Calculus 1

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Calculus 1

Lecture 10

Integrals

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Chapter 5

Integrals

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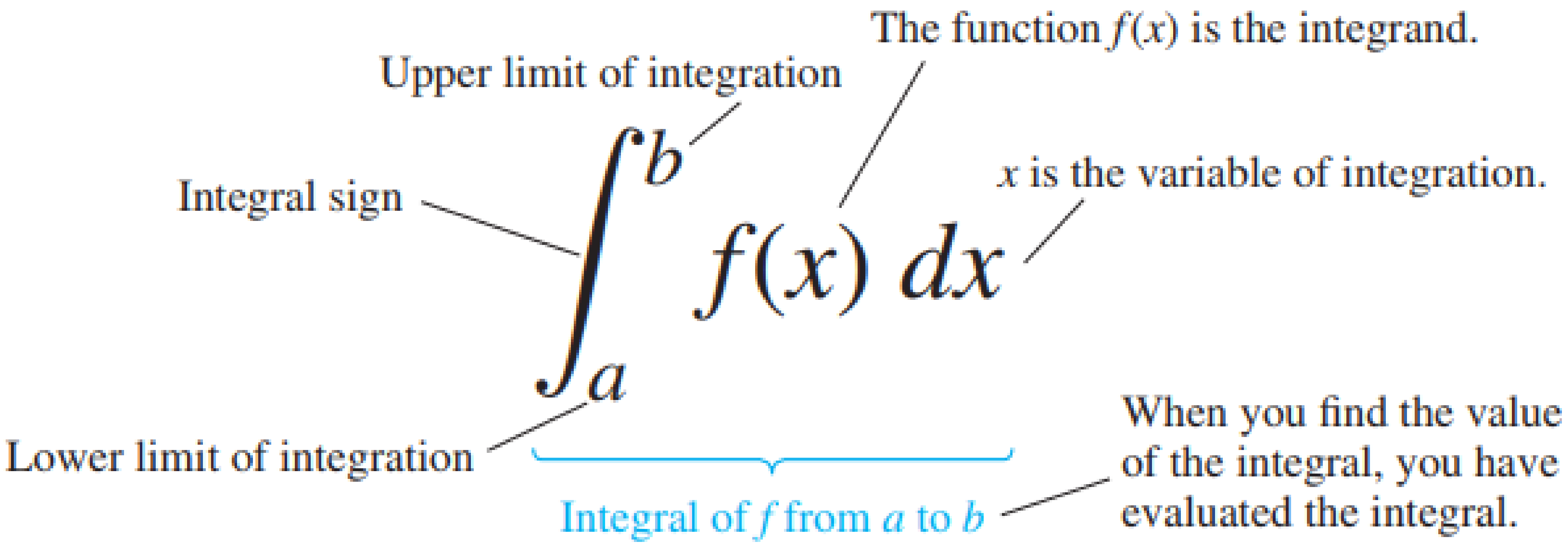
The Definite integral

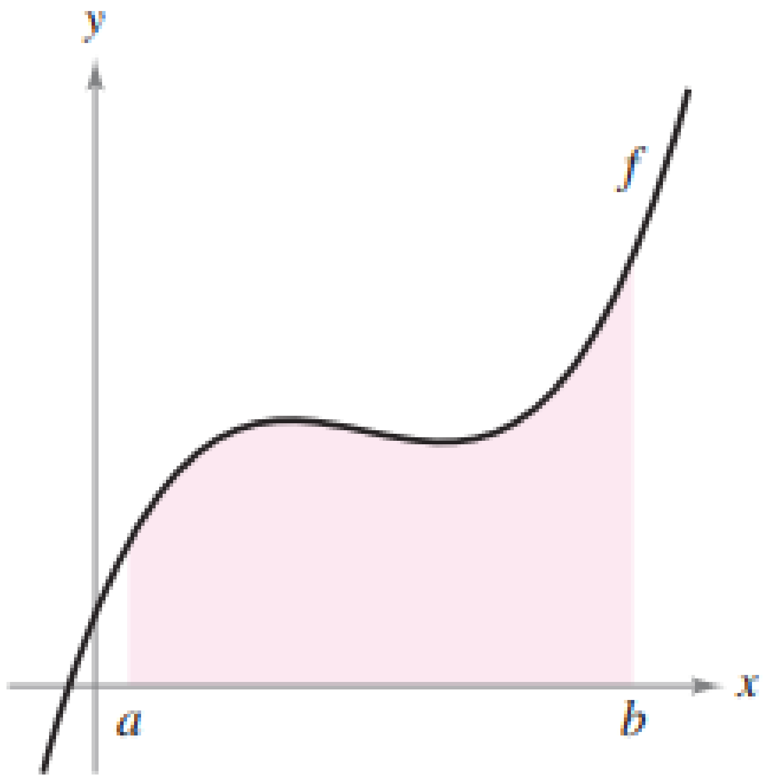


The Definite integral

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx.$$



rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Domination:* If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

   If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)\,dx \geq 0$. — Special case

The Fundamental Theorem of Calculus

THEOREM Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

THEOREM The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1. Find an antiderivative F of f, and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x)\,dx$.

