbf{WN}^*)^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 - a_1 \end{bmatrix}$$

where $\mathbf{K_e}$ is an **nx1** matrix,

$$\mathbf{Q} = (\mathbf{WN}^*)^{-1}$$

Transformation Approach to Obtain State Observer Gain Matrix K*e*

And

$$\mathbf{N} = [\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*]$$

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Direct-Substitution Approach to Obtain State Observer Gain Matrix K*e*

Similar to the case of pole placement, if the system is of low order, then direct substitution of matrix $\mathbf{K_e}$ into the desired characteristic polynomial may be simpler. For example, if $\mathbf{X}$ is a **3-vector**, then write the observer gain matrix $\mathbf{K_e}$ as

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

Substitute this $\mathbf{K_e}$ matrix into the desired characteristic polynomial:

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Direct-Substitution Approach to Obtain State Observer Gain Matrix K*e*

By equating the coefficients of the like powers of ***s*** on both sides of this last equation, we can determine the values of $\boldsymbol{k_{e1}}$, $\boldsymbol{k_{e2}}$, and $\boldsymbol{k_{e3}}$. This approach is convenient if ***n*=1, 2**, or **3**, where ***n*** is the dimension of the state vector $\mathbf{X}$. (Although this approach can be used when ***n*=4, 5, 6,… ,** the computations involved may become very tedious).

Another approach to the determination of the state observer gain matrix $\mathbf{K_e}$ is to use Ackermann's formula. This approach is presented in the following.

Ackermann's Formula

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

we derived Ackermann's formula for pole placement for the system . The result was given by :

$$\mathbf{K} = [0 \quad 0 \cdots 0 \quad 1]\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]^{-1}\phi(\mathbf{A})$$

For the dual of the system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

the preceding Ackermann's formula for pole placement is modified to

$$\mathbf{K} = [0 \quad 0 \cdots 0 \quad 1]\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]^{-1}\phi(\mathbf{A}^*)$$

Ackermann's Formula

As stated earlier, the state observer gain matrix $\mathbf{K_e}$ is given by $\mathbf{K}^*$, where $\mathbf{K}$ is given by last Equation. Thus,

$$\mathbf{K}_e = \mathbf{K}^* = \phi(\mathbf{A}^*)^* \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

where $\emptyset(\boldsymbol{s})$ is the desired characteristic polynomial for the state observer, or

Ackermann's Formula

where $\emptyset(\boldsymbol{s})$ is the desired characteristic polynomial for the state observer, or

$$\phi(s) = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

where $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$,… , $\boldsymbol{\mu_n}$ are the desired eigenvalues. Equation is called Ackermann's formula for the determination of the observer gain matrix $\mathbf{K_e}$.

Comments on Selecting the Best Ke

Referring to Figure, notice that the feedback signal through the observer gain matrix $\mathbf{K_e}$ serves as a correction signal to the plant model to account for the unknowns in the plant. If significant unknowns are involved, the feedback signal through the matrix $\mathbf{K_e}$ should be relatively large. However, if the output signal is contaminated significantly by disturbances and measurement noises, then the output ***y*** is not reliable and the feedback signal through the matrix $\mathbf{K_e}$ should be relatively small. In determining the matrix $\mathbf{K_e}$, we should carefully examine the effects of disturbances and noises involved in the output ***y***.

Remember that the observer gain matrix $\mathbf{K_e}$ depends on the desired characteristic equation

$$(s - \mu_1)(s - \mu_2)\cdots(s - \mu_n) = 0$$

Comments on Selecting the Best K*e*

The choice of $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$,… , $\boldsymbol{\mu_n}$ a set of is, in many instances, not unique. As a general rule, however, the observer poles must be two to five times faster than the controller poles to make sure the observation error (estimation error) converges to zero quickly. This means that the observer estimation error decays two to five times faster than does the state vector **x**. Such faster decay of the observer error compared with the desired dynamics makes the controller poles dominate the system response.

