# Lecture 7: Eigenvalues and Eigenvectors

CECC122: Linear Algebra and Matrix Theory

Manara University

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## ٧.1 Introduction Eigenvalues and Eigenvectors

- **Eigenvalue problem:**

  If $A$ is an $n \times n$ matrix, do there exist nonzero vectors $\mathbf{x}$ in $R^n$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

- **Eigenvalue and eigenvector:**

  $A$: an $n \times n$ matrix

  $\lambda$: a scalar

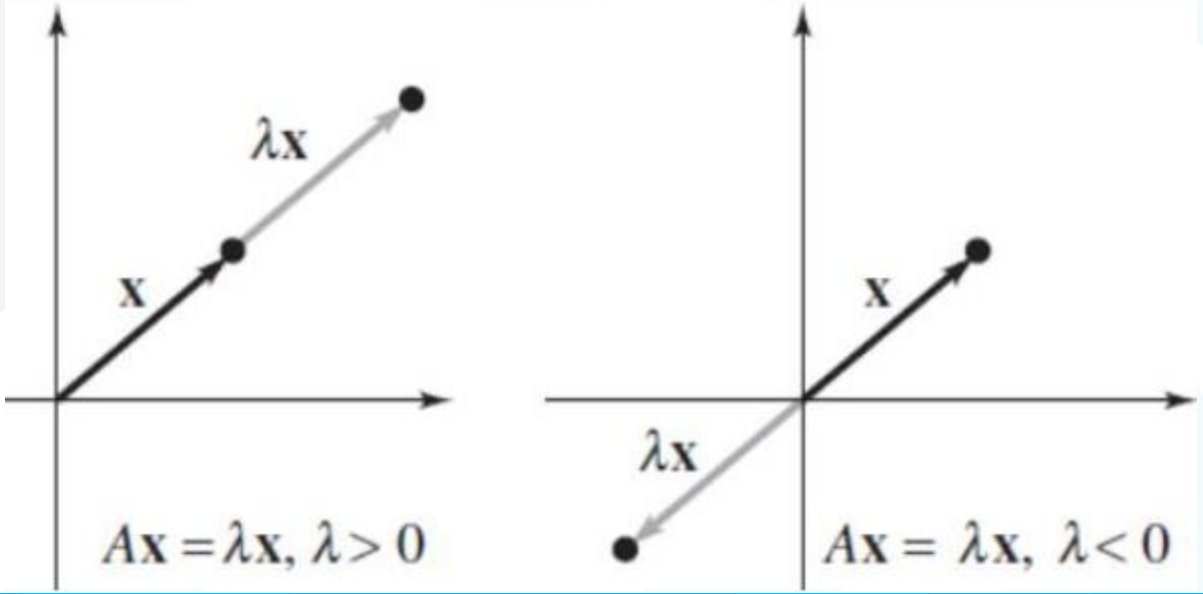
  $\mathbf{x}$: a nonzero vector in $R^n$

Eigenvalue ↓

$$A\mathbf{x} = \lambda \mathbf{x}$$

Eigenvector ↑ ↑

- **Geometrical Interpretation:**



- **Theorem 7.1: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)**

  Let $A$ is an $n\times n$ matrix.

  (1) An eigenvalue of $A$ is a scalar $\lambda$ such that that $\det(\lambda I - A) = 0$

  (2) The eigenvectors of $A$ corresponding to $\lambda$ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$

- **Note:**

  $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (\lambda I - A)\mathbf{x} = \mathbf{0}$ (homogeneous system)

  If $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nonzero solutions iff $\det(\lambda I - A) = 0$

- **Characteristic polynomial of $A \in M_{n\times n}$:**

$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- **Characteristic equation of $A$:** $\det(\lambda I - A) = 0$

- Ex 2: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol:

Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = -2$

$$(1)\ \lambda_1 = -1 \Rightarrow (\lambda_1 I - A)\boldsymbol{x} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2)\ \lambda_2 = -2 \Rightarrow (\lambda_2 I - A)\boldsymbol{x} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Check: $A\boldsymbol{x} = \lambda\boldsymbol{x}$

- **Ex 3: (Finding eigenvalues)**

Find the eigenvalues and corresponding eigenvectors for the matrix $A$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

**Sol:**

Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0 \quad \text{Eigenvalue: } \lambda = 2$$

$\lambda = 2 \Rightarrow$ Eigenvectors:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

- **Note:**

If an eigenvalue $\lambda_1$ occurs as a multiple root ($k$ times) for the characteristic polynomial, then $\lambda_1$ has multiplicity $k$.

- **Theorem 7.2: (Eigenvalues of triangular matrices)**

  If $A$ is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- **Ex 4: (Finding eigenvalues)**

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

**Sol:**

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

- **Diagonalizable matrix:**

A square matrix $A$ is called diagonalizable if there exists an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. ($P$ diagonalizes $A$)

- **Ex 1: (A diagonalizable matrix)**

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4,\ \lambda_2 = -2,\ \lambda_3 = -2$

$$(1)\ \lambda_1 = 4 \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(2)\ \lambda_2 = -2 \Rightarrow \text{Eigenvectors: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- **Notes:**

$$(1)\ P = \begin{bmatrix} p_2 & p_1 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = \begin{bmatrix} p_2 & p_3 & p_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Theorem 7.4: (Condition for diagonalization)

An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

- Note:

If $n$ linearly independent vectors do not exist, then an $n \times n$ matrix $A$ is not diagonalizable.

- Ex 2: (A matrix that is not diagonalizable)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0 \quad \text{Eigenvalue: } \lambda_1 = 1$$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$A$ does not have two ($n = 2$) L. I. eigenvectors, so $A$ is not diagonalizable

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find $n$ linearly independent eigenvectors $p_1, p_2, \ldots p_n$ for $A$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$

Step 2: Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

Step 3: Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where $Ap_i = \lambda_i p_i, \; i = 1,2,\ldots,n$

- Note:

The order of the eigenvalues used to form $P$ will determine the order in which the eigenvalues appear on the main diagonal of $D$.

- Ex 3: (Diagonalizing a matrix)

Find a matrix $P$ such that is $P^{-1}AP$ diagonal $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

$$(1)\ \lambda_1 = 2 \quad \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \quad \text{Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(2)\ \lambda_2 = -2 \quad \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/4)t \\ -(1/4)t \\ t \end{bmatrix} = \frac{1}{4} t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \quad \text{Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$(3)\ \lambda_2 = 3 \quad \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \quad \text{Eigenvector: } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: $k$ is a positive integer

$$(1)\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k=\begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D=P^{-1}AP \Rightarrow D^k=(P^{-1}AP)^k=P^{-1}A^kP$$

$$\Rightarrow A^k=PD^kP^{-1}$$

- **Theorem 7.5: (Sufficient conditions for diagonalization)**

If an $n\times n$ matrix $A$ has $n$ distinct eigenvalues, then the corresponding eigenvectors are linearly independent and $A$ is diagonalizable.

- Ex 4: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol:

Because $A$ is a triangular matrix, its eigenvalues are the main diagonal entries.

$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$

These three values are distinct, so $A$ is diagonalizable.

## ٧.3 Symmetric Matrices and Orthogonal Diagonalization

- **Symmetric matrix:**

A square matrix $A$ is symmetric if it is equal to its transpose: $A = A^T$

- **Theorem 7.6: (Eigenvalues of symmetric matrices)**

If $A$ is an $n \times n$ symmetric matrix, then the following properties are true.

(1) $A$ is diagonalizable.

(2) All eigenvalues of $A$ are real.

(3) If $\lambda$ is an eigenvalue of $A$ with multiplicity $k$, then $\lambda$ has $k$ linearly independent eigenvectors.

- **Orthogonal matrix:**

A square matrix $P$ is called orthogonal if it is invertible and $P^{-1} = P^T$

- **Ex 2: (Orthogonal matrices)**

(a) $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{-4}{5} & 0 & \frac{3}{5} \end{bmatrix}$

- **Theorem 7.7: (Properties of orthogonal matrices)**

An $n \times n$ matrix $P$ is orthogonal if and only if its column vectors form an orthonormal set

- **Ex 3: (An orthogonal matrix)**

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

**Sol:**

If $P$ is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\text{Let } p_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix}, p_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix}, p_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$$

$$\|p_1\| = \|p_2\| = \|p_3\| = 1$$

$\{p_1, p_2, p_3\}$

is an orthonormal set

- **Theorem 7.8: (Properties of symmetric matrices)**

  Let $A$ be an $n \times n$ symmetric matrix. If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $A$, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

- **Ex 4: (Eigenvectors of a symmetric matrix)**

  Show that any two eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

  corresponding to distinct eigenvalues are orthogonal

  Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

$\Rightarrow$ Eigenvalues: $\lambda_1 = 2, \lambda_2 = 4$

$$(1)\ \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = s\begin{bmatrix} -1 \\ 1 \end{bmatrix},\quad s \neq 0$$

$$(2)\ \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad t \neq 0$$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \mathbf{x}_1 \text{ and } \mathbf{x}_2 \text{ are orthogonal}$$

- **Orthogonal Diagonalization**

  matrix $A$ is orthogonally diagonalizable when there exists an orthogonal matrix $P$ such that $P^{-1}AP = D$ is diagonal

- **Theorem 7.9: (Fundamental theorem of symmetric matrices)**

  Let $A$ be an $n \times n$ matrix. Then $A$ is orthogonally diagonalizable and has real eigenvalue if and only if $A$ is symmetric.

- **Orthogonal diagonalization of a symmetric matrix:**

Let $A$ be an $n \times n$ symmetric matrix.

(1) Find all eigenvalues of $A$ and determine the multiplicity of each.

(2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

(3) For each eigenvalue of multiplicity $k \geq 2$, find a set of $k$ linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

(4) The composite of steps 2 and 3 produces an orthonormal set of $n$ eigenvectors. Use these eigenvectors to form the columns of $P$. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- **Ex 6: (Orthogonal diagonalization)**

Find a matrix $P$ that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

Sol: Characteristic equation:

$$(1)\ |\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$$

Eigenvalues: $\lambda_1 = -6$, $\lambda_2 = 3$ (has a multiplicity of 2)

$$(2)\ \lambda_1 = -6,\ \boldsymbol{v}_1 = (1,\ -2,\ 2) \Rightarrow \boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$(3)\ \lambda_2 = 3,\ \boldsymbol{v}_2 = (2,\ 1,\ 0),\ \boldsymbol{v}_3 = (-2,\ 0,\ 1)$$

Linear Independent

## Gram-Schmidt Process:

$$\mathbf{w}_2 = \mathbf{v}_2 = (2,1,0), \quad \mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = (-\frac{2}{5}, \frac{4}{3}, 1)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0), \quad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = (-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}})$$

$$(4)\ P = [p_1\ p_2\ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^T AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$