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Laplace Transformation and Transfer Function

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Contents

Basic Definitions

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots.. + a_{n-1} y' + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \ldots.. + b_{m-1} x' + b_m x$$

- Where *x* is the input of the system and *y* is the output of the system.

Laplace Transformation $$L|f(t)| = F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$G(s) = \frac{L[output]}{L[input]} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots.. + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1} s + a_n}$$

The Laplace Transform of the Unit Step Function $u_0(t)$

We begin with the definition of the Laplace transform, that is,

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

$$\mathscr{L}\{u_0(t)\} = \int_0^{\infty} 1e^{-st}dt = \left.\frac{-e^{-st}}{s}\right|_0^{\infty} = 0-\left(-\frac{1}{s}\right) = \frac{1}{s}$$

Thus, we have obtained the transform pair

$$\boxed{u_0(t) \Leftrightarrow \frac{1}{s}}$$

f(t)	F(s)
$u_0(t)$	$1/s$
t	$1/s^2$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Partial Fraction Expansion

Quite often the Laplace transform expressions are not in recognizable form, but in most cases appear in a rational form of s, that is,

$$F(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0}$$

The coefficients a_k and b_k are real numbers for $k = 1, 2, \ldots, n$, and if the highest power m of $N(s)$ is less than the highest power n of $D(s)$, i.e., $m < n$, $F(s)$ is said to be expressed as a *proper rational function*. If $m \geq n$, $F(s)$ is an *improper rational function*.

In a proper rational function, the roots of N(s) are found by setting N(s) = 0; these are called the *zeros* of F(s). The roots of D(s), found by setting D(s) = 0, are called the *poles* of F(s). We assume that F(s) is a proper rational function. Then, it is customary and very convenient to make the coefficient of s^n unity; thus, we rewrite F(s) as

$$F(s) = \frac{N(s)}{D(s)} = \frac{\frac{1}{a_n}(b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \ldots + b_1 s + b_0)}{s^n + \frac{a_{n-1}}{a_n} s^{n-1} + \frac{a_{n-2}}{a_n} s^{n-2} + \ldots + \frac{a_1}{a_n} s + \frac{a_0}{a_n}}$$

The zeros and poles can be real and distinct, repeated, complex conjugates, or combinations of real and complex conjugates.

If all the poles $p_1, p_2, p_3, \ldots, p_n$ of $F(s)$ are *distinct* (different from each another), we can factor the denominator of $F(s)$ in the form

$$F(s) = \frac{N(s)}{(s-p_1)\cdot(s-p_2)\cdot(s-p_3)\cdot\ldots\cdot(s-p_n)}$$

where p_k is distinct from all other poles. Next, using the *partial fraction expansion method*, we can express as

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

where $r_1, r_2, r_3, \ldots, r_n$ are the *residues*, and $p_1, p_2, p_3, \ldots, p_n$ are the *poles* of $F(s)$.

To evaluate the residue r_k, we multiply both sides by $(s-p_k)$; then, we let $s \to p_k$, that is,

$$r_k = \lim_{s \to p_k} (s-p_k)F(s) = (s-p_k)F(s)\Big|_{s=p_k}$$

Example

Use the partial fraction expansion method to simplify $F_1(s)$, and find the time domain function $f_1(t)$ corresponding to $F_1(s)$.

$$F_1(s) = \frac{3s+2}{s^2+3s+2}$$

Solution:

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{3s+2}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)}$$

The residues are

$$r_1 = \lim_{s \to -1}(s+1)F(s) = \left.\frac{3s+2}{(s+2)}\right|_{s=-1} = -1$$

and

$$r_2 = \lim_{s \to -2}(s+2)F(s) = \left.\frac{3s+2}{(s+1)}\right|_{s=-2} = 4$$

Therefore,

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{-1}{(s+1)} + \frac{4}{(s+2)}$$

and from Table , we find that

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

Therefore,

$$F_1(s) = \frac{-1}{(s+1)} + \frac{4}{(s+2)} \Leftrightarrow (-e^{-t} + 4e^{-2t}) = f_1(t)$$

The residues and poles of a rational function of polynomials , can be found easily using the MATLAB **residue(a,b)** function. For this example, we use the script

```
Ns = [3, 2];
Ds = [1, 3, 2];
[r, p, k] = residue(Ns,Ds)
```

```
r =
    4
   -1
p =
   -2
   -1
k =
    []
```

For the MATLAB script above, we defined **Ns** and **Ds** as two vectors that contain the numerator and denominator coefficients of F(s). When this script is executed, MATLAB displays the r and p vectors that represent the residues and poles respectively. The first value of the vector r is associated with the first value of the vector p, the second value of r is associated with the second value of p, and so on.