Comments on Selecting the Best K*e*

It is important to note that if sensor noise is considerable, we may choose the observer poles to be slower than two times the controller poles, so that the bandwidth of the system will become lower and smooth the noise. In this case the system response will be strongly influenced by the observer poles. If the observer poles are located to the right of the controller poles in the left-half ***s*** plane, the system response will be dominated by the observer poles rather than by the control poles.

Comments on Selecting the Best K*e*

In the design of the state observer, it is desirable to determine several observer gain matrices $\mathbf{K_e}$ based on several different desired characteristic equations. For each of the several different matrices $\mathbf{K_e}$, simulation tests must be run to evaluate the resulting system performance. Then we select the best $\mathbf{K_e}$ from the viewpoint of overall system performance. In many practical cases, the selection of the best matrix $\mathbf{K_e}$ boils down to a compromise between speedy response and sensitivity to disturbances and noises.

EXAMPLE

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 1]$$

We use the observed state feedback such that $u = -\mathbf{K}\tilde{\mathbf{x}}$

Design a full-order state observer, assuming that the system configuration is identical to that shown in Figure. Assume that the desired eigenvalues of the observer matrix are

$$\mu_1 = -10, \qquad \mu_2 = -10$$

The design of the state observer reduces to the determination of an appropriate observer gain matrix $\mathbf{K_e}$.

EXAMPLE

Let us examine the observability matrix. The rank of

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is **2**. Hence, the system is completely observable and the determination of the desired observer gain matrix is possible. We shall solve this problem by three methods.

Method 1*:* We shall determine the observer gain matrix by use of Equation ($\mathbf{K_e}$). The given system is already in the observable canonical form. Hence, the transformation matrix $\mathbf{Q} = (\mathbf{W}\mathbf{N}^*)^{-1}$ is **I**. Since the characteristic equation of the given system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -20.6 \\ -1 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2 = 0$$

EXAMPLE

we have $a_1 = 0, \qquad a_2 = -20.6$

The desired characteristic equation is

$$(s + 10)^2 = s^2 + 20s + 100 = s^2 + \alpha_1 s + \alpha_2 = 0$$

Hence,

$$\alpha_1 = 20, \qquad \alpha_2 = 100$$

Then the observer gain matrix $\mathbf{K_e}$ can be obtained as follows:

$$\mathbf{K}_e = (\mathbf{WN}^*)^{-1}\begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 100 + 20.6 \\ 20 - 0 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

EXAMPLE

Method 2:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

the characteristic equation for the observer becomes

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Define

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}$$

Then the characteristic equation becomes

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} [0 \quad 1] \right| = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix}$$

$$= s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

EXAMPLE

Since the desired characteristic equation is $s^2 + 20s + 100 = 0$

by comparing, we obtain

$$k_{e1} = 120.6, \qquad k_{e2} = 20$$

$$\mathbf{K}_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

***Method 3**:* We shall use Ackermann's formula:

$$\mathbf{K}_e = \phi(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Where

$$\phi(s) = (s - \mu_1)(s - \mu_2) = s^2 + 20s + 100$$

Thus,

$$\phi(\mathbf{A}) = \mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I}$$

EXAMPLE

And

$$\mathbf{K}_e = (\mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

As a matter of course, we get the same $\mathbf{K_e}$ regardless of the method employed.

The equation for the full-order state observer is given by Equation

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -100 \\ 1 & -20 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}y$$