The Fundamental Theorem of Calculus

(a) $\displaystyle\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi}$

$= \sin \pi - \sin 0 = 0 - 0 = 0$

(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{0}$

$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$

$\frac{d}{dx}\sec x = \sec x \tan x$

(c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$

$= [8 + 1] - [5] = 4$

$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$

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EXAMPLE

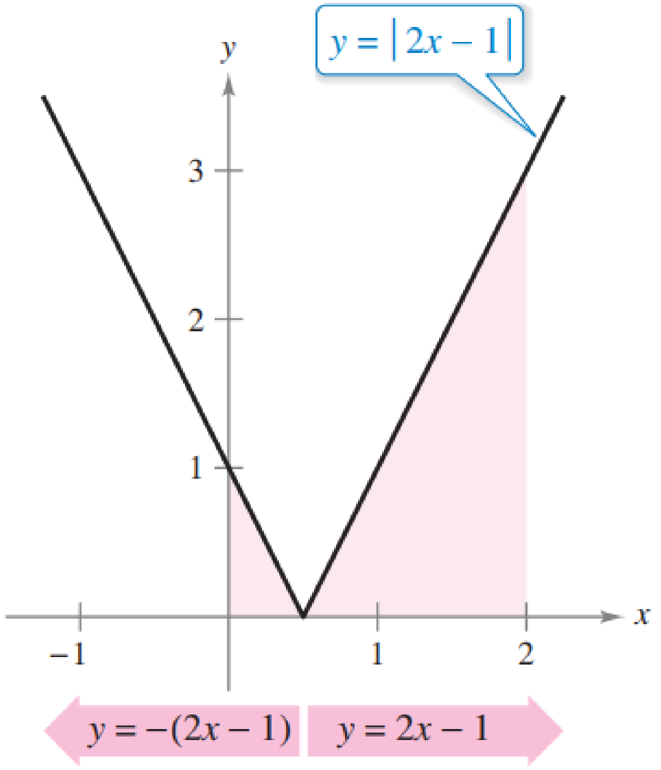
Evaluate $\int_0^2 |2x - 1|\, dx.$

Solution

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned} \int_0^2 |2x - 1|\, dx &= \int_0^{1/2} -(2x - 1)\, dx + \int_{1/2}^2 (2x - 1)\, dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2} \end{aligned}$$



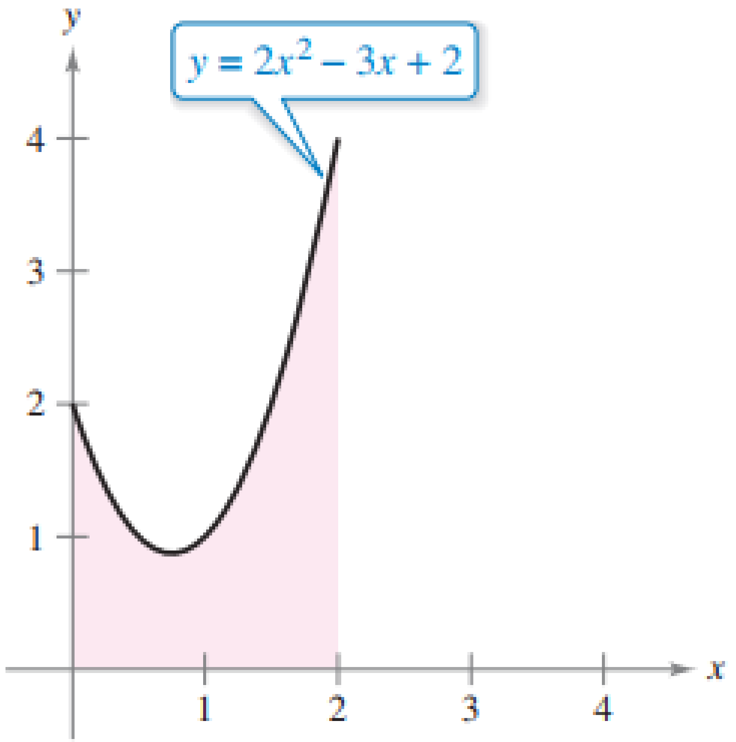
The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.

Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of

$$y = 2x^2 - 3x + 2$$

the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 4.29.



Solution Note that $y > 0$ on the interval $[0, 2]$.

$$\text{Area} = \int_0^2 (2x^2 - 3x + 2)\, dx \qquad \text{Integrate between } x = 0 \text{ and } x = 2.$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2 \qquad \text{Find antiderivative.}$$

$$= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) \qquad \text{Apply Fundamental Theorem.}$$

$$= \frac{10}{3} \qquad \text{Simplify.}$$

Definite integral Substitutions

THEOREM 7—Substitution in Definite Integrals
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx.$

Definite integral Substitutions

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_{0}^{2} \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Definite integral Substitutions

$$\int_{-\pi/4}^{\pi/4} \tan x \, dx$$

$$\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x \, dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\ln |u| \Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval

Definite integral Substitutions

$$\int_0^{\pi/2} \frac{2\sin x \cos x}{(1+\sin^2 x)^3}\,dx$$

$$\int_0^{\pi/2} \frac{2\sin x \cos x}{(1+\sin^2 x)^3}\,dx = \int_1^2 \frac{1}{u^3}\,du$$

$$= -\frac{1}{2u^2}\bigg]_1^2$$

$$= -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}$$

Let $u = 1 + \sin^2 x$, $du = 2\sin x \cos x\,dx$.
When $x = 0$, $u = 1$.
When $x = \pi/2$, $u = 2$.

