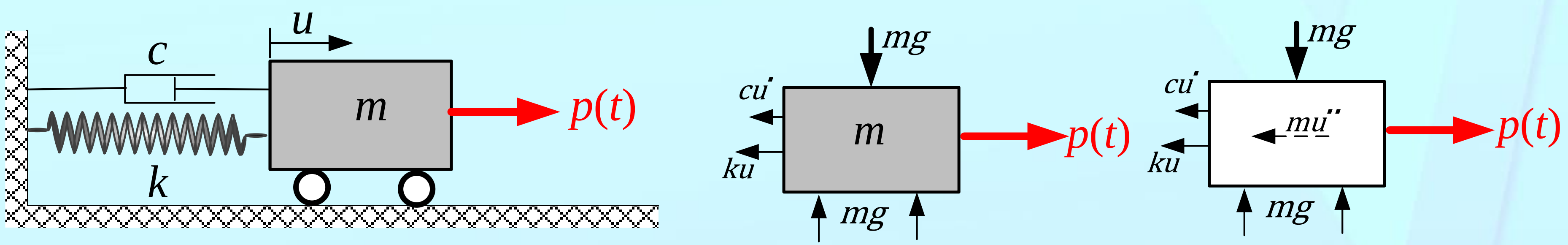
Response to Harmonic Excitation



An harmonic excitation can be described either by means of a sine function: $p(t)=p_0\sin\Omega t$, or by means of a cosine function: $p(t)=p_0\cos\Omega t$.

Equation of Motion (E.o.M.):

$$m\ddot{u}+c\dot{u}+ku=p(t)=\begin{cases} p_0\cos\Omega t \\ p_0\sin\Omega t \end{cases}$$

The complete response (solution) will be the sum of the transient (homogenous) and steady-state (particular)components.

$$u(t)=\underbrace{e^{-\xi\omega t}(A\cos\omega_D t+B\sin\omega_D t)}_{\text{transient}}+\underbrace{C\cos\Omega t+D\sin\Omega t}_{\text{steady state}}$$

Find C & D, for the cosine and sine functions of harmonic excitation

Response to Harmonic Excitation

Undamped harmonic vibrations

$$m\ddot{u} + ku = p_0 \cos\Omega t \qquad \ddot{u} + \omega_n^2 u = (p_0 / m)\cos\Omega t \qquad \ddot{u} + \omega_n^2 u = \omega_n^2 u_{st} \cos\Omega t$$

A steady-state response (A particular solution) can be

$$u_p(t) = C\cos\Omega t \qquad \ddot{u}_p(t) = -C\Omega^2 \cos\Omega t \qquad (-C\Omega^2 + C\omega_n^2)\cos\Omega t = \omega_n^2 u_{st} \cos\Omega t$$

$$(\omega_n^2 - \Omega^2)C = \omega_n^2 u_{st} \qquad C = \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)} \qquad C = \frac{u_{st}}{(1 - r^2)} \quad \text{where} \quad r = \frac{\Omega}{\omega_n}$$

The transient response (The homogeneous solution) is

$$u_h(t) = A\cos\omega_n t + B\sin\omega_n t$$

The complete solution is(A homogeneous solution) is

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)}\cos\Omega t$$

Response to Harmonic Excitation

Undamped harmonic vibrations

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)}\cos\Omega t$$

By means of the initial conditions given by , $u(0) = u_0 \ \& \ \dot{u}(0) = \dot{u}_0$

the constants A and B can be calculated as follows:

$$A = u_0 - \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)} \quad \& \quad B = \frac{\dot{u}_0}{\omega_n}$$

$$u(t) = \left(u_0 - \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)}\right)\cos\omega_n t + \left(\frac{\dot{u}_0}{\omega_n}\right)\sin\omega_n t + \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)}\cos\Omega t$$

For two null initial conditions the complete solution is,

$$u(t) = \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)}(\cos\Omega t - \cos\omega_n t)$$

Response to Harmonic Excitation

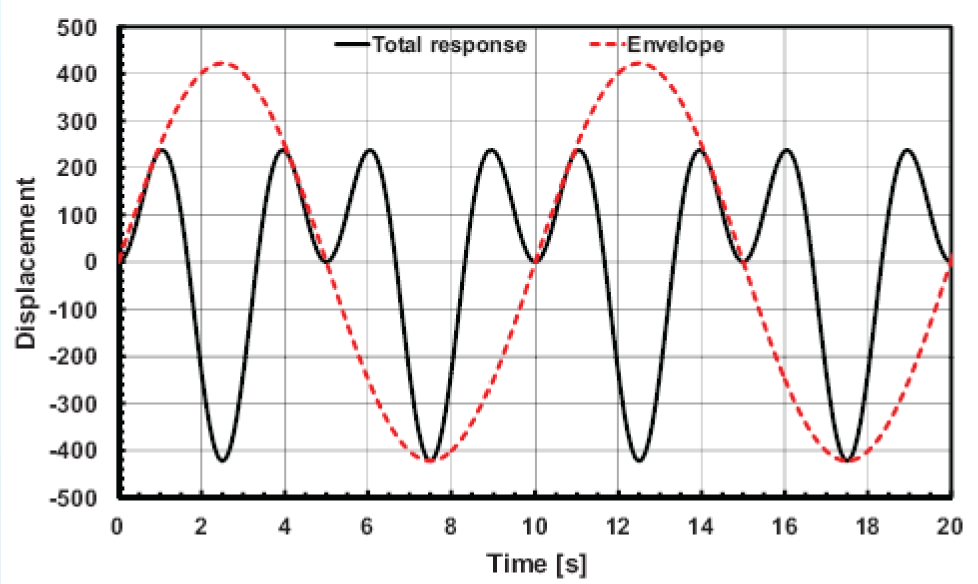
Undamped harmonic vibrations

$$u(t) = \frac{\omega_n^2 u_{st}}{(\omega_n^2 - \Omega^2)} (\cos \Omega t - \cos \omega_n t)$$

