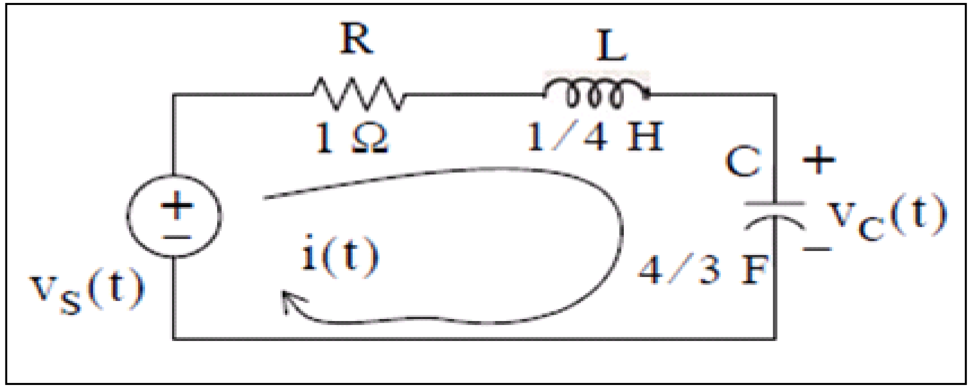
Example

For the circuit of Figure , the initial conditions are $i_L(0\ ) = 0$, and $v_C(0\ ) = 0.5$ V. Use the state variable method to compute $i_L(t)$ and $v_C(t)$.

$v_S(t) = 1$



Solution:

For this example,

$$i = i_L$$

and

$$Ri_L + L\frac{di_L}{dt} + v_C = 1$$

Substitution of given values and rearranging, yields

$$\frac{1}{4}\frac{di_L}{dt} = (-1)i_L - v_C + 1$$

or

$$\frac{di_L}{dt} = -4i_L - 4v_C + 4$$

Next, we define the state variables $x_1 = i_L$ and $x_2 = v_C$. Then,

$$\dot{x}_1 = \frac{di_L}{dt}$$

and

$$\dot{x}_2 = \frac{dv_C}{dt}$$

Also,

$$i_L = C\frac{dv_C}{dt}$$

and thus,

$$x_1 = i_L = C\frac{dv_C}{dt} = C\dot{x}_2 = \frac{4}{3}\dot{x}_2$$

$$\dot{x}_1 = -4x_1 - 4x_2 + 4$$
$$\dot{x}_2 = \frac{3}{4}x_1$$

and in matrix form, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} v_S(t)$

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

where $A = \begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}$ $x_0 = \begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

First, we compute the state transition matrix e^{At}. We find the eigenvalues from

$$\det[A - \lambda I] = 0$$

Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} -4-\lambda & -4 \\ 3/4 & -\lambda \end{bmatrix} = 0 \qquad (-\lambda)(-4-\lambda) + 3 = 0 \qquad \lambda^2 + 4\lambda + 3 = 0$$

Therefore, $\lambda_1 = -1$ and $\lambda_2 = -3$

The next step is to find the coefficients a_i. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A$$

The constants a_0 and a_1 are found from

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$

$$a_0 + a_1\lambda_2 = e^{\lambda_2 t}$$

and with $\lambda_1 = -1$ and $\lambda_2 = -3$, we obtain

$$a_0 - a_1 = e^{-t} \qquad a_0 = 1.5e^{-t} - 0.5e^{-3t}$$

$$a_0 - 3a_1 = e^{-3t} \qquad a_1 = 0.5e^{-t} - 0.5e^{-3t}$$

$$e^{At} = (1.5e^{-t} - 0.5e^{-3t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (0.5e^{-t} - 0.5e^{-2t})\begin{bmatrix} -4 & -4 \\ 3/4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0 \\ 0 & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} + \begin{bmatrix} -2e^{-t} + 2e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix}$$

$$e^{At}x_0 = \begin{bmatrix} -0.5e^{-t} + 1.5e^{-3t} & -2e^{-t} + 2e^{-3t} \\ \frac{3}{8}e^{-t} - \frac{3}{8}e^{-3t} & 1.5e^{-t} - 0.5e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \qquad e^{At}x_0 = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix}$$

We also need to evaluate the integral on the right side

$b = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4$ and denoting this integral as Int, we obtain

$$Int = \int_{t_0}^{t} \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} & -2e^{-(t-\tau)} + 2e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} & 1.5e^{-(t-\tau)} - 0.5e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 4d\tau$$

$$\text{Int} = \int_{t_0}^{t} \begin{bmatrix} -0.5e^{-(t-\tau)} + 1.5e^{-3(t-\tau)} \\ \frac{3}{8}e^{-(t-\tau)} - \frac{3}{8}e^{-3(t-\tau)} \end{bmatrix} 4d\tau \qquad \text{Int} = 4\begin{bmatrix} -0.5e^{-(t-\tau)} + 0.5e^{-3(t-\tau)} \\ 0.375e^{-(t-\tau)} - 0.125e^{-3(t-\tau)} \end{bmatrix}\Bigg|_{\tau=0}^{t}$$

$$\text{Int} = 4\begin{bmatrix} -0.5 + 0.5 \\ 0.375 - 0.125 \end{bmatrix} - 4\begin{bmatrix} -0.5e^{-t} + 0.5e^{-3t} \\ 0.375e^{-t} - 0.125e^{-3t} \end{bmatrix} = 4\begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix}$$

