Exercises 7:Eigenvalues and Eigenvectors

CECC122: Linear Algebra and Matrix Theory

Manara University

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Verify that λ_i is an eigenvalue of A and that x_i is a corresponding eigenvector

① $A=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}$, $\lambda_1=2,\ \boldsymbol{x}_1=(1,0)$, $\lambda_2=-2,\ \boldsymbol{x}_2=(0,1)$

$$A\boldsymbol{x}_1=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=2\begin{bmatrix}1\\ 0\end{bmatrix}=\lambda_1\boldsymbol{x}_1$$

$$A\boldsymbol{x}_2=\begin{bmatrix}2 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -2\end{bmatrix}=-2\begin{bmatrix}0\\ 1\end{bmatrix}=\lambda_2\boldsymbol{x}_2$$

② $A=\begin{bmatrix}4 & -5\\ 2 & -3\end{bmatrix}$, $\lambda_1=-1,\ \boldsymbol{x}_1=(1,1)$, $\lambda_2=2,\ \boldsymbol{x}_2=(5,2)$

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$$Ax_1 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 x_1$$

$$Ax_2 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \lambda_2 x_2$$

③ $$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \quad \begin{aligned} \lambda_1 &= 5, \; x_1 = (1, 2, -1) \\ \lambda_2 &= -3, \; x_2 = (-2, 1, 0) \\ \lambda_3 &= -3, \; x_3 = (3, 0, 1) \end{aligned}$$

$$Ax_1 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda_1 x_1$$

$$A\boldsymbol{x}_2 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \lambda_2 \boldsymbol{x}_2$$

$$A\boldsymbol{x}_3 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \lambda_3 \boldsymbol{x}_3$$

Determine whether $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$

① $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

$(a)\ \boldsymbol{x} = (1, 2)$

$(b)\ \boldsymbol{x} = (2, 1)$

$(c)\ \boldsymbol{x} = (1, -2)$

$(d)\ \boldsymbol{x} = (-1, 0)$

$$(a)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ $\boldsymbol{x}$ is not an eigenvector of A

$$(b)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 8)

$$(c)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 3)

$$(d)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ $\boldsymbol{x}$ is not an eigenvector of A

② $A=\begin{bmatrix}-1 & -1 & 1\\ -2 & 0 & -2\\ 3 & -3 & 1\end{bmatrix}$

$(a)\ \boldsymbol{x}=(2,-4,6)$
$(b)\ \boldsymbol{x}=(2,0,6)$
$(c)\ \boldsymbol{x}=(2,2,0)$
$(d)\ \boldsymbol{x}=(-1,0,1)$

$$(a)\ A\boldsymbol{x}=\begin{bmatrix}-1 & -1 & 1\\ -2 & 0 & -2\\ 3 & -3 & 1\end{bmatrix}\begin{bmatrix}2\\ -4\\ 6\end{bmatrix}=\begin{bmatrix}8\\ -16\\ 24\end{bmatrix}=4\begin{bmatrix}2\\ -4\\ 6\end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 4)

$$(b)\ A\boldsymbol{x}=\begin{bmatrix}-1 & -1 & 1\\ -2 & 0 & -2\\ 3 & -3 & 1\end{bmatrix}\begin{bmatrix}2\\ 0\\ 6\end{bmatrix}=\begin{bmatrix}4\\ -16\\ 12\end{bmatrix}\lambda\begin{bmatrix}2\\ 0\\ 6\end{bmatrix}$$

$\boldsymbol{x}$ is not an eigenvector of A

$$(c)\ A\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

$\mathbf{x}$ is an eigenvector of A (with a corresponding eigenvalue –2)

$$(d)\ A\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{x}$ is an eigenvector of A (with a corresponding eigenvalue –2)

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Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix

① $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0$$

$$(b)\ \lambda_1 = 0,\ \lambda_2 = 7$$

$$\lambda_1 = 0,\quad \begin{bmatrix} \lambda_1 - 6 & 3 \\ 2 & \lambda_1 - 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 2t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is (1, 2)

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$$\lambda_1 = 7, \quad \begin{bmatrix} \lambda_2 - 6 & 3 \\ 2 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(-3t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 7$ is (–3, 1)

② $$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & -1 \\ 0 & \lambda - 3 & -4 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$$

$$(b)\ \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1$$

$$\lambda_1 = 2, \quad \begin{bmatrix} \lambda_1 - 2 & 0 & -1 \\ 0 & \lambda_1 - 3 & -4 \\ 0 & 0 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 0, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is (1, 0, 0)

$$\lambda_2 = 3, \quad \begin{bmatrix} \lambda_2 - 2 & 0 & -1 \\ 0 & \lambda_2 - 3 & -4 \\ 0 & 0 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 3$ is (0,1,0)

$$\lambda_3 = 1,\quad \begin{bmatrix} \lambda_3 - 2 & 0 & -1 \\ 0 & \lambda_3 - 3 & -4 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(-t, -2t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is $(-1, -2, 1)$

③ $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

$$(a)\ \ |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix} = (\lambda + 3)(\lambda - 3)^2 = 0$$

$(b)\ \lambda_1 = -3,\ \lambda_2 = 3$ (repeated)

$$\lambda_1 = -3,\quad \begin{bmatrix} \lambda_1 - 1 & -2 & 2 \\ 2 & \lambda_1 - 5 & 2 \\ 6 & -6 & \lambda_1 + 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, t, 3t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -3$ is (1, 1, 3)

$$\lambda_2 = 3,\quad \begin{bmatrix} \lambda_2 - 1 & -2 & 2 \\ 2 & \lambda_2 - 5 & 2 \\ 6 & -6 & \lambda_2 + 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(s - t, s, t): s, t \in R\}$. So, two eigenvectors corresponding to $\lambda_2 = 3$ are (1,1,0) and (1, 0, −1)

Find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal

① $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0$$

$\lambda_1 = 0, \lambda_2 = 7$

$$\lambda_1 = 0, \quad \begin{bmatrix} \lambda_1 - 6 & 3 \\ 2 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 2t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, 2)$

$$\lambda_2 = 7, \quad \begin{bmatrix} \lambda_2 - 6 & 3 \\ 2 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(-3t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 7$ is $(-3, 1)$

$$P = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{7} \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$$

(2) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & -1 \\ 0 & \lambda-3 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

$$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1$$

$$\lambda_1 = 2, \quad \begin{bmatrix} \lambda_1-2 & 0 & -1 \\ 0 & \lambda_1-3 & -4 \\ 0 & 0 & \lambda_1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(t, 0, 0) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is (1, 0, 0)

$$\lambda_2 = 3, \quad \begin{bmatrix} \lambda_2-2 & 0 & -1 \\ 0 & \lambda_2-3 & -4 \\ 0 & 0 & \lambda_2-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$S = \{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 3$ is (0, 1, 0)

$$\lambda_3 = 1, \quad \begin{bmatrix} \lambda_3 - 2 & 0 & -1 \\ 0 & \lambda_3 - 3 & -4 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(-t, -2t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is (–1, –2, 1)

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix} = (\lambda + 3)(\lambda - 3)^2 = 0$$

$\lambda_1 = -3, \lambda_2 = 3$ (repeated)

$$\lambda_1 = -3, \quad \begin{bmatrix} \lambda_1 - 1 & -2 & 2 \\ 2 & \lambda_1 - 5 & 2 \\ 6 & -6 & \lambda_1 + 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(t, t, 3t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -3$ is (1, 1, 3)

$$\lambda_2 = 3, \begin{bmatrix} \lambda_2 - 1 & -2 & 2 \\ 2 & \lambda_2 - 5 & 2 \\ 6 & -6 & \lambda_2 + 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(s-t, s, t): s, t \in R\}$. So, two eigenvectors corresponding to $\lambda_2 = 3$ are $(1,1,0)$ and $(1, 0, -1)$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 4 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 4 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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④ $A=\begin{bmatrix}1&0&0\\1&2&1\\1&0&2\end{bmatrix}$

$$|\lambda I-A|=\begin{vmatrix}\lambda-1&0&0\\-1&\lambda-2&-1\\-1&0&\lambda-2\end{vmatrix}=(\lambda-1)(\lambda-2)^2=0$$

$\lambda_1=1,\ \lambda_2=2$ (repeated)

$$\lambda_1=1,\quad \begin{bmatrix}\lambda_1-1&0&0\\-1&\lambda_1-2&-1\\-1&0&\lambda_1-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}0&0&0\\-1&-1&-1\\-1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$S=\{(-t,0,t): t\in R\}$. So, an eigenvector corresponding to $\lambda_1=1$ is $(-1, 0, 1)$

$$\lambda_2 = 2, \quad \begin{bmatrix} \lambda_2 - 1 & 0 & 0 \\ -1 & \lambda_2 - 2 & -1 \\ -1 & 0 & \lambda_2 - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 2$ is $(0, 1, 0)$.

There are just two linearly independent eigenvectors of A. So, A is not diagonalizable

Find A^7

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Find eigenvalues of the matrix A

(b) Find eigenvectors for each eigenvalue of A

(c) Diagonalize the matrix A

(d) Calculate A^7

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 5$$

$$(b)\ \lambda_1 = -1, \quad \begin{bmatrix} \lambda_1 - 1 & -2 \\ -4 & \lambda_1 - 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(t, -t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -1$ is $(1, -1)$

$$\lambda_2 = 5, \quad \begin{bmatrix} \lambda_2 - 1 & -2 \\ -4 & \lambda_2 - 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 2t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 5$ is $(1, 2)$

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$$(c)\ P=\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix} \Rightarrow P^{-1}=\frac{1}{3}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}$$

$$D=P^{-1}AP=\frac{1}{3}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 5\end{bmatrix}$$

$$(d)\ A=PDP^{-1} \Rightarrow A^{7}=PD^{7}P^{-1}$$

$$A^{7}=\frac{1}{3}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}(-1)^{7} & 0\\ 0 & 5^{7}\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}26041 & 26042\\ 52084 & 52083\end{bmatrix}$$

Find a matrix P such that P^TAP orthogonally diagonalizes A. Verify that P^TAP gives the correct diagonal form

① $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 1 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

$$\lambda_1 = 0, \begin{bmatrix} \lambda_1 - 1 & -1 \\ -1 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(t, -t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, -1)$

$$\lambda_2 = 2, \begin{bmatrix} \lambda_2 - 1 & -1 \\ -1 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Determine whether the matrix is orthogonal

(1) $A=\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Because the column vectors of the matrix form an orthonormal set, the matrix is orthogonal

(2) $A=\begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

Because the column vectors of the matrix do not form an orthonormal set [(–4, 0, 3) and (3, 0, 4) are not unit vectors], the matrix is not orthogonal

vectors

Find a matrix P such that $P^T AP$ orthogonally diagonalizes A. Verify that $P^T AP$ gives the correct diagonal form

① $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 1 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

$$\lambda_1 = 0, \begin{bmatrix} \lambda_1 - 1 & -1 \\ -1 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(t, -t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, -1)$

$$\lambda_2 = 2, \begin{bmatrix} \lambda_2 - 1 & -1 \\ -1 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is (1, 1)

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \quad P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

② $$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & -2 \\ 1 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 4) = 0$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = 4$, with corresponding eigenvectors $(-1, -1, 1)$, $(-1, 1, 0)$, and $(1, 1, 2)$, respectively. Normalizing:

$$P = \begin{bmatrix} -\sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \\ -\sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix

① $A=\begin{bmatrix}1 & -4\\ -2 & 8\end{bmatrix}$

② $A=\begin{bmatrix}2 & -2 & 3\\ 0 & 3 & -2\\ 0 & -1 & 2\end{bmatrix}$

③ $A=\begin{bmatrix}3 & 2 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}$

Find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal

① $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

④ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Find A^7

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Find eigenvalues of the matrix A

(b) Find eigenvectors for each eigenvalue of A

(c) Diagonalize the matrix A (d) Calculate A^7

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Determine whether the matrix is orthogonal

① $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

② $A = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$

Find a matrix P such that $P^T A P$ orthogonally diagonalizes A. Verify that $P^T A P$ gives the correct diagonal form

① $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$