Definite integral Substitutions

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11} \frac{du}{\sqrt{u}}$$

$u = x^2 - 3x + 1,\ du = (2x-3)\,dx;$
$u = 1$ when $x = 3$, $u = 11$ when $x = 5$

$$= \int_1^{11} u^{-1/2}\,du$$

$$= 2\sqrt{u}\,\Big]_1^{11} = 2\left(\sqrt{11}-1\right) \approx 4.63.$$

Table 8.1, Formula 2 ■

Evaluating Definite Integrals by Parts

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Evaluating Definite Integrals by Parts

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

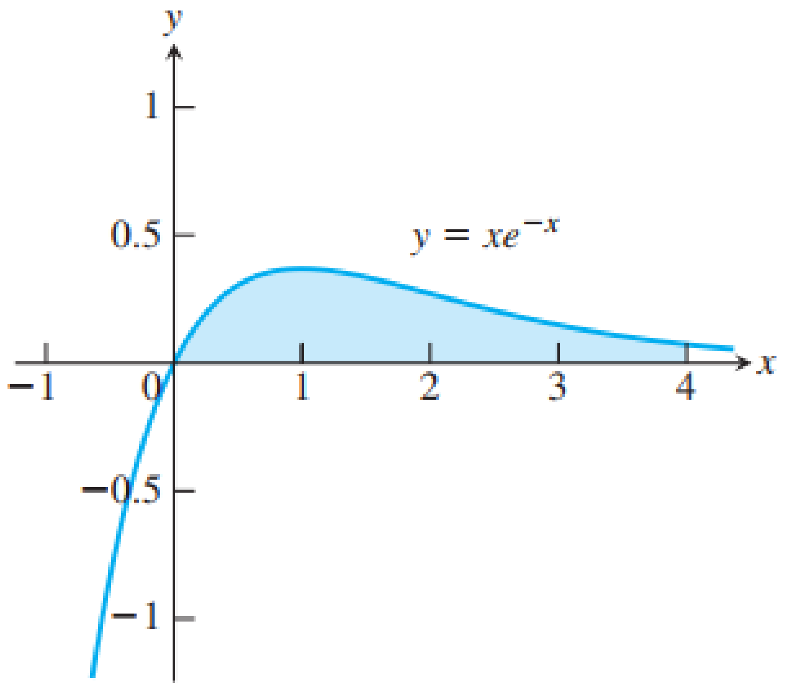
Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$

$$= \left[-4e^{-4} - (-0e^{-0})\right] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4}$$



The Definite integral

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \cos 2x \geq 0 \text{ on } [0, \pi/4]$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

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$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(1 - \sin^2 2\theta\right) \cos 2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta \, d\theta = \left[\frac{1}{2} \cdot \frac{\sin^3 2\theta}{3} - \frac{1}{2} \cdot \frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$$

The Definite integral

Evaluate $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx.$

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x \cos x}{\sqrt{\sin x}}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{(1 - \sin^2 x)(\cos x)}{\sqrt{\sin x}}\, dx$$

$$= \int_{\pi/6}^{\pi/3} [(\sin x)^{-1/2} - (\sin x)^{3/2}] \cos x\, dx$$

$$= \left[\frac{(\sin x)^{1/2}}{1/2} - \frac{(\sin x)^{5/2}}{5/2}\right]_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2} + \frac{\sqrt{32}}{80}$$

The Definite integral

Evaluate $\int_0^2 \frac{x}{1 + e^{-x^2}}\, dx.$

$$\int \frac{du}{1 + e^u} = u - \ln(1 + e^u) + C.$$

Let $u = -x^2$. Then $du = -2x\,dx$, and you have

$$\int \frac{x}{1 + e^{-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x\,dx}{1 + e^{-x^2}}$$

$$= -\frac{1}{2}\left[-x^2 - \ln\left(1 + e^{-x^2}\right)\right] + C$$

$$= \frac{1}{2}\left[x^2 + \ln\left(1 + e^{-x^2}\right)\right] + C.$$

So, the value of the definite integral is

$$\int_0^2 \frac{x}{1 + e^{-x^2}}\, dx = \frac{1}{2}\left[x^2 + \ln\left(1 + e^{-x^2}\right)\right]_0^2 = \frac{1}{2}[4 + \ln(1 + e^{-4}) - \ln 2]$$