Using the trigonometric identity $\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$

$$u(t) = \frac{2\omega_n^2 u_{st}}{(\Omega^2 - \omega_n^2)} \sin\left(\frac{\Omega - \omega_n}{2} t\right) \sin\left(\frac{\Omega + \omega_n}{2} t\right)$$

Case 1: Natural frequency SDoF 0.2 Hz, excitation frequency 0.4 Hz. $\Omega / \omega_n = 2$

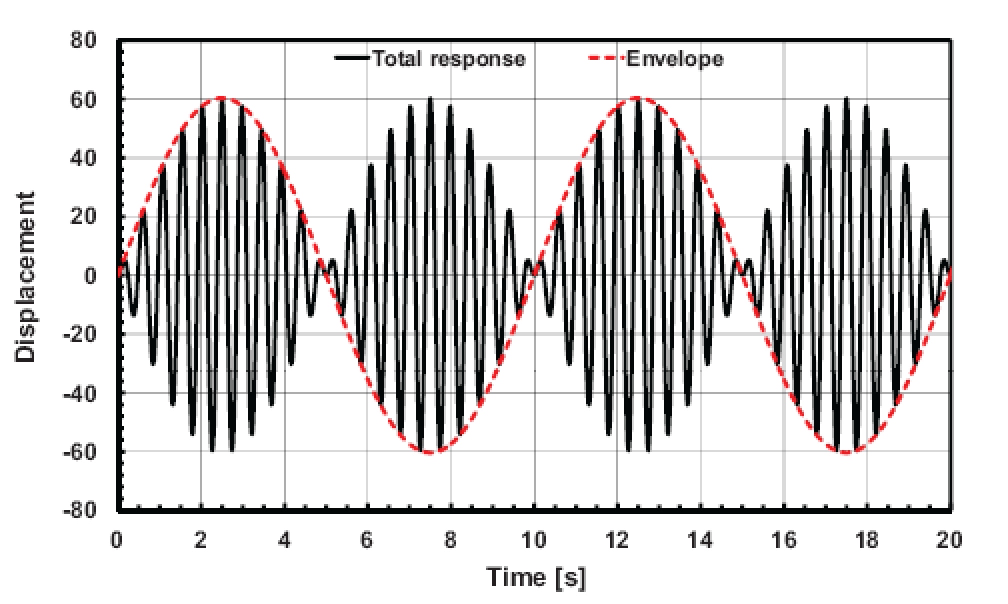


Response to Harmonic Excitation

Undamped harmonic vibrations

$$u(t)=\frac{2\omega_n^2 u_{st}}{(\Omega^2-\omega_n^2)}\sin\left(\frac{\Omega-\omega_n}{2}t\right)\sin\left(\frac{\Omega+\omega_n}{2}t\right)$$

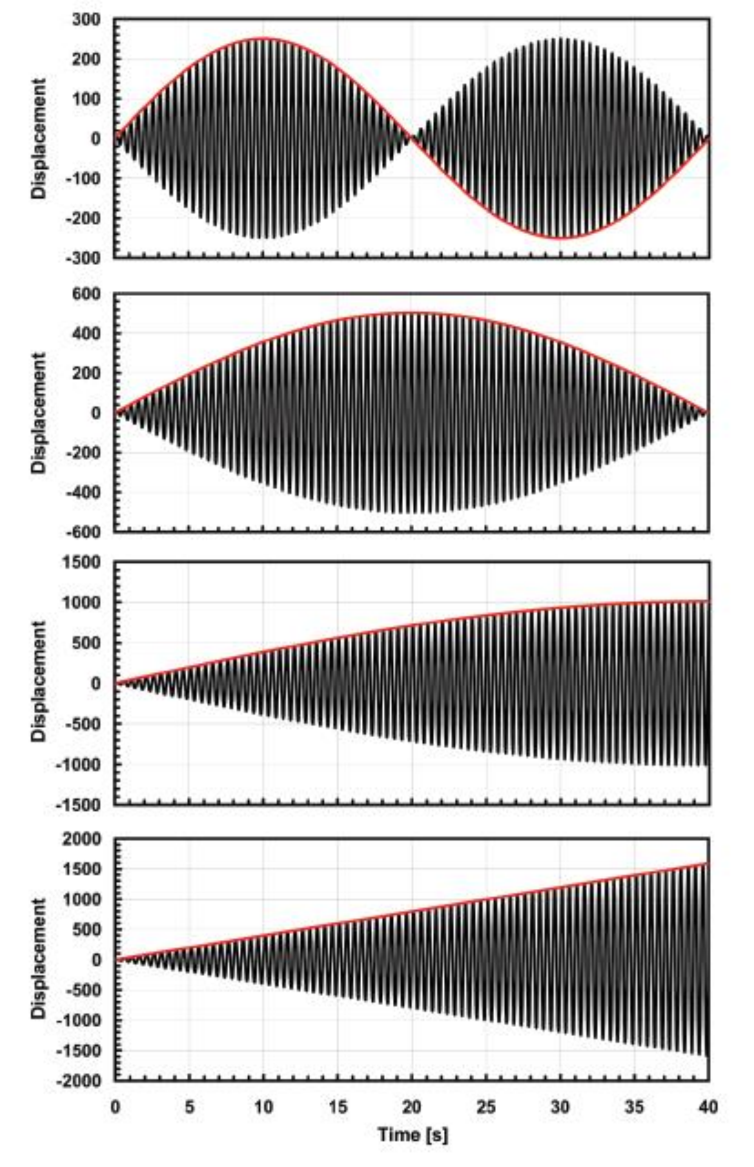
Case 2: Natural frequency SDoF 2.0 Hz, excitation frequency 2.2 Hz. $\Omega/\omega_n=1.1$



A beat is always present, but is more evident when the natural frequency of the SDoF system and the excitation frequency are close