EXAMPLE

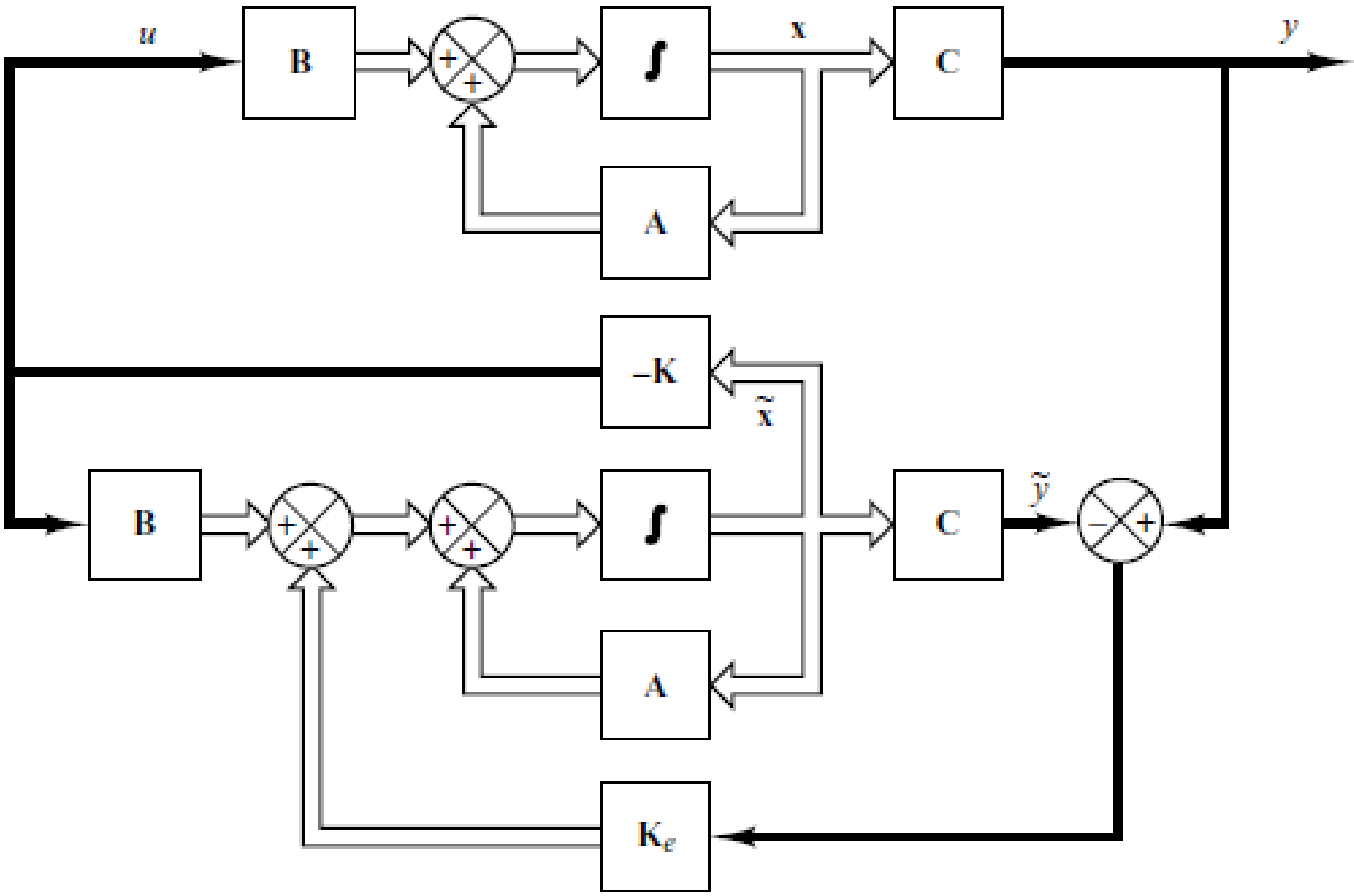
Finally, it is noted that, similar to the case of pole placement, if the system order ***n*** is **4** or higher, methods **1** and **3** are preferred, because all matrix computations can be carried out by a computer, while method **2** always requires hand computation of the characteristic equation involving unknown parameters $\boldsymbol{k_{e1}}, \boldsymbol{k_{e2}}, \ldots, \boldsymbol{k_{en}}$.