Definite Integrals of Symmetric Functions

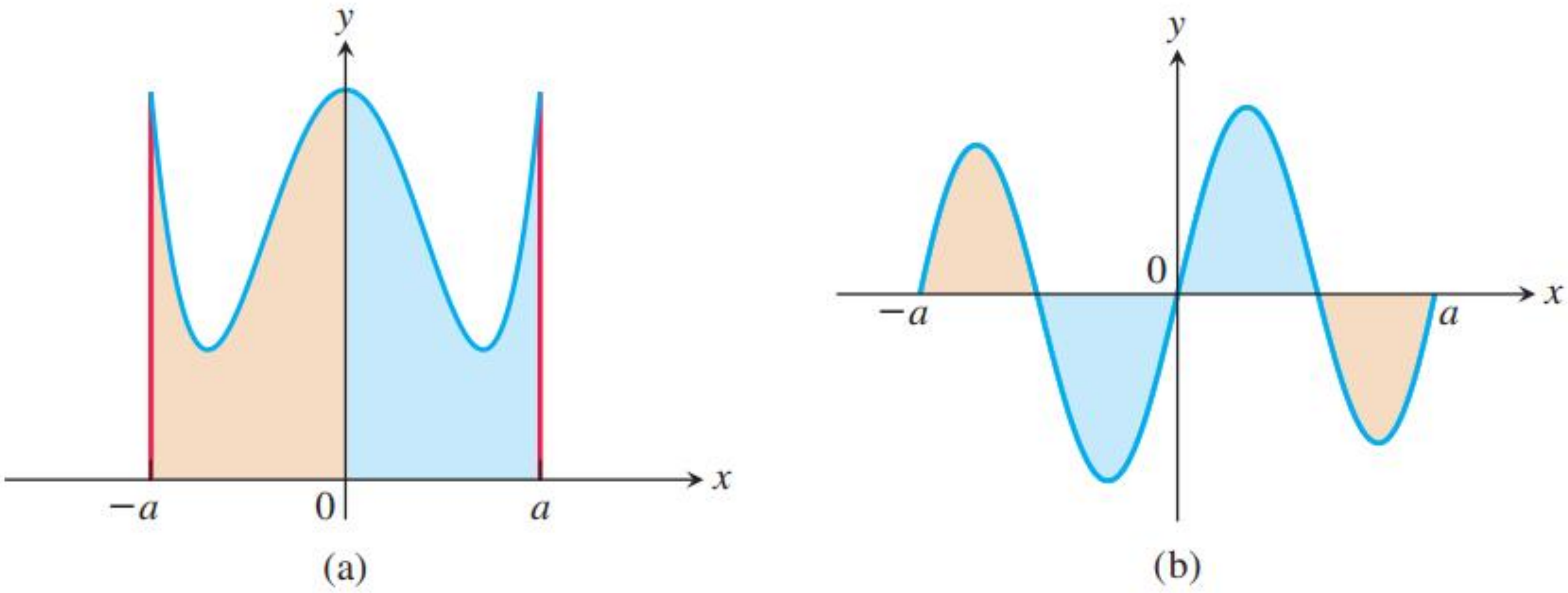


FIGURE (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

Definite Integrals of Symmetric Functions

THEOREM Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

Definite Integrals of Symmetric Functions

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx$$

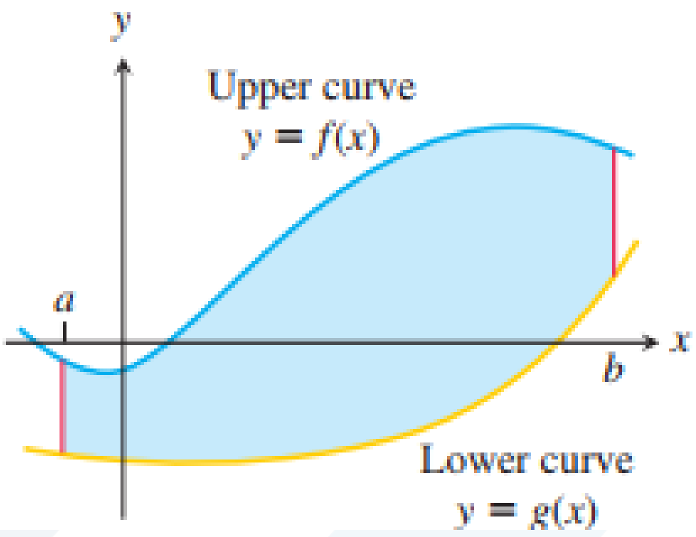
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

Areas Between Curves

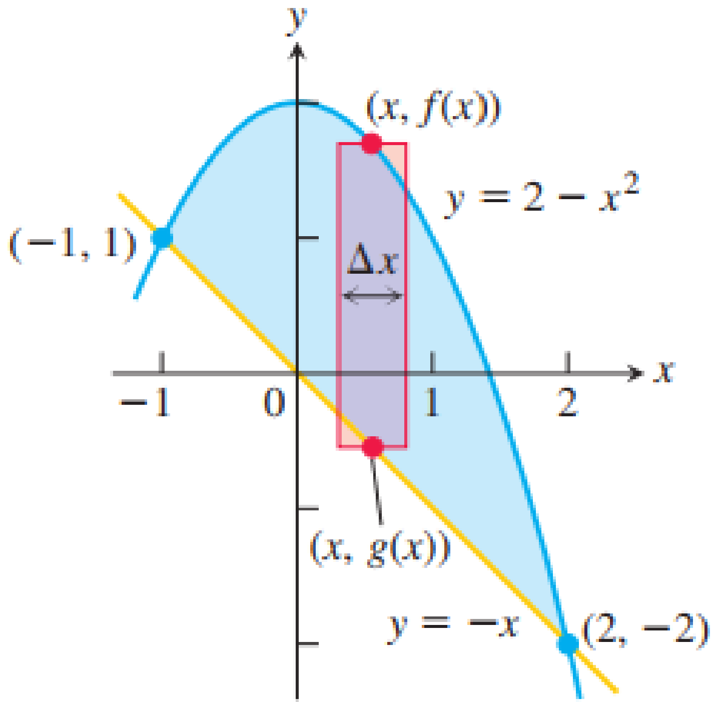
DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



EXAMPLE 4 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.28). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.



$$2 - x^2 = -x \quad \text{Equate } f(x) \text{ and } g(x).$$

$$x^2 - x - 2 = 0 \quad \text{Rewrite.}$$

$$(x + 1)(x - 2) = 0 \quad \text{Factor.}$$

$$x = -1, \quad x = 2. \quad \text{Solve.}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

Areas Between Curves

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}.$$

EXAMPLE 5 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution

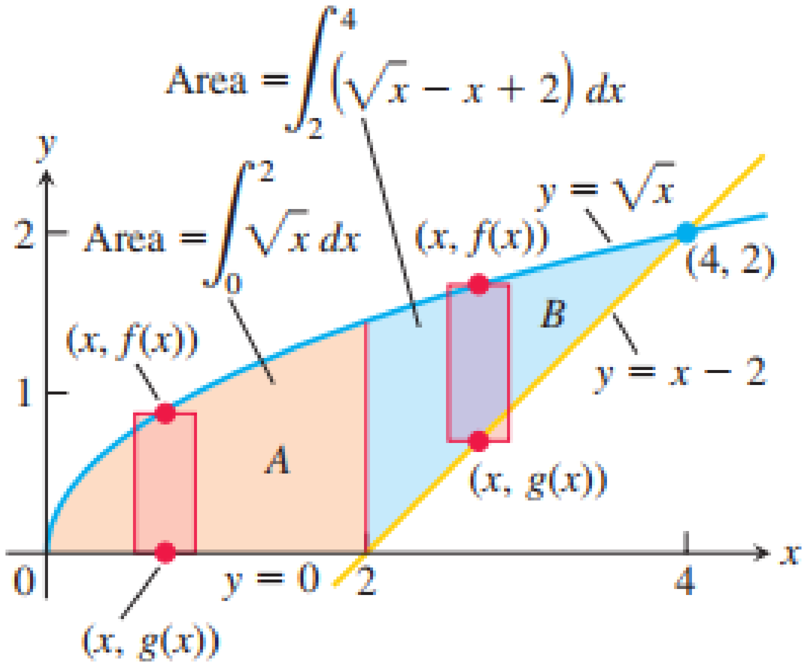
$$\sqrt{x} = x - 2 \qquad \text{Equate } f(x) \text{ and } g(x).$$

