إشارات ونظم

جلسة خامسة + سادسة

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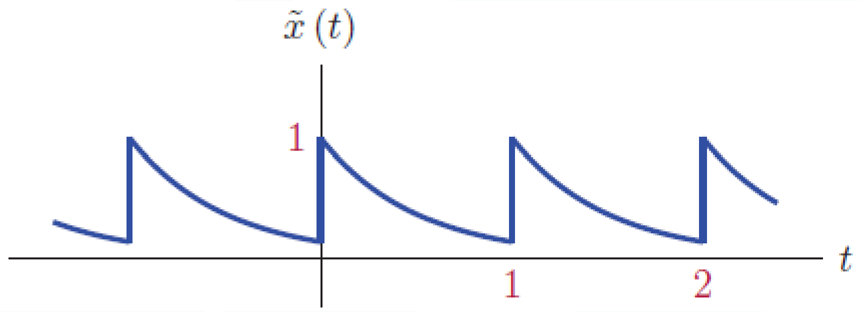
1. **Determine the TFS coefficients for the periodic signal shown. One period of the signal is $\tilde{x}(t) = e^{-2t}$ for $0 < t < 1$ s.**

The fundamental period is $T_0 = 1$ second which corresponds to a fundamental frequency of $f_0 = 1$ Hz or $\omega_0 = 2\pi$ rad/s.

$$a_0 = \int_0^1 e^{-2t} dt = 0.4323$$

$$a_k = 2\int_0^1 e^{-2t} \cos(2\pi kt) dt = \frac{0.8647}{1+\pi^2 k^2}$$

$$b_k = 2\int_0^1 e^{-2t} \sin(2\pi kt) dt = \frac{0.8647\pi k}{1+\pi^2 k^2}$$



$$\tilde{x}(t) = 0.4323 + \sum_{k=1}^{\infty} \frac{0.8647}{1+\pi^2 k^2} \cos(2\pi kt) + \sum_{k=1}^{\infty} \frac{0.8647\pi k}{1+\pi^2 k^2} \sin(2\pi kt)$$

2. Consider the periodic signal $\tilde{x}(t)$ of exercise 1

a. Determine the EFS coefficients from the TFS coefficients obtained in exercise 1.

b. Determine the EFS coefficients by direct application of the analysis equation.

c. Sketch the line spectrum (magnitude and phase).