Effects of the Addition of the Observer on a Closed-Loop System

In the pole-placement design process, we assumed that the actual state $\mathbf{X}(t)$ was available for feedback. In practice, however, the actual state $\mathbf{X}(t)$ may not be measurable, so we will need to design an observer and use the observed state $\widetilde{\mathbf{X}}(t)$ for feedback. The design process, therefore, becomes a two-stage process, the first stage being the determination of the feedback gain matrix $\mathbf{K}$ to yield the desired characteristic equation and the second stage being the determination of the observer gain matrix $\mathbf{K_e}$ to yield the desired observer characteristic equation.

Let us now investigate the effects of the use of the observed state $\widetilde{\mathbf{X}}(t)$, rather than the actual state $\mathbf{X}(t)$, on the characteristic equation of a closed-loop control system.

Effects of the Addition of the Observer on a Closed-Loop System



Effects of the Addition of the Observer on a Closed-Loop System

Consider the completely state controllable and completely observable system defined by the equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

For the state-feedback control based on the observed state $\widetilde{\mathbf{X}}$

$$u = -\mathbf{K}\widetilde{\mathbf{x}}$$

With this control, the state equation becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\widetilde{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}(\mathbf{x} - \widetilde{\mathbf{x}})$$

The difference between the actual state $\mathbf{X}(t)$ and the observed state $\widetilde{\mathbf{X}}(t)$ has been defined as the error $\mathbf{e}(t)$:

$$\mathbf{e}(t) = \mathbf{x}(t) - \widetilde{\mathbf{x}}(t)$$

Effects of the Addition of the Observer on a Closed-Loop System

Substitution of the error vector $\mathbf{e}(t)$ gives

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe}$$

Note that the observer error equation was given by :

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

Combining Equations, we obtain

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

This Equation describes the dynamics of the observed-state feedback control system.

The characteristic equation for the system is

$$\begin{vmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} \end{vmatrix} = 0$$

Effects of the Addition of the Observer on a Closed-Loop System

Or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}||s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Notice that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole-placement design alone and the poles due to the observer design alone. This means that the pole-placement design and the observer design are independent of each other. They can be designed separately and combined to form the observed-state feedback control system. Note that, if the order of the plant is ***n***, then the observer is also of ***n***th order (if the full-order state observer is used), and the resulting characteristic equation for the entire closed-loop system becomes of order ***2n***.

Transfer Function of the Observer-Based Controller

Consider the plant defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

Assume that the plant is completely observable. Assume that we use observed-state feedback control

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

Then, the equations for the observer are given by

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{B}\mathbf{K})\tilde{\mathbf{x}} + \mathbf{K}_e y$$

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

By taking the Laplace transform of Equation, assuming a zero initial condition, and solving for $\tilde{\mathbf{X}}(s)$ we obtain

Transfer Function of the Observer-Based Controller

$$\tilde{\mathbf{X}}(s) = (s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e Y(s)$$

By substituting this $\tilde{\mathbf{X}}(s)$ into the Laplace transform of Equation $u = -\mathbf{K}\tilde{\mathbf{x}}$