Response to Harmonic Excitation

Undamped harmonic vibrations



$\Omega / \omega_n = 1.025$

$\Omega / \omega_n = 1.0125$

$\Omega / \omega_n = 1.00625$

$\Omega / \omega_n = 1$
Resonance

Response to Harmonic Excitation

Undamped harmonic vibrations

Resonant excitation ($\Omega = \omega_n$)

$$\ddot{u} + \omega_n^2 u = \omega_n^2 u_{st} \cos \Omega t \qquad\qquad \ddot{u} + \omega_n^2 u = \omega_n^2 u_{st} \cos \omega_n t$$

A steady-state response (A particular solution) could not be $u_p(t) = C \cos \omega_n t$

Another possible choice is $u_p(t) = Ct \sin \omega_n t$

$$\dot{u}_p(t) = C \sin \omega_n t + Ct\,\omega_n \cos \omega_n t$$

$$\ddot{u}_p(t) = 2C\,\omega_n \cos \omega_n t - Ct\,\omega_n^2 \sin \omega_n t$$

Substituting into the E. o. M.

$$2C\,\omega_n \cos \omega_n t - Ct\,\omega_n^2 \sin \omega_n t + Ct\,\omega_n^2 \sin \omega_n t = \omega_n^2 u_{st} \cos \omega_n t \qquad\qquad 2C = \omega_n u_{st}$$

So the particular solution is $u_p(t) = \left(\dfrac{\omega_n u_{st}}{2} \right) t \sin \omega_n t$

Response to Harmonic Excitation

Undamped harmonic vibrations

The transient response (The homogeneous solution) is

$$u_h(t) = A\cos\omega_n t + B\sin\omega_n t$$

The complete solution is(A homogeneous solution) is

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \left(\frac{\omega_n u_{st}}{2}\right) t\sin\omega_n t$$

By means of the initial conditions given by , $u(0) = u_0 \ \& \ \dot{u}(0) = \dot{u}_0$

the constants A and B can be calculated as follows: $A = u_0 \ \& \ B = \dot{u}_0/\omega_n$

$$u(t) = u_0\cos\omega_n t + \left(\frac{\dot{u}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{\omega_n u_{st}}{2}\right) t\sin\omega_n t$$

For two null initial conditions the homogeneous part of the solution falls away and the complete solution reduces to the particular solution,

$$u(t) = \left(\frac{\omega_n u_{st}}{2}\right) t\sin\omega_n t$$

Response to Harmonic Excitation

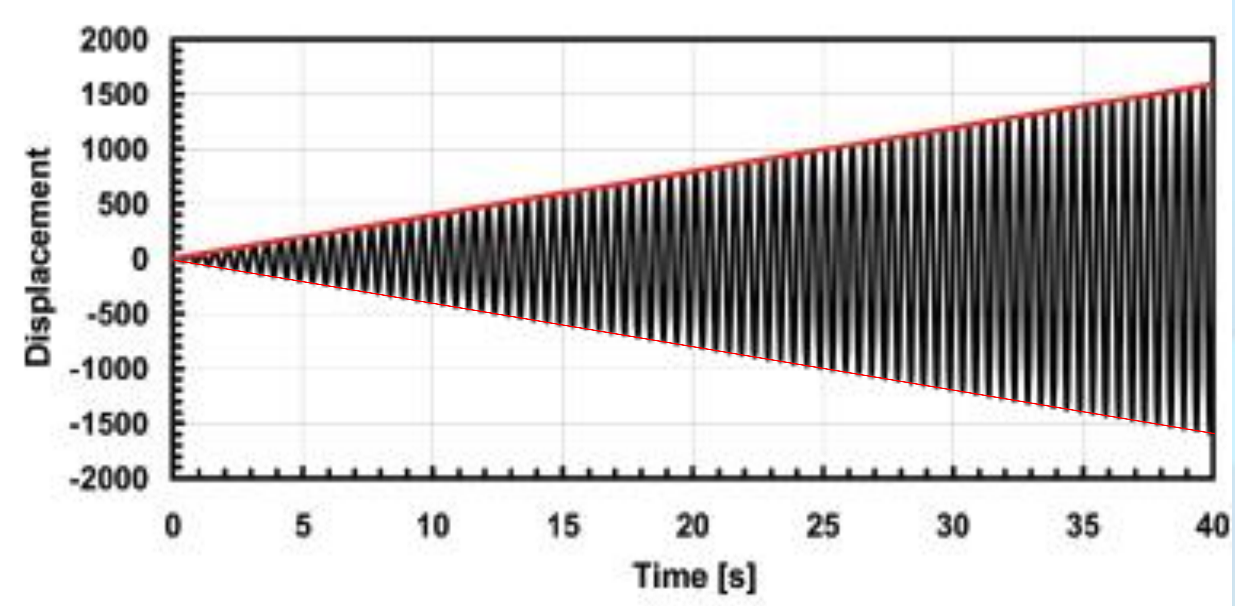
Undamped harmonic vibrations

$$u(t) = \left(\frac{\omega_n u_{st}}{2} \right) t \sin \omega_n t$$

This is a sinusoidal vibration with increasing amplitude: $C = (\omega_n u_{st}/2)t$.

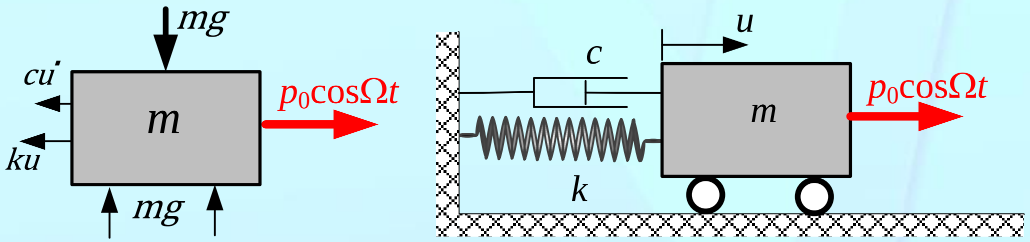
The amplitude grows linearly with time and when: $t \to \infty$, $C \to \infty$.