By substitution of these values, the solution of

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}bu(\tau)d\tau$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^{-3t} \\ 0.75e^{-t} - 0.25e^{-3t} \end{bmatrix} + 4\begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ 0.25 - 0.375e^{-t} + 0.125e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-3t} \\ 1 - 0.75e^{-t} + 0.25e^{-3t} \end{bmatrix}$$

$$x_1 = i_L = e^{-t} - e^{-3t}$$

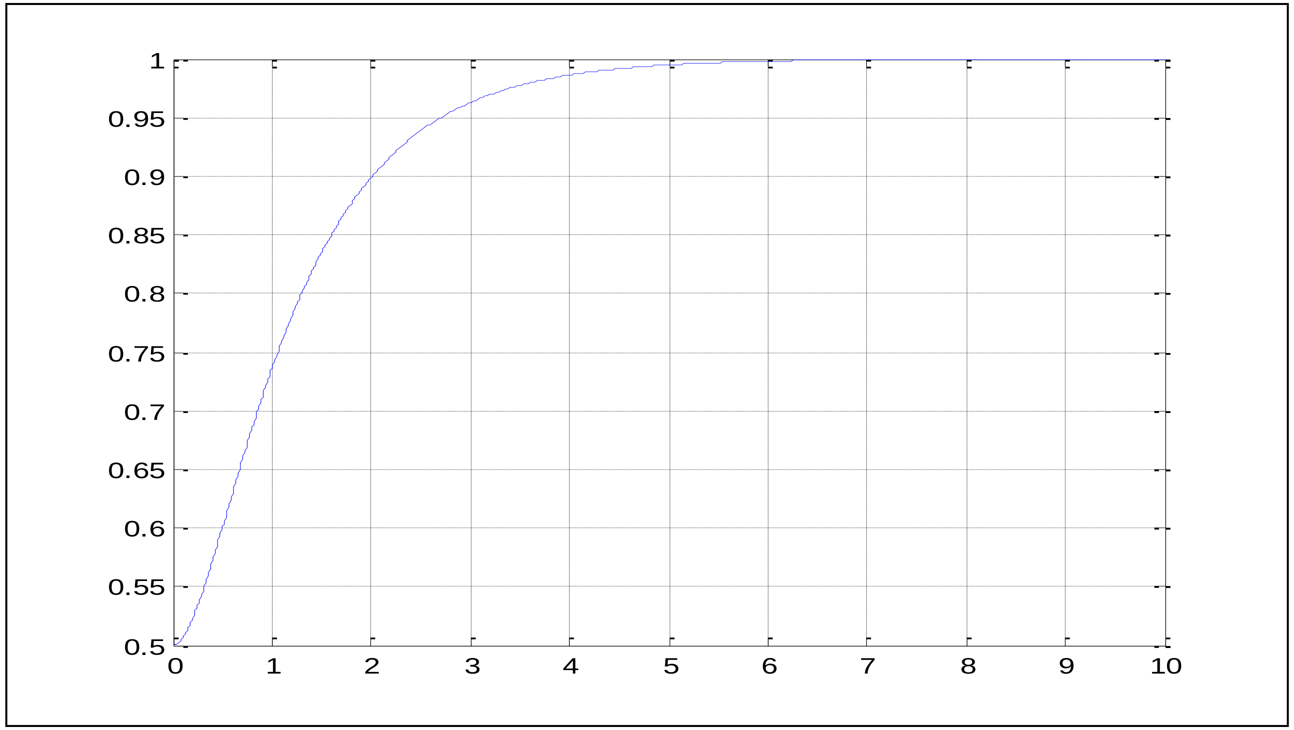
$$x_2 = v_C = 1 - 0.75e^{-t} + 0.25e^{-3t}$$

Other variables of the circuit can now be computed. For example, the voltage across the inductor is

$$v_L = L\frac{di_L}{dt} = \frac{1}{4}\frac{d}{dt}(e^{-t} - e^{-3t}) = -\frac{1}{4}e^{-t} + \frac{3}{4}e^{-3t}$$

We use the MATLAB script below to plot the relation

```
t=0:0.01:10;
x2=1-0.75.*exp(-t)+0.25.*exp(-3.*t);
plot(t,x2); grid
```



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