we obtain

$$U(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e Y(s)$$

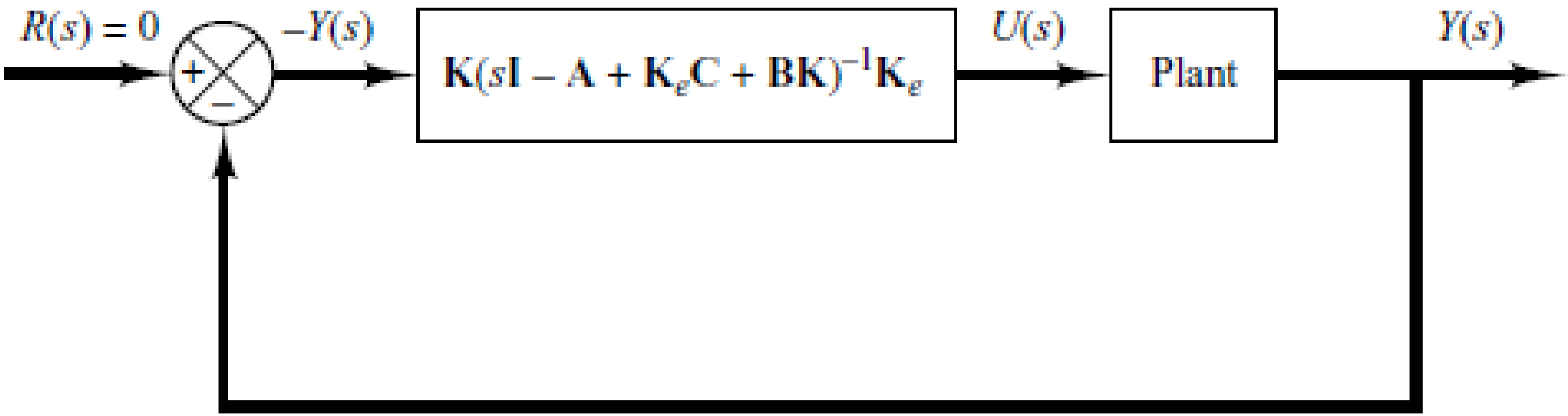
Then the transfer function ***U(s)/Y(s)*** can be obtained as

$$\frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e$$

Figure shows the block diagram representation for the system. Notice that the transfer function

$$\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e$$

Transfer Function of the Observer-Based Controller



acts as a controller for the system. Hence, we call the transfer function

$$\frac{U(s)}{-Y(s)} = \frac{\text{num}}{\text{den}} = \mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e$$

Transfer Function of the Observer-Based Controller

the observer-based controller transfer function or, simply, the observer-controller transfer function.

Note that the observer-controller matrix

$$\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK}$$

may or may not be stable, although $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{K}_e\boldsymbol{C}$ are chosen to be stable. In fact, in some cases the matrix $\mathbf{A} - \mathbf{K}_e\boldsymbol{C} - \mathbf{BK}$ may be poorly stable or even unstable.

EXAMPLE

Consider the design of a regulator system for the following plant:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0]$$

Suppose that we use the pole-placement approach to the design of the system and that the desired closed-loop poles for this system are at $s = \mu_i$**(*i*=1, 2)**, where μ_1**=–1.8+j2.4** and μ_2**=–1.8-j2.4**. The state-feedback gain matrix $\mathbf{K}$ for this case can be obtained as follows:

$$\mathbf{K} = [29.6 \quad 3.6]$$

EXAMPLE

Using this state-feedback gain matrix $\mathbf{K}$, the control signal u is given by

$$u = -\mathbf{K}\mathbf{x} = -[29.6 \quad 3.6]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Suppose that we use the observed-state feedback control instead of the actual-state feedback control, or

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -[29.6 \quad 3.6]\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

where we choose the observer poles to be at

$$s = -8, \qquad s = -8$$

Obtain the observer gain matrix $\mathbf{K}_e$ and draw a block diagram for the observed-state feedback control system.

EXAMPLE

Then obtain the transfer function $\boldsymbol{U(s)/[-Y(s)]}$ for the observer controller, and draw another block diagram with the observer controller as a series controller in the feedforward path. Finally, obtain the response of the system to the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}(0) = \mathbf{x}(0) - \tilde{\mathbf{x}}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

For the system, the characteristic polynomial is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ -20.6 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2$$

Thus,

$$a_1 = 0, \qquad a_2 = -20.6$$

EXAMPLE

The desired characteristic polynomial for the observer is

$$(s-\mu_1)(s-\mu_2) = (s+8)(s+8) = s^2 + 16s + 64$$
$$= s^2 + \alpha_1 s + \alpha_2$$