The vector k is referred to as the *direct term* and it is always empty (has no value) whenever F(s) is a proper rational function, that is, when the highest degree of the denominator is larger than that of the numerator. For this example, we observe that the highest power of the denominator is s^2, whereas the highest power of the numerator is s and therefore the direct term is empty.

We can also use the MATLAB **ilaplace(f)** function to obtain the time domain function directly from F(s). This is done with the script that follows.

```
syms s t;
Fs=(3*s+2)/(s^2+3*s+2);
ft=ilaplace(Fs)
```

```
ft =
4*exp(-2*t) - exp(-t)
```

Complex Poles

The partial fraction expansion method can also be used in this case, but it may be necessary to manipulate the terms of the expansion in order to express them in a recognizable form.

Example

Use the partial fraction expansion method to simplify F(s), and find the time domain function f(t) corresponding to F(s).

$$F(s) = \frac{s+3}{s^3 + 5s^2 + 12s + 8}$$

Solution:

Let us first express the denominator in factored form to identify the poles of F(s) using the MATLAB **factor(s)** symbolic function. Then,

```
syms s
factor(s^3 + 5*s^2 + 12*s + 8)
ans =
(s+1)*(s^2+4*s+8)
```

The **factor(s)** function did not factor the quadratic term. We will use the **roots(p)** function.

```
p=[1 4 8];
roots_p=roots(p)
roots_p =
-2.0000 + 2.0000i
-2.0000 - 2.0000i
```

Then,

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{s+3}{(s+1)(s+2+j2)(s+2-j2)}$$

or

$$F_3(s) = \frac{s+3}{s^3+5s^2+12s+8} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2+j2)} + \frac{r_2^*}{(s+2-j2)}$$

The residues are

$$r_1 = \left.\frac{s+3}{s^2+4s+8}\right|_{s=-1} = \frac{2}{5}$$

$$r_2 = \left.\frac{s+3}{(s+1)(s+2-j2)}\right|_{s=-2-j2} = \frac{1-j2}{(-1-j2)(-j4)} = \frac{1-j2}{-8+j4}$$

$$= \frac{(1-j2)}{(-8+j4)}\frac{(-8-j4)}{(-8-j4)} = \frac{-16+j12}{80} = -\frac{1}{5} + \frac{j3}{20}$$

$$r_2^* = \left(-\frac{1}{5} + \frac{j3}{20}\right)^* = -\frac{1}{5} - \frac{j3}{20} \qquad F(s) = \frac{2/5}{(s+1)} + \frac{-1/5+j3/20}{(s+2+j2)} + \frac{-1/5-j3/20}{(s+2-j2)}$$

The last two terms on the right side , do not resemble any Laplace transform pair that we derived Therefore, we will express them in a different form. We combine them into a single term , and now is written as

$$F(s) = \frac{2/5}{(s+1)} - \frac{1}{5}\cdot\frac{(2s+1)}{(s^2+4s+8)}$$

For convenience, we denote the first term on the right side as $F_1(s)$, and the second as $F_2(s)$.

Then,

$$F_1(s) = \frac{2/5}{(s+1)} \Leftrightarrow \frac{2}{5}e^{-t} = f_1(t)$$

Next, for $F_2(s)$

$$F_2(s) = -\frac{1}{5}\cdot\frac{(2s+1)}{(s^2+4s+8)}$$

From Table

$$e^{-at}\sin\omega t \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$$

Accordingly, we express $F_2(s)$ as

$$F_2(s) = -\frac{2}{5}\left(\frac{s+\frac{1}{2}+\frac{3}{2}-\frac{3}{2}}{(s+2)^2+2^2)}\right) = -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)} + \frac{-3/2}{(s+2)^2+2^2)}\right)$$

$$= -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{6/10}{2}\left(\frac{2}{(s+2)^2+2^2)}\right) \quad = -\frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2)}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2)}\right)$$

$$F_3(s) = F_{31}(s) + F_{32}(s) = \frac{2/5}{(s+1)} - \frac{2}{5}\left(\frac{s+2}{(s+2)^2+2^2}\right) + \frac{3}{10}\left(\frac{2}{(s+2)^2+2^2}\right)$$

$$\Leftrightarrow \frac{2}{5}e^{-t} - \frac{2}{5}e^{-2t}\cos 2t + \frac{3}{10}e^{-2t}\sin 2t = f_3(t)$$

Check with MATLAB:

```
syms s
Fs=(s + 3)/(s^3 + 5*s^2 + 12*s + 8);
ft=ilaplace(Fs)

ft =
(2*exp(-t))/5 - (2*exp(-2*t)*(cos(2*t) - (3*sin(2*t))/4))/5
```

Multiple Poles

In this case, $F(s)$ has simple poles, but one of the poles, say p_1, has a multiplicity m. For this condition, we express it as

$$F(s) = \frac{N(s)}{(s-p_1)^m(s-p_2)\ldots(s-p_{n-1})(s-p_n)}$$

Denoting the m residues corresponding to multiple pole p_1 as $r_{11}, r_{12}, r_{13}, \ldots, r_{1m}$, the partial fraction expansion is expressed as

$$F(s) = \frac{r_{11}}{(s-p_1)^m} + \frac{r_{12}}{(s-p_1)^{m-1}} + \frac{r_{13}}{(s-p_1)^{m-2}} + \ldots + \frac{r_{1m}}{(s-p_1)}$$
$$+ \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

For the simple poles $p_2, \ldots, p_n$, we proceed as before, that is, we find the residues from

$$r_k = \lim_{s \to p_k}(s-p_k)F(s) = (s-p_k)F(s)\Big|_{s=p_k}$$

The residues $r_{11}, r_{12}, r_{13}, \ldots, r_{1m}$ corresponding to the repeated poles, are found by multiplication of both sides by $(s-p)^m$. Then,

$$(s-p_1)^m F(s) = r_{11} + (s-p_1)r_{12} + (s-p_1)^2 r_{13} + \ldots + (s-p_1)^{m-1} r_{1m}$$

$$+ (s-p_1)^m \left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)} \right)$$

Next, taking the limit as $s \to p_1$ on both sides, we obtain

$$\lim_{s \to p_1} (s-p_1)^m F(s) = r_{11} + \lim_{s \to p_1} [(s-p_1)r_{12} + (s-p_1)^2 r_{13} + \ldots + (s-p_1)^{m-1} r_{1m}]$$

$$+ \lim_{s \to p_1} \left[(s-p_1)^m \left(\frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)} \right) \right]$$

$$r_{11} = \lim_{s \to p_1} (s-p_1)^m F(s)$$

and thus yields the residue of the first repeated pole.

The residue r_{12} for the second repeated pole p_1, is found by differentiating with respect to s and again, we let $s \to p_1$, that is,

$$r_{12} = \lim_{s \to p_1} \frac{d}{ds}[(s-p_1)^m F(s)]$$

In general, the residue r_{1k} can be found from

$$r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}}[(s-p_1)^m F(s)]$$

Example

Use the partial fraction expansion method to simplify F(s), and find the time domain function f(t) corresponding to F(s).

$$F(s) = \frac{s+3}{(s+2)(s+1)^2}$$

Solution:

We observe that there is a pole of multiplicity 2 at $s = -1$, and thus in partial fraction expansion form, $F(s)$ is written as

$$F(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{r_1}{(s+2)} + \frac{r_{21}}{(s+1)^2} + \frac{r_{22}}{(s+1)}$$

The residues are

$$r_1 = \left.\frac{s+3}{(s+1)^2}\right|_{s=-2} = 1$$

$$r_{21} = \left.\frac{s+3}{s+2}\right|_{s=-1} = 2$$

$$r_{22} = \left.\frac{d}{ds}\left(\frac{s+3}{s+2}\right)\right|_{s=-1} = \left.\frac{(s+2)-(s+3)}{(s+2)^2}\right|_{s=-1} = -1$$

$$F(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{1}{(s+2)} + \frac{2}{(s+1)^2} + \frac{-1}{(s+1)} \Leftrightarrow e^{-2t} + 2te^{-t} - e^{-t} = f_4(t)$$

```
syms s;
Fs=(s+3)/((s+2)*(s+1)^2);
ft=ilaplace(Fs)
exp(-2*t) - exp(-t) + 2*t*exp(-t)
```

We can use the following script to check the partial fraction expansion.

```
syms s
Ns = [1 3];              % Coefficients of the numerator N(s) of F(s)
expand((s + 1)^2);       % Expands (s + 1)^2 to s^2 + 2*s + 1;
d1 = [1 2 1];            % Coefficients of (s + 1)^2 = s^2 + 2*s + 1 term in D(s)
d2 = [0 1 2];            % Coefficients of (s + 2) term in D(s)
Ds=conv(d1,d2);          % Multiplies polynomials d1 and d2 to express the
                         % denominator D(s) of F(s) as a polynomial
[r,p,k]=residue(Ns,Ds)
```

```
r =
    1.0000
   -1.0000
    2.0000
p =
   -2.0000
   -1.0000
   -1.0000
k =
     []
```

Example

Use the partial fraction expansion method to simplify F(s), and find the time domain function f(t) corresponding to the given F(s).

$$F(s) = \frac{s^2 + 3s + 1}{(s+1)^3 (s+2)^2}$$

Solution:

We observe that there is a pole of multiplicity 3 at $s = -1$, and a pole of multiplicity 2 at $s = -2$. Then, in partial fraction expansion form, F(s) is written as

$$F(s) = \frac{r_{11}}{(s+1)^3} + \frac{r_{12}}{(s+1)^2} + \frac{r_{13}}{(s+1)} + \frac{r_{21}}{(s+2)^2} + \frac{r_{22}}{(s+2)}$$

The residues are

$$r_{11} = \left.\frac{s^2 + 3s + 1}{(s+2)^2}\right|_{s=-1} = -1$$

$$r_{12} = \left.\frac{d}{ds}\left(\frac{s^2 + 3s + 1}{(s+2)^2}\right)\right|_{s=-1}$$

$$= \left.\frac{(s+2)^2(2s+3) - 2(s+2)(s^2+3s+1)}{(s+2)^4}\right|_{s=-1} = \left.\frac{s+4}{(s+2)^3}\right|_{s=-1} = 3$$

$$r_{13} = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{s^2+3s+1}{(s+2)^2}\right)\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s^2+3s+1}{(s+2)^2}\right)\right]\Bigg|_{s=-1}$$

$$= \frac{1}{2}\frac{d}{ds}\left(\frac{s+4}{(s+2)^3}\right)\Bigg|_{s=-1} = \frac{1}{2}\left[\frac{(s+2)^3 - 3(s+2)^2(s+4)}{(s+2)^6}\right]\Bigg|_{s=-1}$$

$$= \frac{1}{2}\left(\frac{s+2-3s-12}{(s+2)^4}\right)\Bigg|_{s=-1} = \frac{-s-5}{(s+2)^4}\Bigg|_{s=-1} = -4$$

Next, for the pole at $s = -2$,

$$r_{21} = \frac{s^2+3s+1}{(s+1)^3}\Bigg|_{s=-2} = 1$$

and

$$r_{22} = \frac{d}{ds}\left(\frac{s^2+3s+1}{(s+1)^3}\right)\Bigg|_{s=-2} = \frac{(s+1)^3(2s+3) - 3(s+1)^2(s^2+3s+1)}{(s+1)^6}\Bigg|_{s=-2}$$

$$= \frac{(s+1)(2s+3) - 3(s^2+3s+1)}{(s+1)^4}\Bigg|_{s=-2} = \frac{-s^2-4s}{(s+1)^4}\Bigg|_{s=-2} = 4$$

By substitution of the residues, we obtain

$$F(s) = \frac{-1}{(s+1)^3} + \frac{3}{(s+1)^2} + \frac{-4}{(s+1)} + \frac{1}{(s+2)^2} + \frac{4}{(s+2)}$$

We will check the values of these residues with the MATLAB

```
syms s; % The function collect(s) below multiplies (s+1)^3 by (s+2)^2
% and we use it to express the denominator D(s) as a polynomial so that we can
% use the coefficients of the resulting polynomial with the residue function
Ds=collect(((s+1)^3)*((s+2)^2))
Ds =
s^5+7*s^4+19*s^3+25*s^2+16*s+4

Ns=[1 3 1]; Ds=[1 7 19 25 16 4]; [r,p,k]=residue(Ns,Ds)
```

```
r =
    4.0000
    1.0000
   -4.0000
    3.0000
   -1.0000
p =
   -2.0000
   -2.0000
   -1.0000
   -1.0000
   -1.0000
k =
    []
```

$$e^{-at} \Leftrightarrow \frac{1}{s+a} \qquad te^{-at} \Leftrightarrow \frac{1}{(s+a)^2} \qquad t^{n-1}e^{-at} \Leftrightarrow \frac{(n-1)!}{(s+a)^n}$$

and with these, we derive f(t) as

$$f(t) = -\frac{1}{2}t^2e^{-t} + 3te^{-t} - 4e^{-t} + te^{-2t} + 4e^{-2t}$$

We can verify with MATLAB as follows:

```
syms s;
Fs=-1/((s+1)^3) + 3/((s+1)^2) - 4/(s+1) + 1/((s+2)^2) + 4/(s+2);
ft=ilaplace(Fs)
ft =
4*exp(-2*t) - 4*exp(-t) + 3*t*exp(-t) + t*exp(-2*t) - (t^2*exp(-t))/2
```

Improper Rational Function

Our discussion thus far, was based on the condition that $F(s)$ is a proper rational function. However, if $F(s)$ is an improper rational function, that is, if $m \geq n$, we must first divide the numerator $N(s)$ by the denominator $D(s)$ to obtain an expression of the form

$$F(s) = k_0 + k_1 s + k_2 s^2 + \ldots + k_{m-n} s^{m-n} + \frac{N(s)}{D(s)}$$

where $N(s)/D(s)$ is a proper rational function.

Example

Derive the Inverse Laplace transform $f(t)$ of

$$F(s) = \frac{s^2 + 2s + 2}{s + 1}$$

Solution:

For this example, $F(s)$ is an improper rational function.

$$F(s) = \frac{s^2 + 2s + 2}{s + 1} = \frac{1}{s + 1} + 1 + s$$

Now, we recognize that

$$\frac{1}{s+1} \Leftrightarrow e^{-t}$$

and

$$1 \Leftrightarrow \delta(t)$$
$$s \Leftrightarrow \delta'(t)$$

and thus,

$$F(s) = \frac{s^2 + 2s + 2}{s + 1} = \frac{1}{s + 1} + 1 + s \Leftrightarrow e^{-t} + \delta(t) + \delta'(t) = f(t)$$

In general,

$$\frac{d^n}{dt^n}\delta(t) \Leftrightarrow s^n$$

We verify with MATLAB as follows:

```
Ns = [1 2 2];
 Ds = [1 1];
[r, p, k] = residue(Ns, Ds)
```

```
r =
   1
p =
  -1
k =
   1    1
```

The direct terms k= [1 1] above are the coefficients of $\delta(t)$ and $\delta'(t)$ respectively.

Example

$$F(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4}$$

```
p=[1 8 10 4]; r=roots(p) % Find the roots of D(s)
r =
  -6.5708
  -0.7146 + 0.3132i
  -0.7146 - 0.3132i
syms s
y=expand((s + 0.7146 - 0.3132j)*(s + 0.7146 + 0.3132j)) % Find quadratic form
y =
s^2 + (3573*s)/2500 + 3043737/5000000
```

```
3573/2500                        % Simplify coefficient of s

ans =
      1.4292

3043737/5000000                  % Simplify constant term

ans =
      0.6087
```

Therefore,

$$F(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4} = \frac{2s(s+3)}{(s+6.57)(s^2 + 1.43s + 0.61)}$$

Next, we perform partial fraction expansion.

$$F(s) = \frac{2s(s+3)}{(s+6.57)(s^2 + 1.43s + 0.61)} = \frac{r_1}{s+6.57} + \frac{r_2 s + r_3}{s^2 + 1.43s + 0.61}$$

$$r_1 = \left.\frac{2s(s+3)}{s^2+1.43s+0.61}\right|_{s=-6.57} = 1.36$$

The residues r_2 and r_3 are found from the equality

$$2s(s+3) = r_1(s^2+1.43s+0.61)+(r_2 s+r_3)(s+6.57)$$

Equating constant terms

$$0 = 0.61r_1 + 6.57r_3$$

and by substitution of the known value of r_1

$$r_3 = -0.12$$

Similarly, equating coefficients of s^2, we obtain

$$2 = r_1 + r_2$$

and using the known value of r_1, we obtain

$$r_2 = 0.64$$

By substitution

$$F(s) = \frac{1.36}{s+6.57} + \frac{0.64s-0.12}{s^2+1.43s+0.61}$$

$$e^{-at}\sin\omega t \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$$

or

$$F(s) = \frac{1.36}{s+6.57} + (0.64)\frac{s+0.715-0.91}{(s+0.715)^2+(0.316)^2}$$

$$= \frac{1.36}{s+6.57} + \frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2} - \frac{0.58}{(s+0.715)^2+(0.316)^2}$$

$$= \frac{1.36}{s+6.57} + \frac{0.64(s+0.715)}{(s+0.715)^2+(0.316)^2} - \frac{1.84\times0.316}{(s+0.715)^2+(0.316)^2}$$

Taking the Inverse Laplace

$$F(t) = (1.36e^{-6.57t} + 0.64e^{-0.715t}\cos0.316t - 1.84e^{-0.715t}\sin0.316t)u_0(t)$$

انتهت المحاضرة