After infinite time the amplitude of the vibration is infinite as well.



Response to Harmonic Excitation

Damped harmonic vibrations



$$m\ddot{u} + c\dot{u} + ku = p_0 \cos\Omega t \qquad \ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = (p_0/m)\cos\Omega t$$

Canonical E. o. M. $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = \omega_n^2 u_{st} \cos\Omega t$

Particular solution $u_p(t) = C\cos\Omega t + D\sin\Omega t$

$$\dot{u}_p(t) = -C\Omega\sin\Omega t + D\Omega\cos\Omega t$$

$$\ddot{u}_p(t) = -C\Omega^2\cos\Omega t - D\Omega^2\sin\Omega t$$

Sub. Into E. o. M.

$$-C\Omega^2\cos\Omega t - D\Omega^2\sin\Omega t + 2\xi\omega_n\left(-C\Omega\sin\Omega t + D\Omega\cos\Omega t\right) + \omega_n^2 u = \omega_n^2 u_{st}\cos\Omega t$$

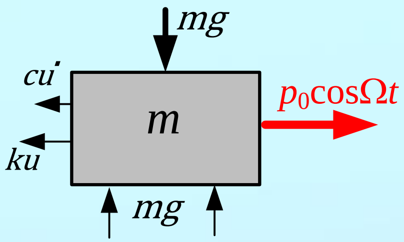
$$\left(\left(\omega_n^2 - \Omega^2\right)C + 2\xi\omega_n\Omega D\right)\cos\Omega t + \left(-2\xi\omega_n\Omega C + \left(\omega_n^2 - \Omega^2\right)D\right)\sin\Omega t = \omega_n^2 u_{st}\cos\Omega t$$

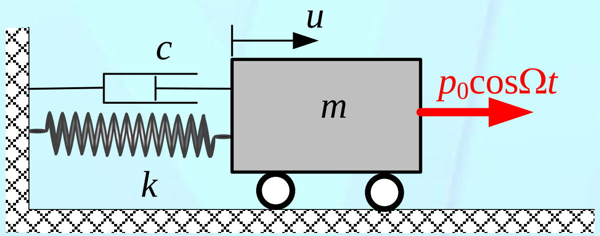
This is true at any time t, So

$$\left.\begin{aligned} &\left(\omega_n^2 - \Omega^2\right)C + 2\xi\omega_n\Omega D = \omega_n^2 u_{st} \\ &-2\xi\omega_n\Omega C + \left(\omega_n^2 - \Omega^2\right)D = 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} C &= \omega_n^2 u_{st}\frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2} \\ D &= \omega_n^2 u_{st}\frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2} \end{aligned}\right.$$

Response to Harmonic Excitation

Damped harmonic vibrations





Canonical E. o. M. $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = \omega_n^2 u_{st}\cos\Omega t$

Particular solution $u_p(t) = C\cos\Omega t + D\sin\Omega t$

$$C = \omega_n^2 u_{st}\frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2} \quad \text{and} \quad D = \omega_n^2 u_{st}\frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}$$

$$u_h(t) = e^{-\xi\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) \quad \text{with} \quad \omega_D = \omega_n\sqrt{1-\xi^2}$$

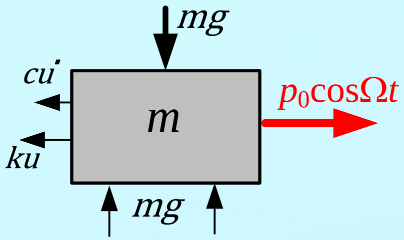
By means of the initial conditions the constants A and B, can be determined

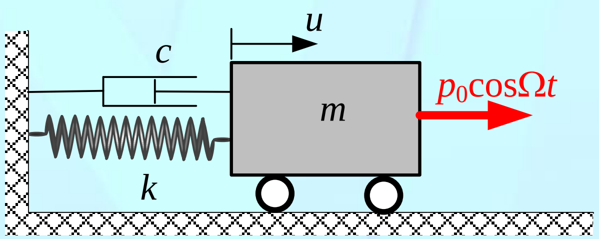
Denominations:

- Homogeneous part of the solution: **“transient”**
- Particular part of the solution: **“steady-state”**