Hence,

$$\alpha_1 = 16, \qquad \alpha_2 = 64$$

For the determination of the observer gain matrix, we use Equation

$$\mathbf{K}_e = (\mathbf{W}\mathbf{N}^*)^{-1}\begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix}$$

Where

$$\mathbf{N} = [\mathbf{C}^* \; \vdots \; \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

EXAMPLE

Hence,

$$\mathbf{K}_e = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 64 + 20.6 \\ 16 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 84.6 \\ 16 \end{bmatrix} = \begin{bmatrix} 16 \\ 84.6 \end{bmatrix}$$

This Equation gives the observer gain matrix $\mathbf{K}_e$. The observer equation is given by Equation

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

Since $u = -\mathbf{K}\tilde{\mathbf{x}}$

becomes

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{B}\mathbf{K})\tilde{\mathbf{x}} + \mathbf{K}_e y$$

EXAMPLE

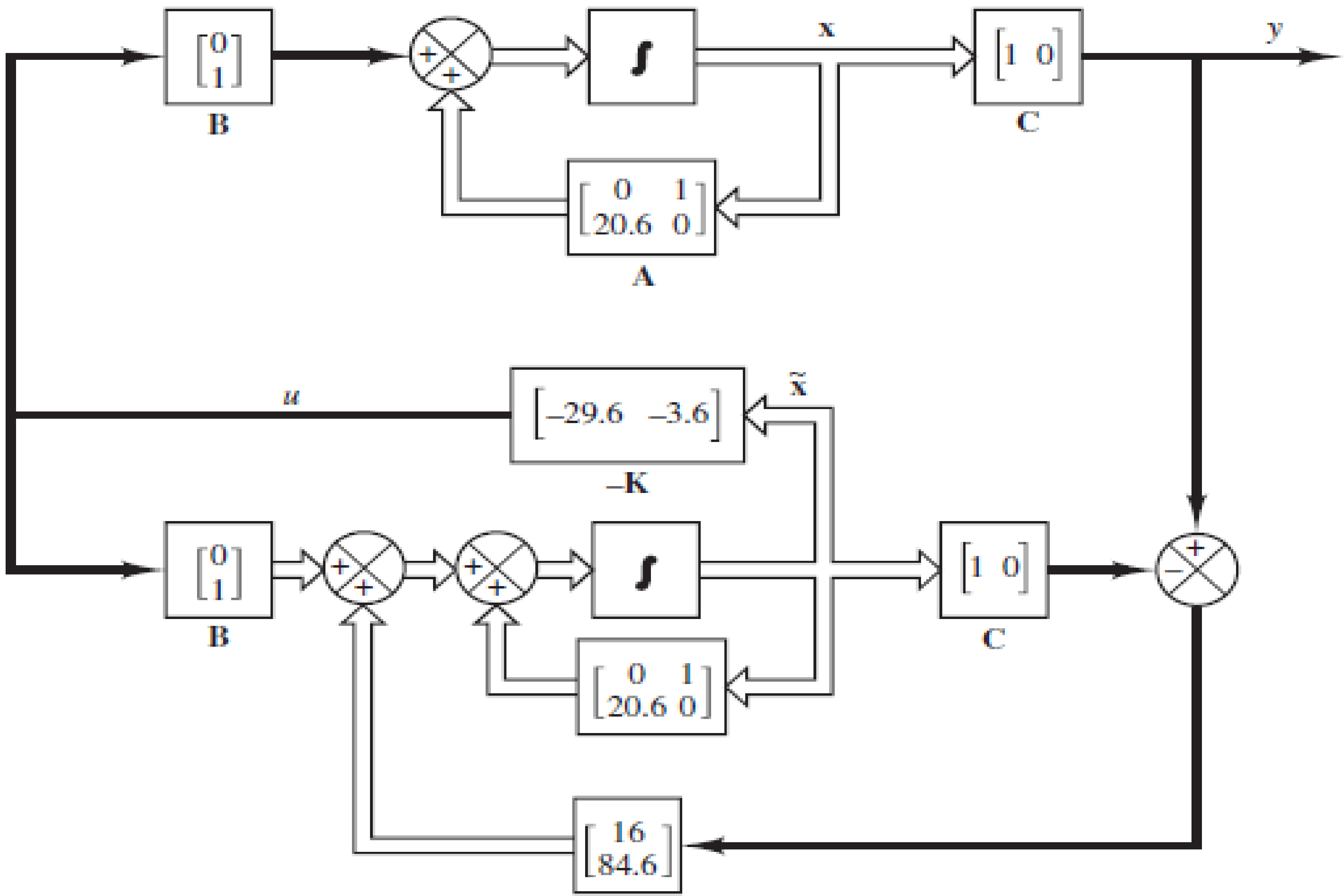
Or

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} [1 \quad 0] - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [29.6 \quad 3.6] \right\} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$