$$x = (x-2)^2 = x^2 - 4x + 4 \qquad \text{Square both sides.}$$

$$x^2 - 5x + 4 = 0 \qquad \text{Rewrite.}$$

$$(x-1)(x-4) = 0 \qquad \text{Factor.}$$

$$x = 1, \qquad x = 4. \qquad \text{Solve.}$$



Areas Between Curves

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

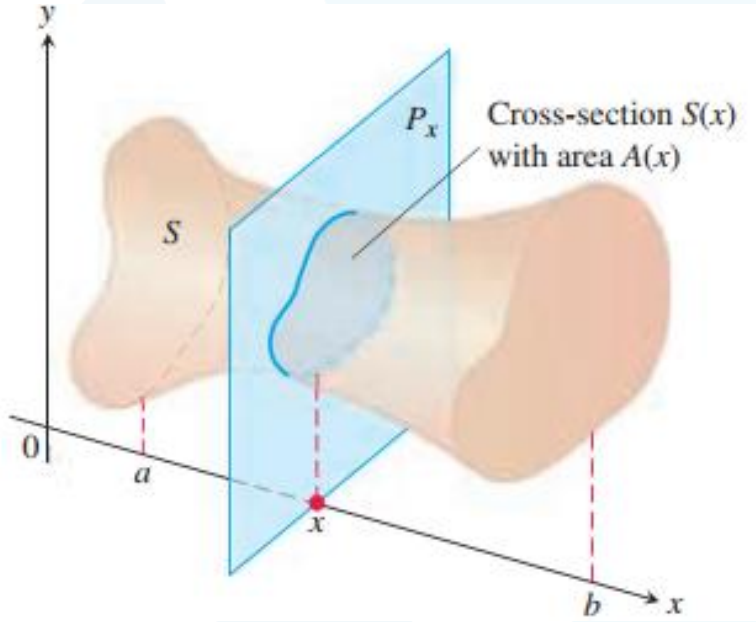
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Volumes

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

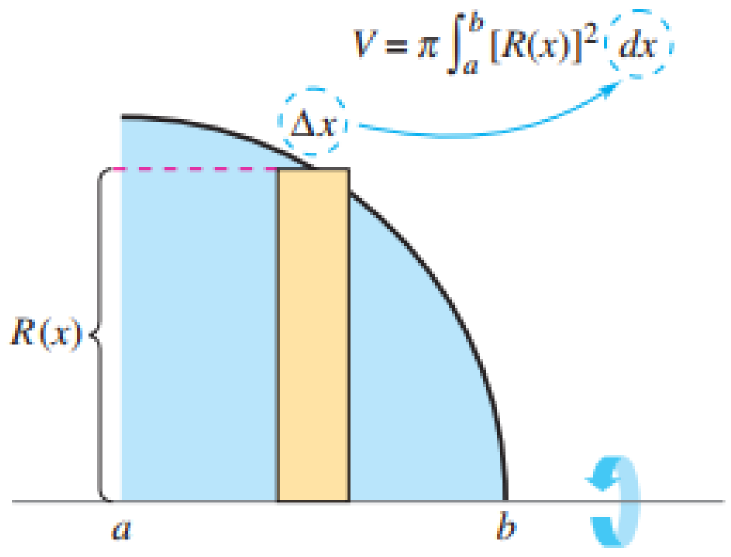
$$V = \int_a^b A(x)\,dx.$$



Solids of Revolution: the Disk Method

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$



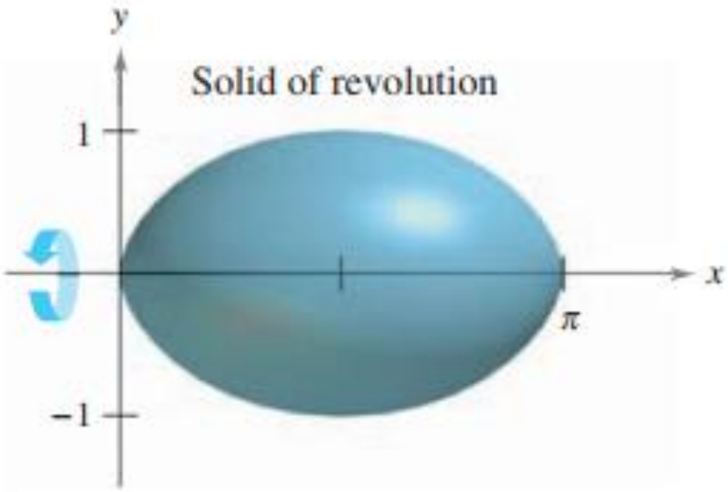
Horizontal axis of revolution

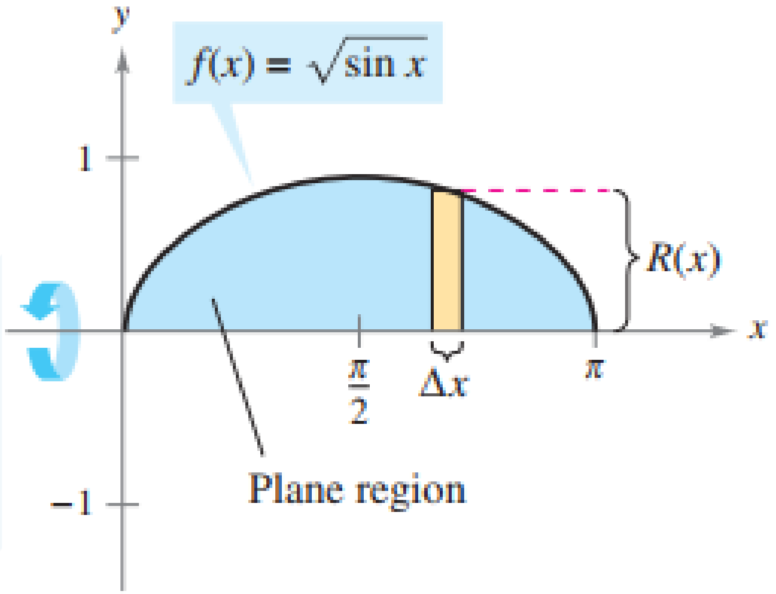
EXAMPLE

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x-axis $(0 \le x \le \pi)$ about the x-axis.





Solids of Revolution: the Disk Method

Solution From the representative rectangle in the upper graph in Figure , you can see that the radius of this solid is

$$R(x) = f(x)$$
$$= \sqrt{\sin x}.$$

So, the volume of the solid of revolution is

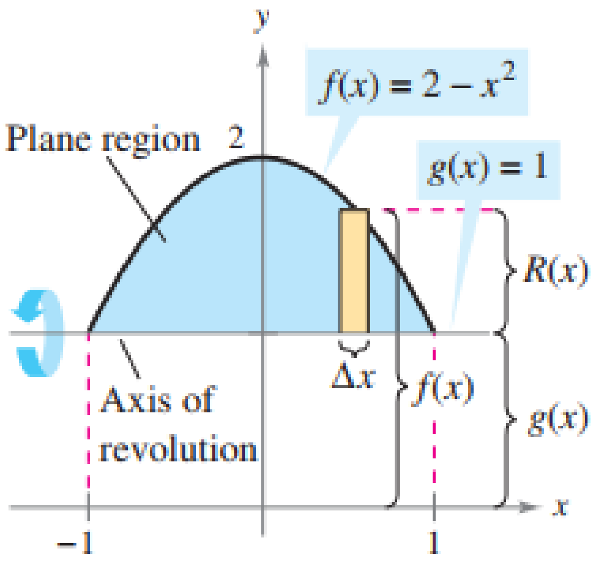
$$V = \pi\int_a^b [R(x)]^2\,dx = \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx \qquad \text{Apply disk method.}$$
$$= \pi\int_0^{\pi} \sin x\,dx \qquad \text{Simplify.}$$
$$= \pi\Big[-\cos x\Big]_0^{\pi} \qquad \text{Integrate.}$$
$$= \pi(1 + 1)$$
$$= 2\pi.$$

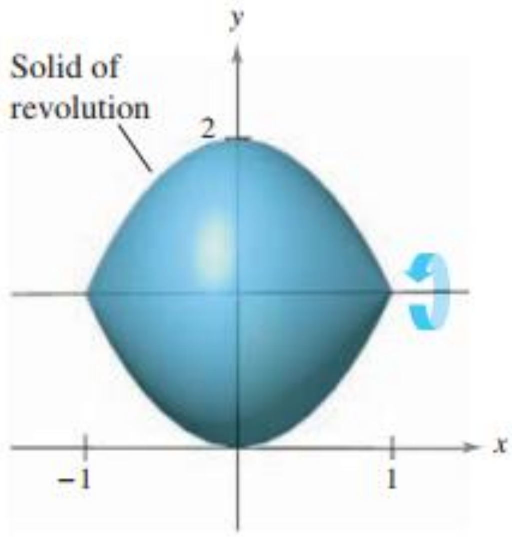
EXAMPLE

Find the volume of the solid formed by revolving the region bounded by

$$f(x) = 2 - x^2$$

and $g(x) = 1$ about the line $y = 1$, as shown in Figure . . . .





Solids of Revolution: the Disk Method

Solution By equating $f(x)$ and $g(x)$, you can determine that the two graphs intersect when $x = \pm 1$. To find the radius, subtract $g(x)$ from $f(x)$.

$$\begin{aligned} R(x) &= f(x) - g(x) \\ &= (2 - x^2) - 1 \\ &= 1 - x^2 \end{aligned}$$

Finally, integrate between -1 and 1 to find the volume.

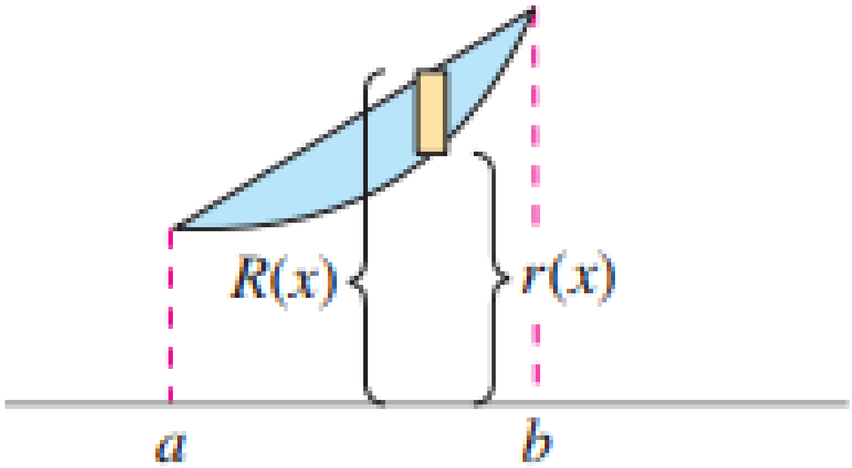
$$\begin{aligned} V = \pi \int_a^b [R(x)]^2\,dx &= \pi \int_{-1}^{1} (1 - x^2)^2\,dx && \text{Apply disk method.} \\ &= \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\,dx && \text{Simplify.} \\ &= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} && \text{Integrate.} \\ &= \frac{16\pi}{15} \end{aligned}$$

■

Volume by Washers for Rotation About the x-Axis

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx.$$

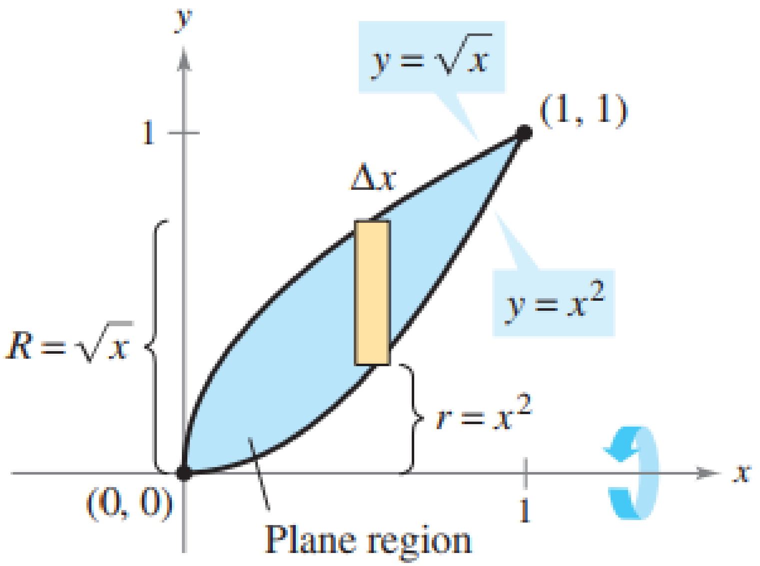


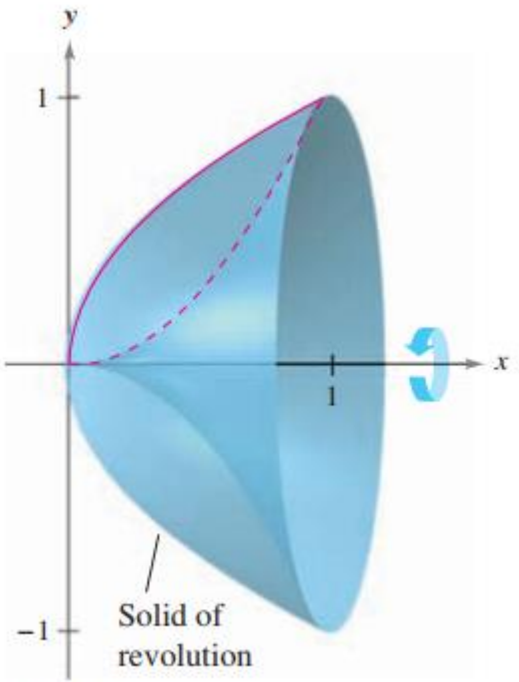
Plane region

Volume by Washers for Rotation About the x-Axis

EXAMPLE

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown in Figure





Solid of revolution

Volume by Washers for Rotation About the x-Axis

Solution In Figure ', you can see that the outer and inner radii are as follows.

$$R(x) = \sqrt{x} \qquad \text{Outer radius}$$

$$r(x) = x^2 \qquad \text{Inner radius}$$

Integrating between 0 and 1 produces

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx \qquad \text{Apply washer method.}$$

$$= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2\right] dx$$

$$= \pi \int_0^1 (x - x^4)\, dx \qquad \text{Simplify.}$$

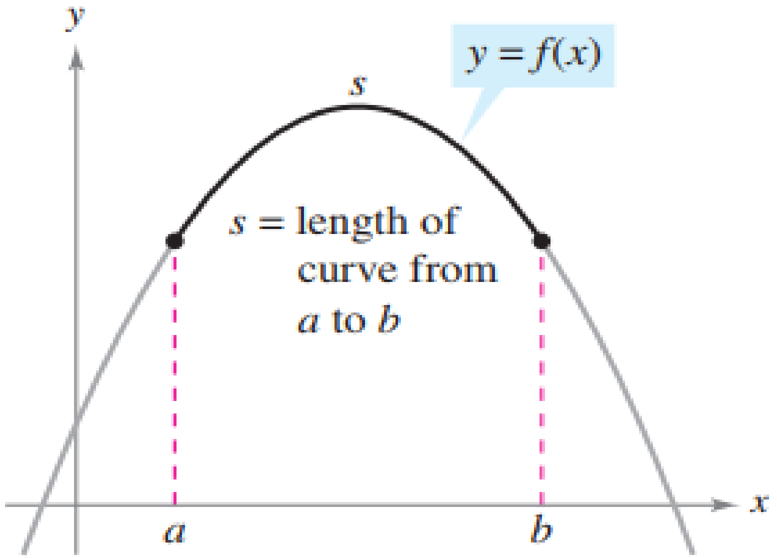
$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \qquad \text{Integrate.}$$