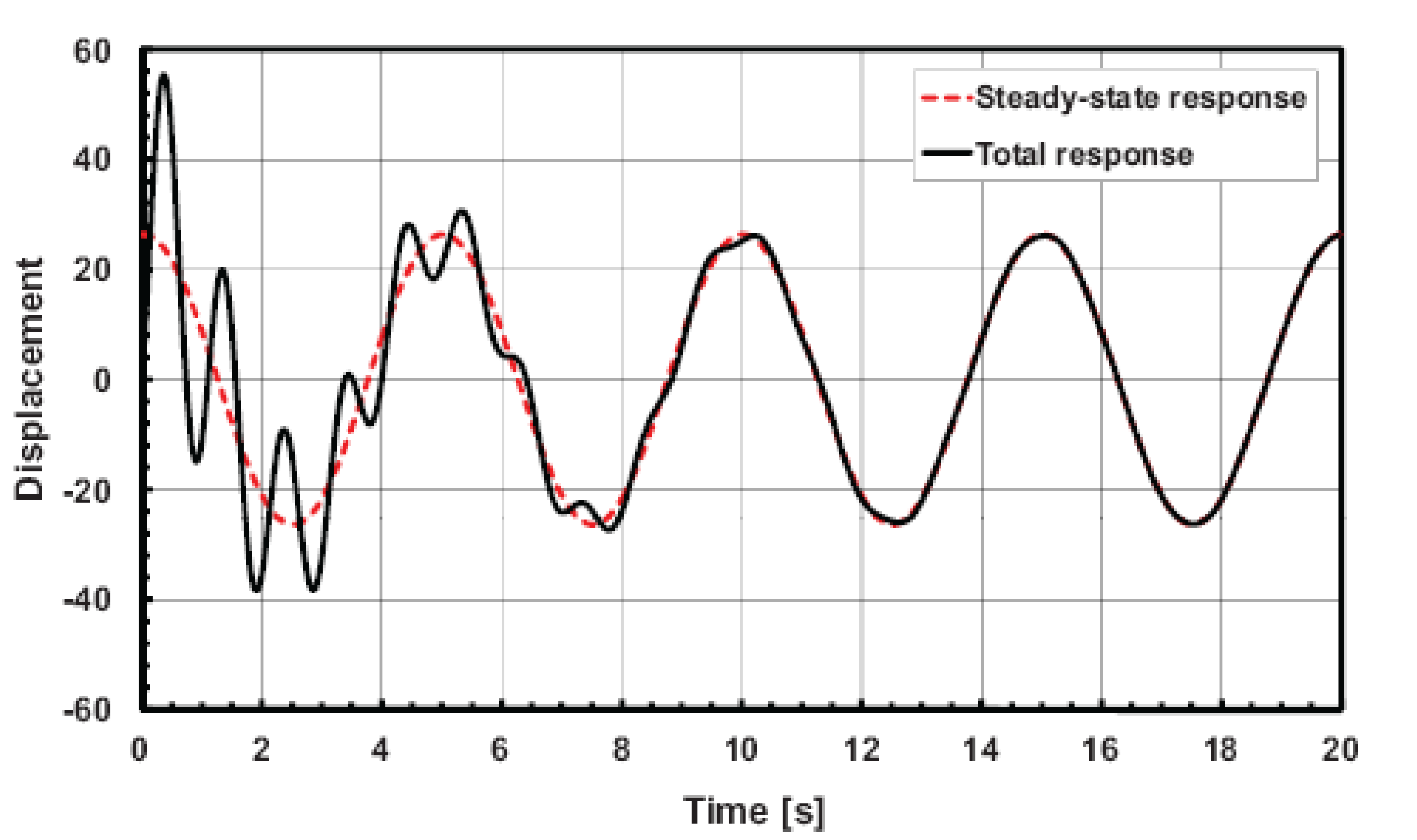
Visualization of the solution is illustrated in the next example

Damped harmonic vibrations



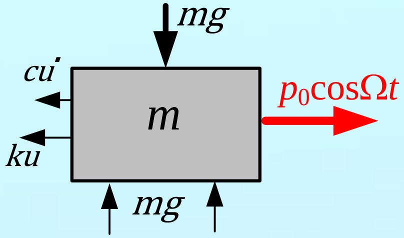


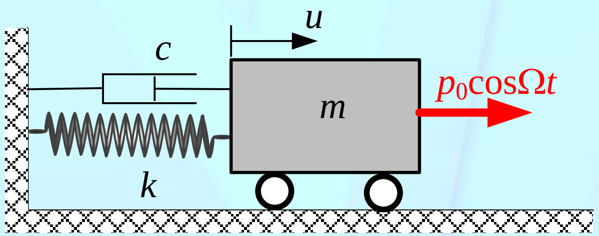
Example 1: $\omega_n = 2\pi$ [rad/sec], $\Omega = 0.4\pi$ [rad/sec], $\xi = 5\%$, u_{st}=25mm, u_0=0, $\dot{u}_0 = u_{st}\,\omega_n$



Response to Harmonic Excitation

Damped harmonic vibrations





Resonant excitation ($\Omega = \omega_n$)

Canonical E. o. M. $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = \omega_n^2 u_{st}\cos\Omega t$

$$u_p(t) = C\cos\Omega t + D\sin\Omega t$$

$$C = \omega_n^2 u_{st}\frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2} \quad \text{and} \quad D = \omega_n^2 u_{st}\frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}$$

By substituting **($\Omega = \omega_n$)** in thee two expressions constants C and D, becomes:

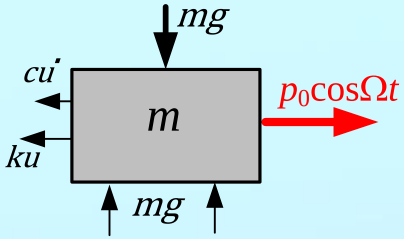
$$C = 0 \quad \text{and} \quad D = \frac{u_{st}}{2\xi}$$

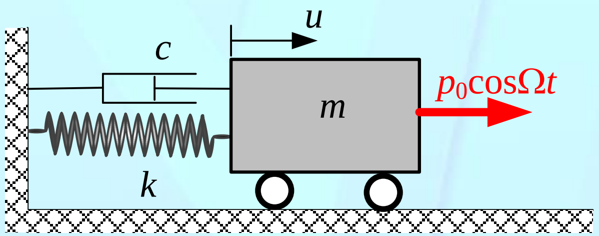
This means that if damping is present, the resonant excitation is not a special case any more, and the complete solution of the differential equation is:

$$u(t) = e^{-\xi\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + \frac{u_{st}}{2\xi}\sin\omega_n t$$

Response to Harmonic Excitation

Damped harmonic vibrations





Resonant excitation ($\Omega = \omega_n$)

Canonical E. o. M. $$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = \omega_n^2 u_{st}\cos\Omega t$$

$$u_p(t) = C\cos\Omega t + D\sin\Omega t$$

$$C = \omega_n^2 u_{st}\frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2} \quad \text{and} \quad D = \omega_n^2 u_{st}\frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}$$