$$= \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$

The block diagram of the system with observed-state feedback is shown in Figure (a).

EXAMPLE



(a)

EXAMPLE

The transfer function of the observer-controller is

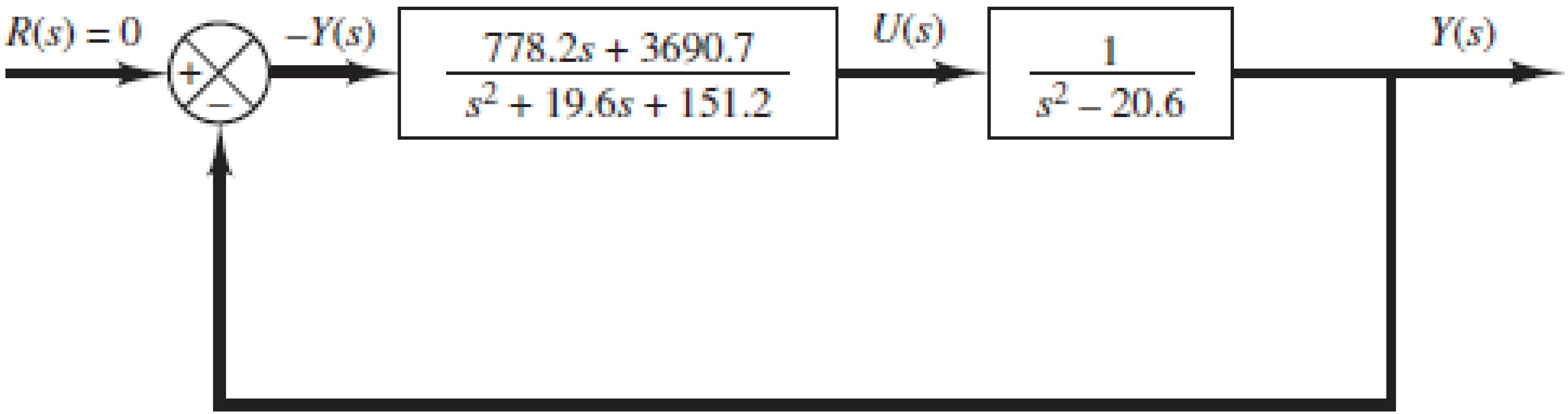
$$\frac{U(s)}{-Y(s)} = \mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e$$

$$= [29.6 \quad 3.6]\begin{bmatrix} s+16 & -1 \\ 93.6 & s+3.6 \end{bmatrix}^{-1}\begin{bmatrix} 16 \\ 84.6 \end{bmatrix}$$

$$= \frac{778.2s + 3690.7}{s^2 + 19.6s + 151.2}$$

Figure (b) shows a block diagram of the system.

EXAMPLE



(b)

EXAMPLE

The dynamics of the observed-state feedback control system just designed can be described by the following equations: For the plant,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the observer,

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$

$$u = -[29.6 \quad 3.6] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The system, as a whole, is of fourth order.