$$= \frac{3\pi}{10}.$$

Length of a Curve $y = f(x)$

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$



Length of a Curve $y = f(x)$

Find the length of the curve

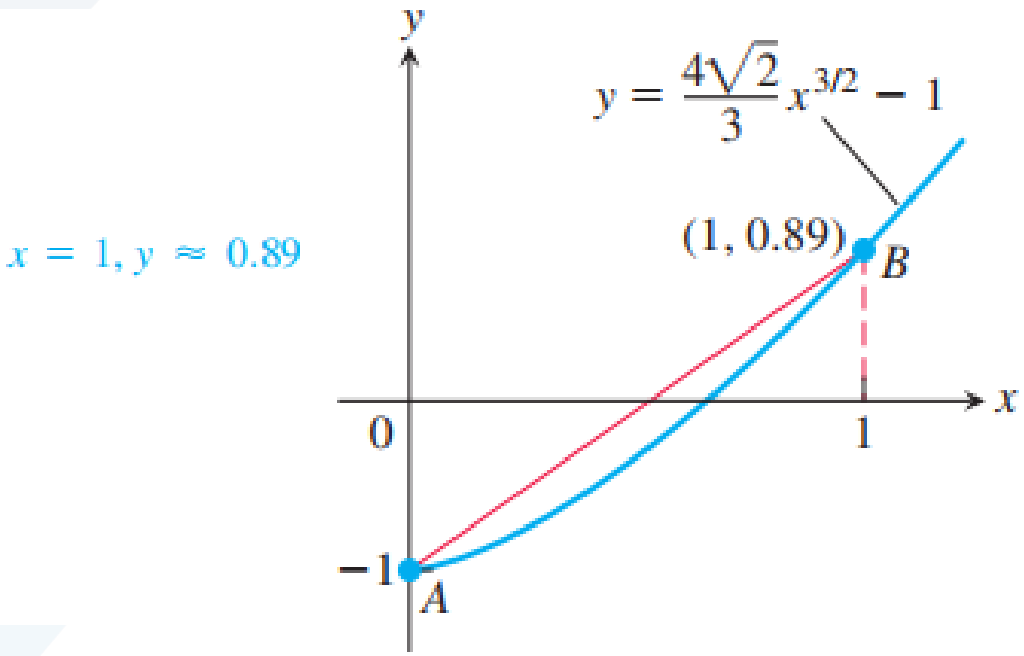
$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$



Length of a Curve y = f(x)

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17.$$

Eq. (3) with $a = 0, b = 1$.

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.

EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4.$$

Length of a Curve $y = f(x)$

Solution A graph of the function is shown in Figure . To use Equation (3), we find

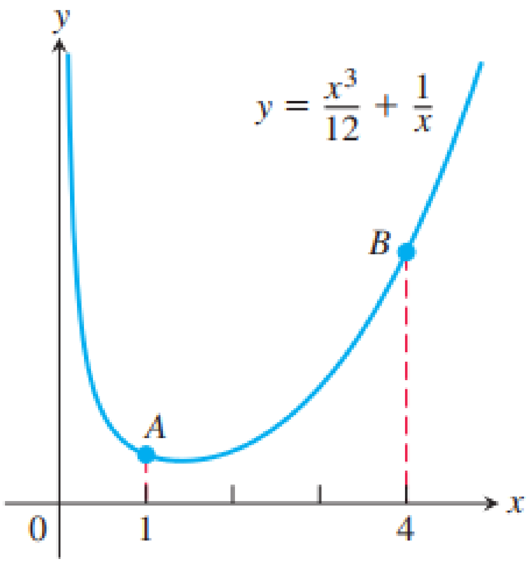
$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

so

$$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$
$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

The length of the graph over $[1, 4]$ is

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$
$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6.$$



Length of a Curve y = f(x)

EXAMPLE Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \qquad 0 \le x \le 2.$$

Solution We use Equation (3) with $a = 0$, $b = 2$, and

$$y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \left[\frac{1}{2}(e^x + e^{-x})\right]^2.$$

Length of a Curve y = f(x)

The length of the curve from $x = 0$ to $x = 2$ is

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \frac{1}{2}\left(e^x + e^{-x}\right) dx \qquad \text{Eq. (3) with } a = 0,\ b = 2.$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - e^{-2}\right)$$

■

Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

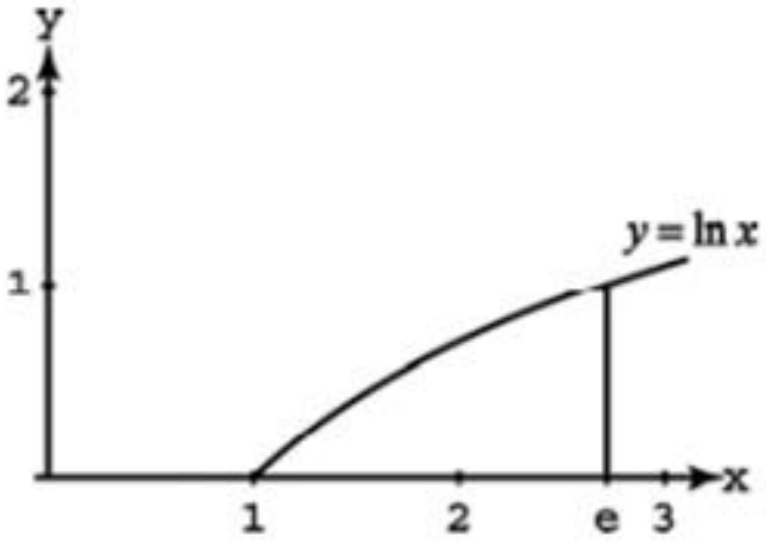
(a) $A=\int_1^e \ln x\,dx=[x\ln x]_1^e-\int_1^e dx$

$=(e\ln e-1\ln 1)-[x]_1^e=e-(e-1)=1$

(b) $V=\int_1^e \pi(\ln x)^2\,dx=\pi\left(\left[x(\ln x)^2\right]_1^e-\int_1^e 2\ln x\,dx\right)$

$=\pi\left[\left(e(\ln e)^2-1(\ln 1)^2\right)-\left([2x\ln x]_1^e-\int_1^e 2\,dx\right)\right]$

$=\pi\left[e-\left((2e\ln e-2(1)\ln 1)-[2x]_1^e\right)\right]=\pi\left[e-(2e-(2e-2))\right]=\pi(e-2)$



Thank you for your attention