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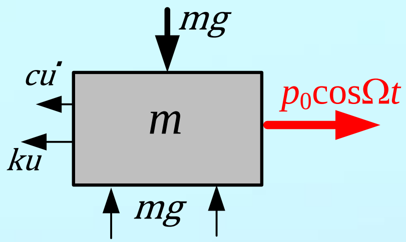
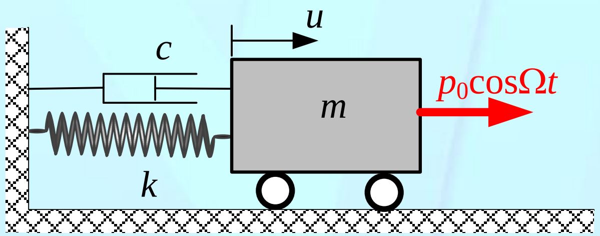
Response to Harmonic Excitation

Damped harmonic vibrations

Resonant excitation ($\Omega = \omega_n$)





$$u(t) = e^{-\xi\omega_n t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{u_{st}}{2\xi}\sin\omega_n t$$

By means of the initial conditions the constants *A* and *B*, can be determined. For example in the special case, $u_0 = 0$ & $\dot{u}_0 = 0$, *A* & *B*, are

$$A = 0 \quad \text{and} \quad B = -\frac{u_{st}}{2\xi\sqrt{1-\xi^2}}$$

$$u(t) = \frac{u_{st}}{2\xi}\left(\sin\omega_n t - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\omega_D t\right)$$

After a certain time, the homogeneous part of the solution vanishes and what remains is a sinusoidal oscillation of the maximum limited amplitude: ($u_{max} = u_{st}/2\xi$)

For small damping ratios ($\xi < 0.2$), $\omega_n \approx \omega_D$ and $(1-\xi)^{1/2} \approx 1$, hence $u(t)$ becomes:

$$u(t) = u_{max}\left(1 - e^{-\xi\omega_n t}\right)\sin\omega_n t$$

Response to Harmonic Excitation

Damped harmonic vibrations

Dynamic Amplification Factor

The steady-state displacement of a system due to harmonic excitation is the dominant part of its response. This steady-state response is given by

$$u_p(t) = C\cos\Omega t + D\sin\Omega t$$

Where $C = \omega_n^2 u_{st} \dfrac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}$ and $D = \omega_n^2 u_{st} \dfrac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}$

By means of the trigonometric identity:

$$a\cos\alpha + b\sin\alpha = (a^2 + b^2)^{1/2}\cos(\alpha - \beta) \quad \text{with} \quad \tan\beta = b/a$$

The steady-state response can be transformed as follows

$$u_p(t) = u_{\max}\cos(\Omega t - \varphi)$$

It is a cosine vibration with the maximum dynamic amplitude $u_{\max}$, given by

$$u_{\max} = (C^2 + D^2)^{1/2}$$

and the phase angle φ obtained from:

$$\tan\varphi = D/C$$

Response to Harmonic Excitation

Damped harmonic vibrations

Dynamic Amplification Factor

Substitution of C and D, in u_{max} expression gives

$$u_{max} = \sqrt{\left[\omega_n^2 u_{st} \frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}\right]^2 + \left[\omega_n^2 u_{st} \frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}\right]^2}$$

$$u_{max} = \omega_n^2 u_{st} \sqrt{\left[\frac{\left(\omega_n^2 - \Omega^2\right)}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}\right]^2 + \left[\frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}\right]^2}$$

$$u_{max} = \omega_n^2 u_{st} \sqrt{\frac{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}{\left[\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2\right]^2}}$$

$$u_{max} = \omega_n^2 u_{st} \frac{1}{\sqrt{\left(\omega_n^2 - \Omega^2\right)^2 + \left(2\xi\omega_n\Omega\right)^2}}$$

$$u_{max} = u_{st} \frac{1}{\sqrt{\left(1 - \left(\Omega/\omega_n\right)^2\right)^2 + \left(2\xi\left(\Omega/\omega_n\right)\right)^2}} \qquad \mathrm{DAF} = \frac{u_{max}}{u_{st}} = \frac{1}{\sqrt{\left(1 - \left(\Omega/\omega_n\right)^2\right)^2 + \left(2\xi\left(\Omega/\omega_n\right)\right)^2}}$$