EXAMPLE

The characteristic equation for the system is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}||s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = (s^2 + 3.6s + 9)(s^2 + 16s + 64)$$
$$= s^4 + 19.6s^3 + 130.6s^2 + 374.4s + 576 = 0$$

The characteristic equation can also be obtained from the block diagram for the system shown in Figure (b). Since the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{778.2s + 3690.7}{(s^2 + 19.6s + 151.2)(s^2 - 20.6) + 778.2s + 3690.7}$$

the characteristic equation is

EXAMPLE

$$(s^2 + 19.6s + 151.2)(s^2 - 20.6) + 778.2s + 3690.7$$
$$= s^4 + 19.6s^3 + 130.6s^2 + 374.4s + 576 = 0$$

As a matter of course, the characteristic equation is the same for the system in state-space representation and in transfer-function representation.

Finally, we shall obtain the response of the system to the following initial condition:

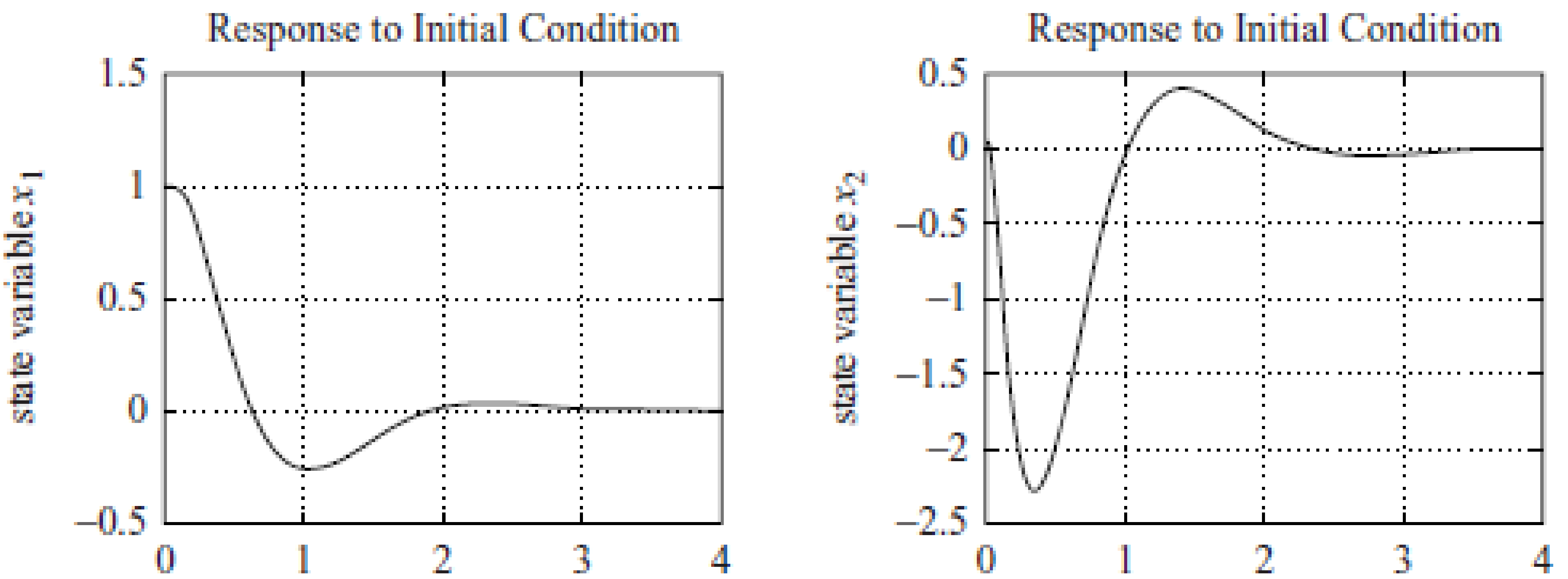
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The response to the initial condition can be determined from

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{e}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

EXAMPLE

The resulting response curves are shown in Figure:



EXAMPLE

The resulting response curves are shown in Figure: