Numerical Analysis and Programming

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Ordinary Differential Equations

Learning Objectives of Lesson 1

- Recall basic definitions of ODEs:
  - Order
  - Linearity
  - Initial conditions
  - Solution
- Classify ODEs based on:
  - Order, linearity, and conditions.
- Classify the solution methods.

Objectives

- Solve Ordinary Differential Equations (**ODEs**).
- Appreciate the importance of numerical methods in solving ODEs.
- Assess the reliability of the different techniques.
- Select the appropriate method for any particular problem.

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

v is a function of one independent variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of more than one independent variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2v}{dt^2} + 6tv = 1$$

involve one or more
Ordinary derivatives of
unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more
partial derivatives of
unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more ordinary derivatives of unknown functions with respect to one independent variable

Examples :

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Example of ODE - Model of Falling Parachutist

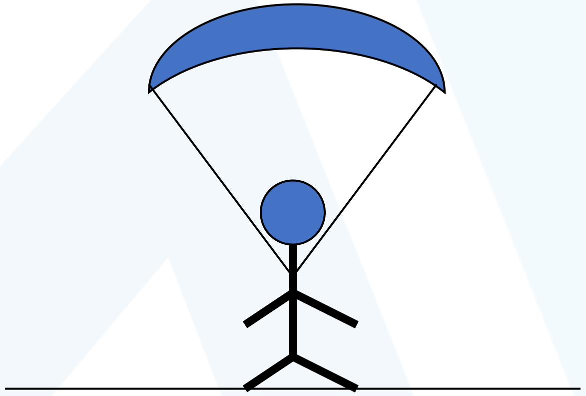
The velocity of a falling parachutist is given by:

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

$M : mass$

$c : drag\ coefficient$

$v : velocity$



Definitions

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

Ordinary differential equation

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(Dependent variable) unknown function to be determined

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(independent variable)
the variable with respect to which other variables are differentiated

Order of Differential Equation

The **order** of an ordinary differential equations is the order of the highest order derivative.

Examples :

$$\frac{dx(t)}{dt} - x(t) = e^{t}$$

First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

Solution of Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

Example:

$$\frac{dx(t)}{dt} + x(t) = 0$$

Solution $\quad x(t) = e^{-t}$

<u>Proof :</u>

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples :

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2 x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Nonlinear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt}x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

Solution of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form $x(t) = \cos(2t + c)$ (where c is a real constant) are solutions.

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

$$\frac{d^2y(x)}{dx^2} + 4y(x) = 0$$

Second order ODE

$$y(0) = a$$
$$\dot{y}(0) = b$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions

Auxiliary Conditions

Initial Conditions

- All conditions are at one point of the independent variable

Boundary Conditions

- The conditions are not at one point of the independent variable

Boundary Value and Initial Value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Classification of ODEs

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

Analytical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.

Numerical Solutions

- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

Classification of the Methods

Numerical Methods for Solving ODE

Single-Step Methods

Estimates of the solution at a particular step are entirely based on information on the previous step

Multiple-Step Methods

Estimates of the solution at a particular step are based on information on more than one step

Taylor Series Methods

Learning Objectives of Lesson 2

- Derive Euler formula using the Taylor series expansion.
- Solve the first order ODEs using Euler method.
- Assess the error level when using Euler method.
- Appreciate different types of errors in the numerical solution of ODEs.
- Improve Euler method using higher-order Taylor Series.

Taylor Series Method

The problem to be solved is a first order ODE:

$$\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0$$

Estimates of the solution at different base points:

$$y(x_0 + h),\ y(x_0 + 2h),\ y(x_0 + 3h),\ \ldots.$$

are computed using the truncated Taylor series expansions.

Taylor Series Expansion

Truncated Taylor Series Expansion

$$y(x_0+h) \approx \sum_{k=0}^{n} \frac{h^k}{k!}\left(\left.\frac{d^k y}{dx^k}\right|_{x=x_0,\,y=y_0} \right)$$

$$\approx y(x_0) + h \left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{h^2}{2!} \left.\frac{d^2 y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \ldots + \frac{h^n}{n!} \left.\frac{d^n y}{dx^n}\right|_{\substack{x=x_0,\\ y=y_0}}$$

The n^{th} order Taylor series method uses the n^{th} order Truncated Taylor series expansion.

Euler Method

- First order Taylor series method is known as Euler Method.
- Only the constant term and linear term are used in the Euler method.
- The error due to the use of the truncated Taylor series is of order $O(h^2)$.

First Order Taylor Series Method (Euler Method)

$$y(x_0+h) = y(x_0) + h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + O(h^2)$$

$Notation:$

$$x_n = x_0 + nh, \qquad y_n = y(x_n),$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_i,\\ y=y_i}} = f(x_i, y_i)$$

$Euler\ \ Method$

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

Euler Method

Problem :

Given the first order ODE : $\dot{y}(x) = f(x, y)$

with the initial condition : $y_0 = y(x_0)$

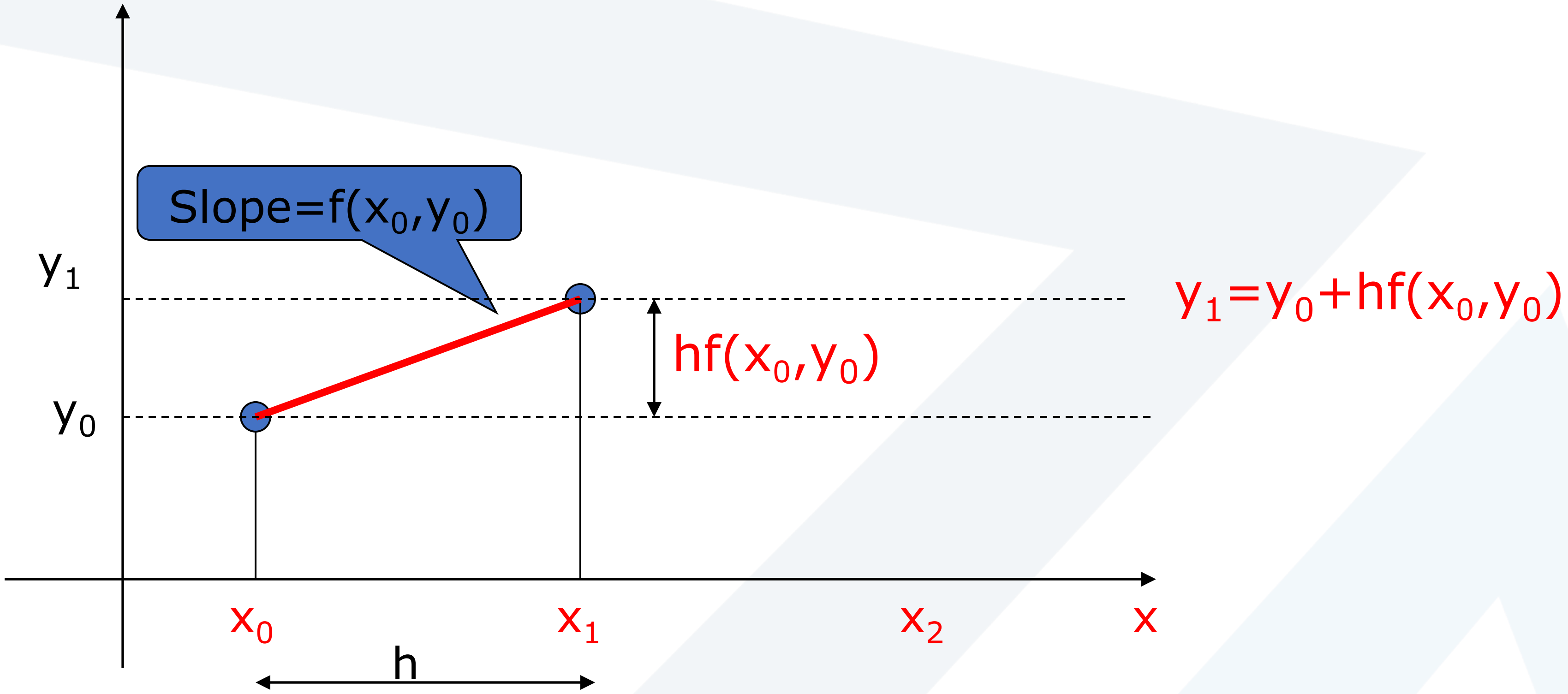
Determine : $y_i = y(x_0 + ih) \quad for\ i = 1,2,\ldots$

Euler Method :

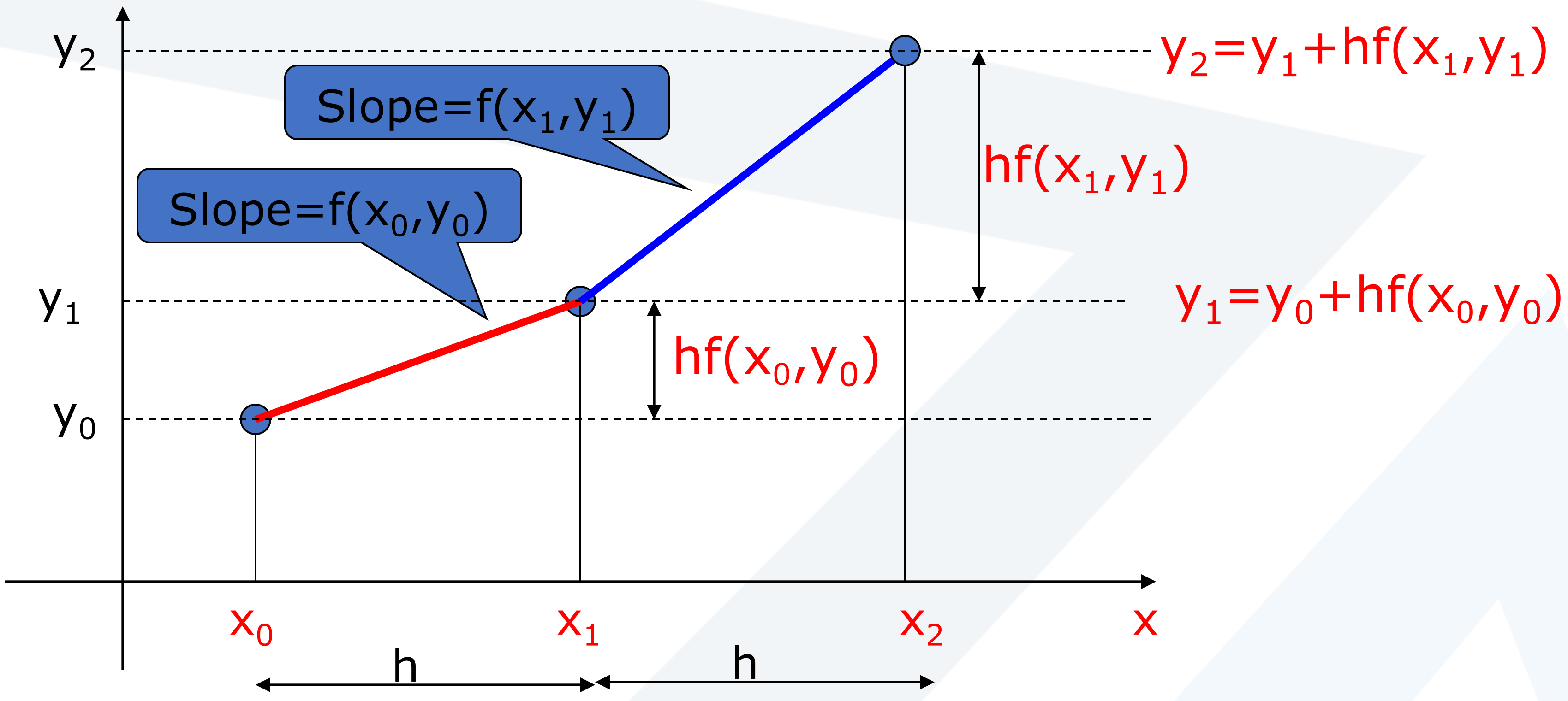
$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i, y_i) \quad for\ i = 1,2,\ldots$$

Interpretation of Euler Method



Interpretation of Euler Method



Example 1

Use Euler method to solve the ODE:

$$\frac{dy}{dx} = 1 + x^2, \qquad y(1) = -4$$

to determine y(1.01), y(1.02) and y(1.03).

Example 1

$$f(x,y)=1+x^2\,,\quad x_0=1,\ \ y_0=-4,\ \ h=0.01$$

Euler Method

$$y_{i+1}=y_i+h\,f(x_i,y_i)$$

$$Step1:\quad y_1=y_0+h\,f(x_0,y_0)=-4+0.01(1+(1)^2)=-3.98$$

$$Step2:\quad y_2=y_1+h\,f(x_1,y_1)=-3.98+0.01\left(1+(1.01)^2\right)=-3.9598$$

$$Step3:\quad y_3=y_2+h\,f(x_2,y_2)=-3.9598+0.01\left(1+(1.02)^2\right)=-3.9394$$

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Example 1

$$f(x,y)=1+x^2, \quad x_0=1, \quad y_0=-4, \quad h=0.01$$

Summary of the result:

i	xi	yi
0	1.00	-4.00
1	1.01	-3.98
2	1.02	-3.9595
3	1.03	-3.9394

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$$

Comparison with true value:

i	xi	yi	True value of yi
0	1.00	-4.00	-4.00
1	1.01	-3.98	-3.97990
2	1.02	-3.9595	-3.95959
3	1.03	-3.9394	-3.93909

Second Order Taylor Series Method

$$Given\ \ \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

Second order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + O(h^3)$$

$\dfrac{d^2y}{dx^2}$ needs to be derived analytically.

Third Order Taylor Series Method

$$\text{Given} \quad \frac{dy(x)}{dx} = f(y, x), \quad y(x_0) = y_0$$

Third order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \frac{h^3}{3!}\frac{d^3y}{dx^3} + O(h^4)$$

$\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ need to be derived analytically.

High Order Taylor Series Method

$$Given \quad \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

n th order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \dots + \frac{h^n}{n!}\frac{d^ny}{dx^n} + O(h^{n+1})$$

$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$ need to be derived analytically.

High Order Taylor Series Method

- High order Taylor series methods are more accurate than Euler method.

- But, the 2^{nd}, 3^{rd}, and higher order derivatives need to be derived analytically which may not be easy.

Example 2 – Second Order Taylor Series Method

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \; x(0) = 1, \qquad use \;\; h = 0.01$$

What is: $\dfrac{d^2x(t)}{dt^2}$?

Example 2 – Second Order Taylor Series Method

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1,\ x(0) = 1, \qquad use \quad h = 0.01$$

$$\frac{dx}{dt} = 1 - 2x^2 - t$$

$$\frac{d^2x(t)}{dt^2} = 0 - 4x\frac{dx}{dt} - 1 = -4x(1 - 2x^2 - t) - 1$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\,2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\,2} - t_i))$$

Example 2 – Second Order Taylor Series Method

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \quad x_0 = 1, \quad h = 0.01$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\,2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\,2} - t_i))$$

Step 1:

$$x_1 = 1 + 0.01(1 - 2(1)^2 - 0) + \frac{(0.01)^2}{2}(-1 - 4(1)(1 - 2 - 0)) = 0.9901$$

Step 2:

$$x_2 = 0.9901 + 0.01(1 - 2(0.9901)^2 - 0.01) + \frac{(0.01)^2}{2}(-1 - 4(0.9901)(1 - 2(0.9901)^2 - 0.01)) = 0.9807$$

Step 3:

$$x_3 = 0.9716$$

Example 2 – Second Order Taylor Series Method

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \quad x_0 = 1, \quad h = 0.01$$

Summary of the results:

i	t_i	x_i
0	0.00	1
1	0.01	0.9901
2	0.02	0.9807
3	0.03	0.9716