Response to Harmonic Excitation

Damped harmonic vibrations

Dynamic Amplification Factor

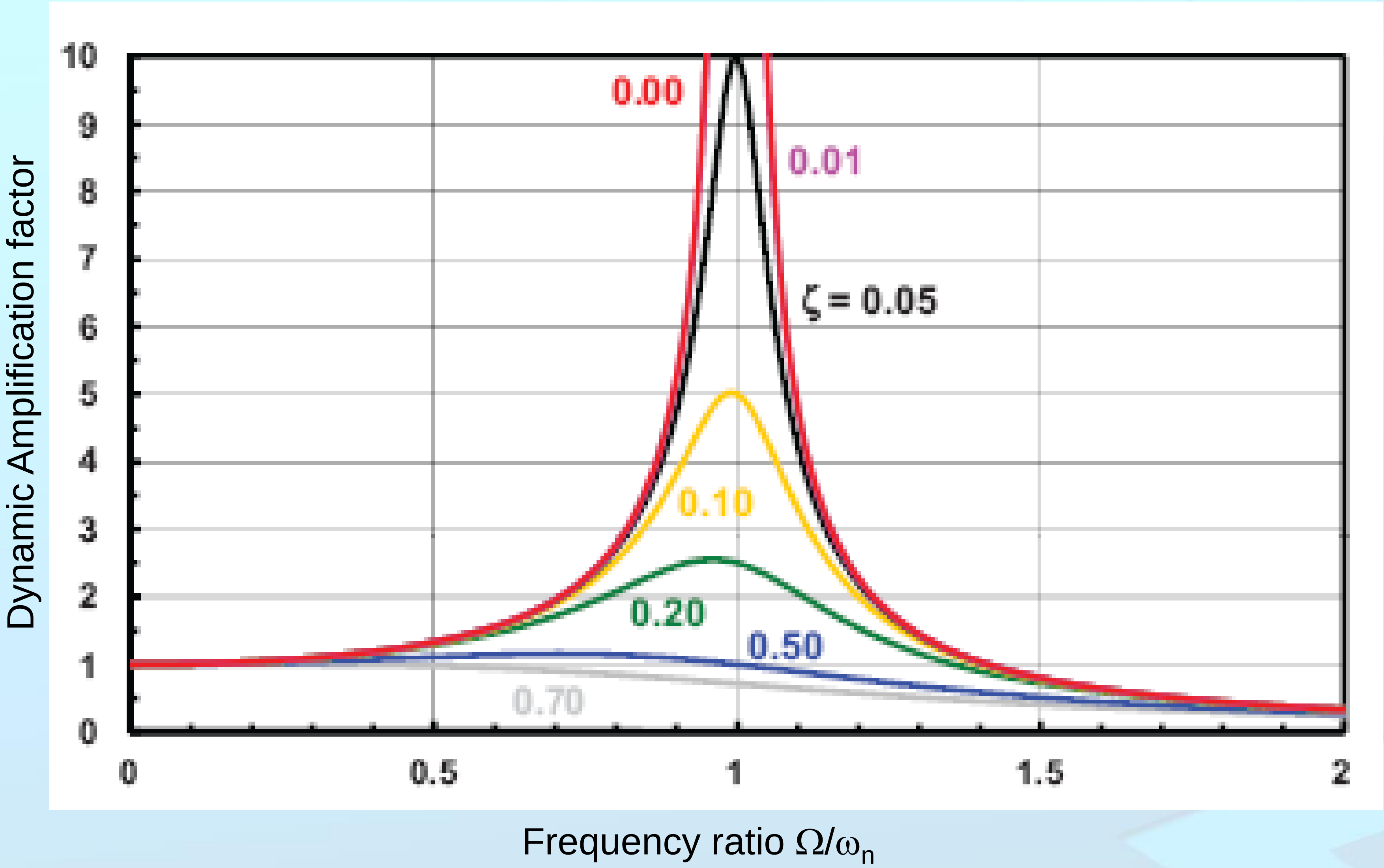
Substitution of C and D, in $\tan\varphi$ expression gives

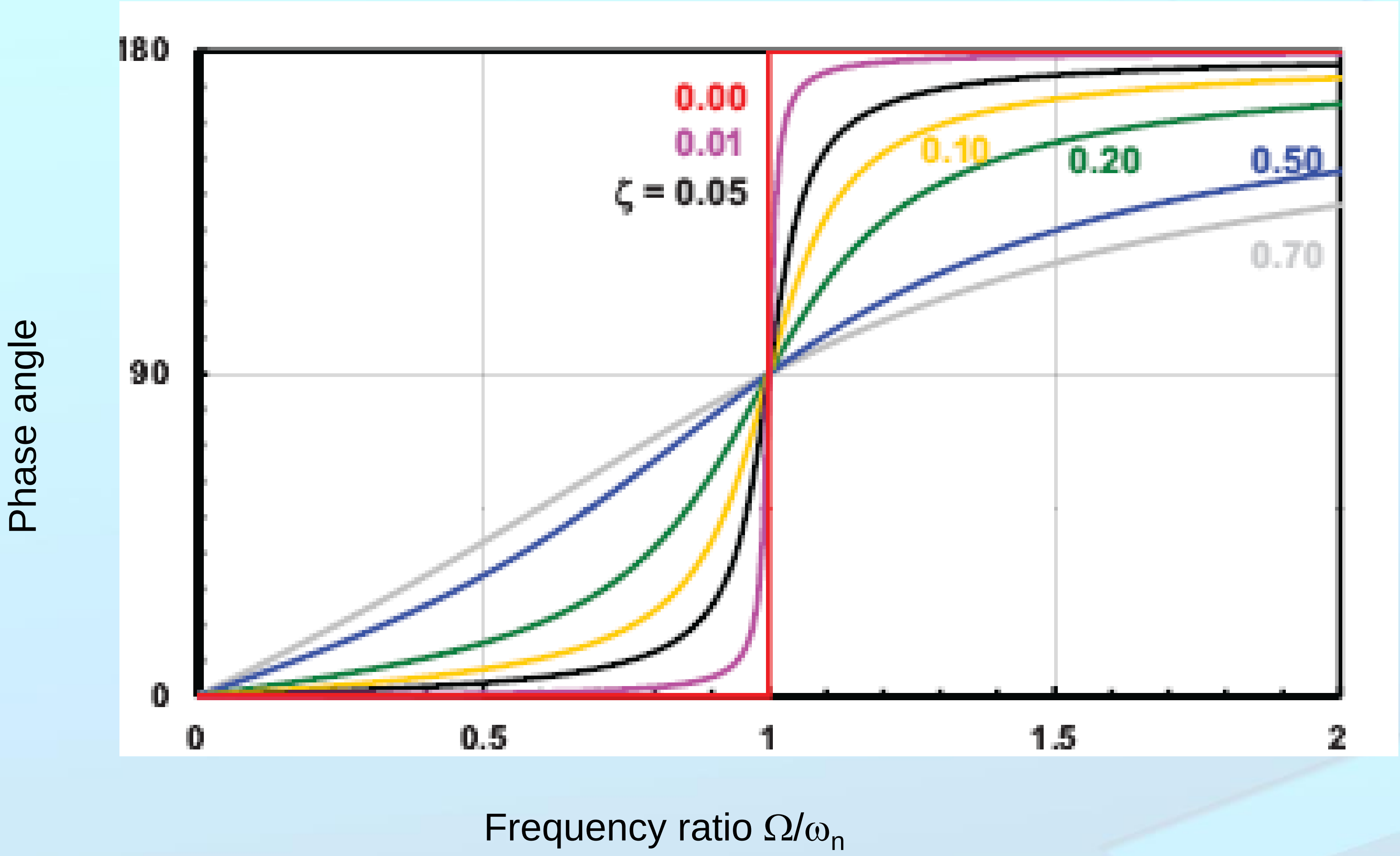
$$\tan\varphi = \frac{D}{C} = \frac{2\xi\omega_n\Omega}{\left(\omega_n^2 - \Omega^2\right)} = \frac{2\xi\left(\Omega/\omega_n\right)}{1-\left(\Omega/\omega_n\right)^2}$$

Defining the ratio $r = \Omega/\omega_n$, the two expressions simplify to

$$\text{DAF} = \frac{u_{\max}}{u_{st}} = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

$$\tan\varphi = \frac{2\xi r}{1-r^2}$$


Dynamic Amplification factor
Frequency ratio Ω/ωn
0.00
0.01
ζ = 0.05
0.10
0.20
0.50
0.70
0 0.5 1 1.5 2
0 1 2 3 4 5 6 7 8 9 10


180
90
0
0
0.5
1
1.5
2
0.00
0.01
ζ = 0.05
0.10
0.20
0.50
0.70
Phase angle
Frequency ratio Ω/ωn

Ex. 1. An undamped oscillator is driven by an harmonic loading. If the static displacement is u_{st}= 0.05m, determine the displacement response amplitude for the following frequency ratios: r = 0.2, 0.9, 1.1, 1.8 & 3.0.

Ex. 2 An undamped system consisting of a 10 kg mass and a spring of stiffness k =4 kN/m is acted upon by a harmonic force of magnitude P_0=0.5kN. The displacement amplitude of the steady-state response was observed to be 11 cm. Determine the frequency of the excitation force.

Ex. 3. An undamped system having a mass of 50 kg is excited by a harmonic force with magnitude P_0=100 N and an operating frequency of 10 Hz. The displacement amplitude of the steady-state response was observed to be 3.2 mm. Determine the spring constant k of the system.

Ex. 4. An undamped system having a mass of 10 kg and a spring of constant of k =8 N/mm is excited by a harmonic force with magnitude F_0=200 N and an operating frequency of 35 rad/sec. If the initial displacement is 21 mm and the initial velocity is 175mm/sec, determine the total displacement, velocity and acceleration of the mass at (a) t =2 sec, (b) t =4 sec and (c) t =6 sec.

Ex. 5. A portable eccentric mass shaker is sometimes used to evaluate the *in situ* dynamic properties of a structure, using two different frequencies and measuring the displacement amplitudes as well as the phase angles. Such a test was carried out on a single story building and the following responses were recorded:

(1) at Ω_1=18.30 rad/s, P_{o1}=837 kN, u_{max1}= 1.39mm & φ_1=8°;

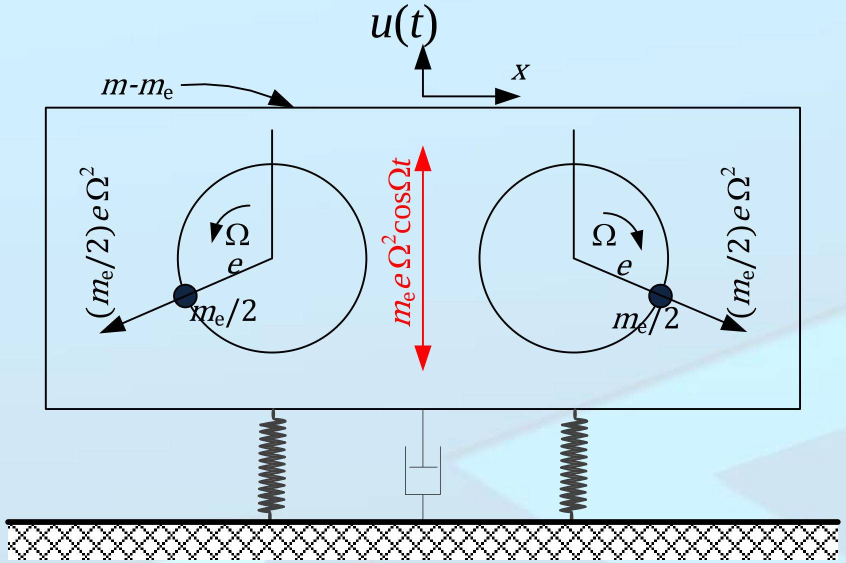
(2) at Ω_2=60.99 rad/s, P_{o2}= 9300 kN, u_{max2}= 3.32mm & φ_2=174.29°.

Compute the natural frequency ω_n & the damping ratio ξ for the structure

$$\text{DAF} = \frac{u_{max}}{u_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad \tan\varphi = \frac{2\xi r}{1-r^2}$$

$$u_{max} = \frac{u_{st}}{(1-r^2)} \frac{1}{\sqrt{1+\left[(2\xi r)/(1-r^2)\right]^2}}$$

$$u_{max} = \frac{u_{st}\cos\varphi}{(1-r^2)} = \frac{u_{st}\,\omega_n^2\cos\varphi}{(\omega_n^2-\Omega^2)} = \frac{P_0\cos\varphi}{m(\omega_n^2-\Omega^2)}$$



Ex.6. An undamped spring-mass system having a mass of 4.5 kg and a spring of constant of k =3.5 N/mm is excited by a harmonic force with magnitude F_0=100 N and an operating frequency of 18 rad/sec. If the initial displacement is 15 mm and the initial velocity is 150 mm/sec, determine

(a) The frequency ratio

(b) The amplitude o the forced response

(c) The displacement of the mass at t =2 sec

Ex.7. A Structure having a mass of 100 kg and a translational stiffness of 40000 N/m is excited by a harmonic force with magnitude F_0=500 N and an operating frequency of 2.5 Hz. The damping ratio for the structure is 0.10. For the steady-state vibration determine

(a) The amplitude of the steady-state displacement

(b) Its phase with respect to the exciting force, and

(c) The maximum velocity of the response