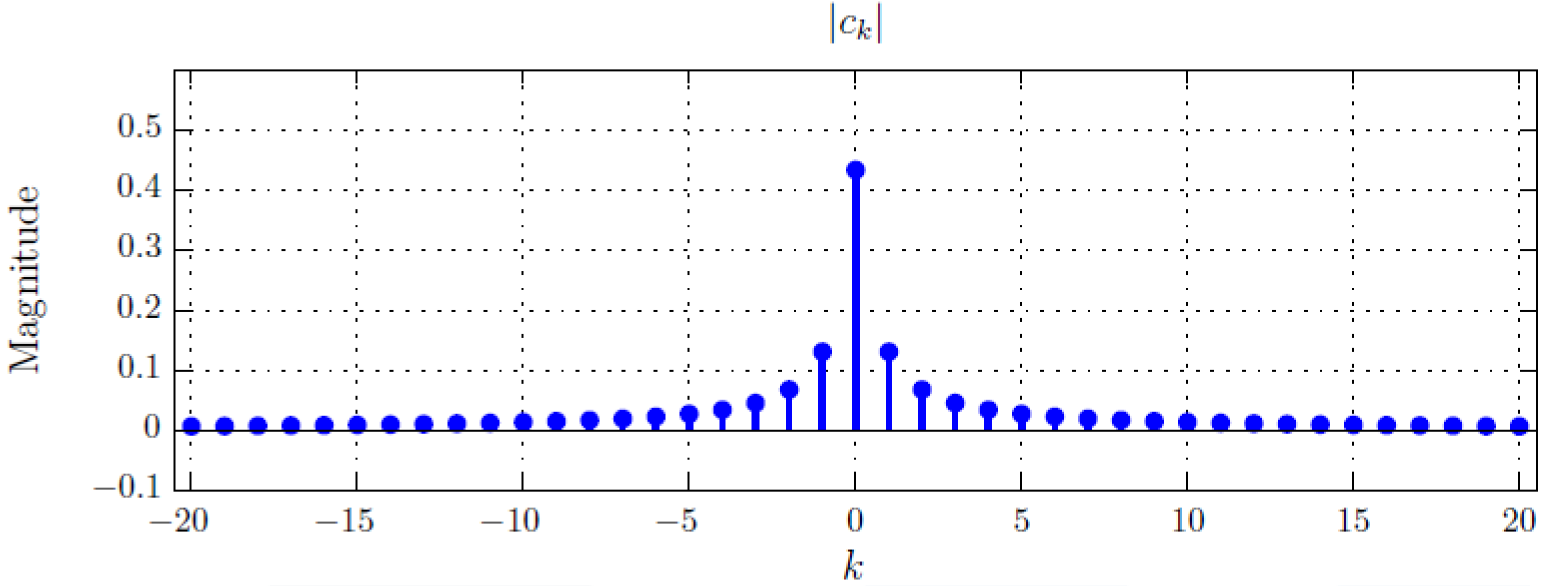
$$a.\ c_0 = a_0 = 0.4323$$

$$c_k = \frac{1}{2}(a_k - jb_k) = \frac{0.4323}{1 + j2\pi k}, \quad k > 0$$

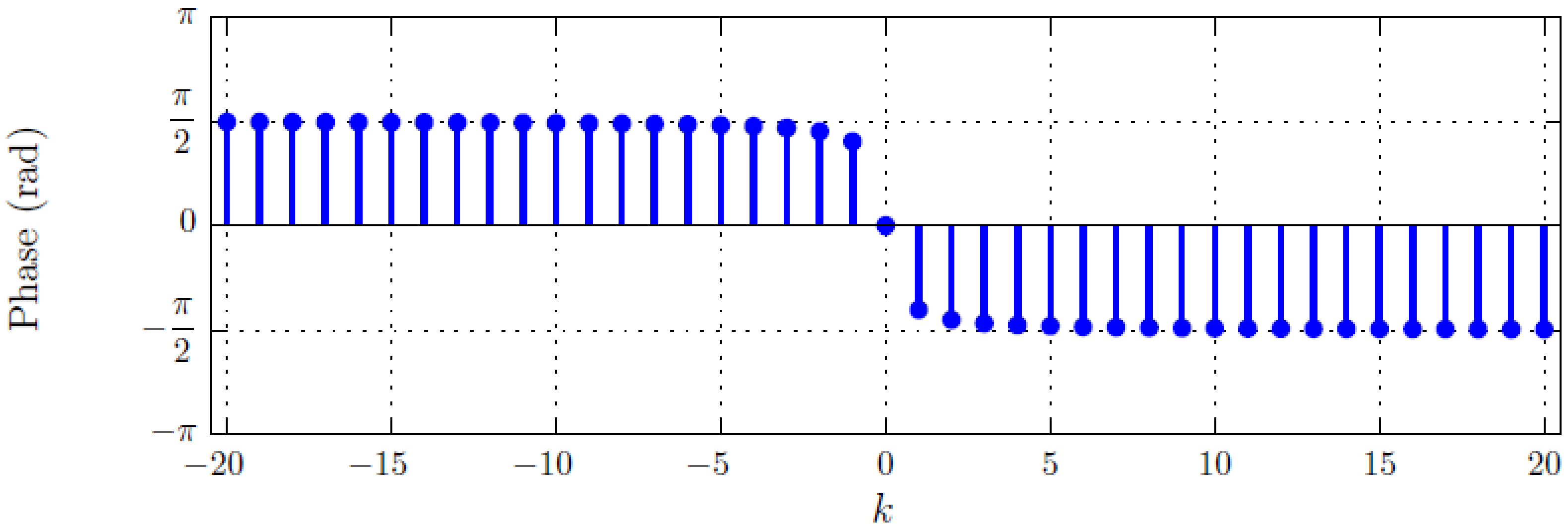
$$c_{-k} = \frac{1}{2}(a_k + jb_k) = \frac{0.4323}{1 - j2\pi k}, \quad k > 0$$

$$b.\ c_k = \int_0^1 e^{-2t} e^{-j2\pi kt} dt = \frac{0.4323}{1 + j2\pi k}, \quad \text{all } k$$

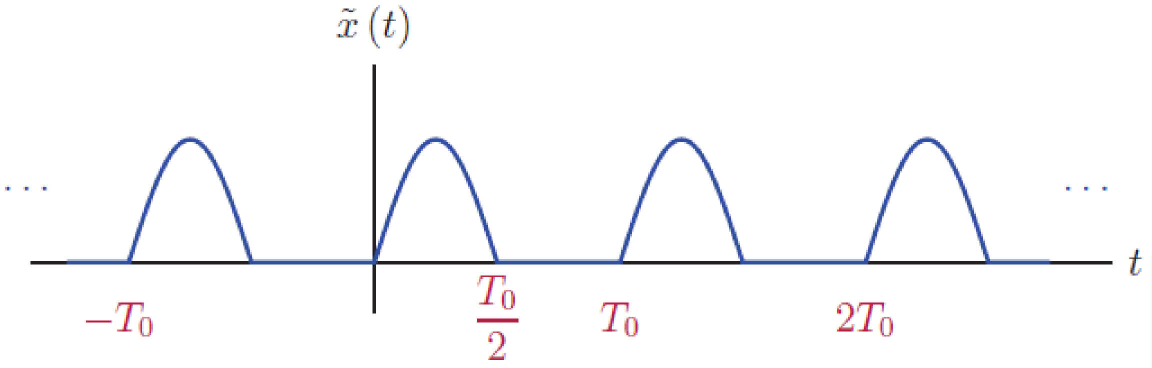
c.



$$\theta_k = \measuredangle\, c_k$$



3. Determine the EFS coefficients of the half-wave rectified signal shown:



$$c_k = \frac{1}{T_0}\int_0^{T_0/2} \sin(2\pi t/T_0) e^{-j2\pi kt/T_0} dt$$

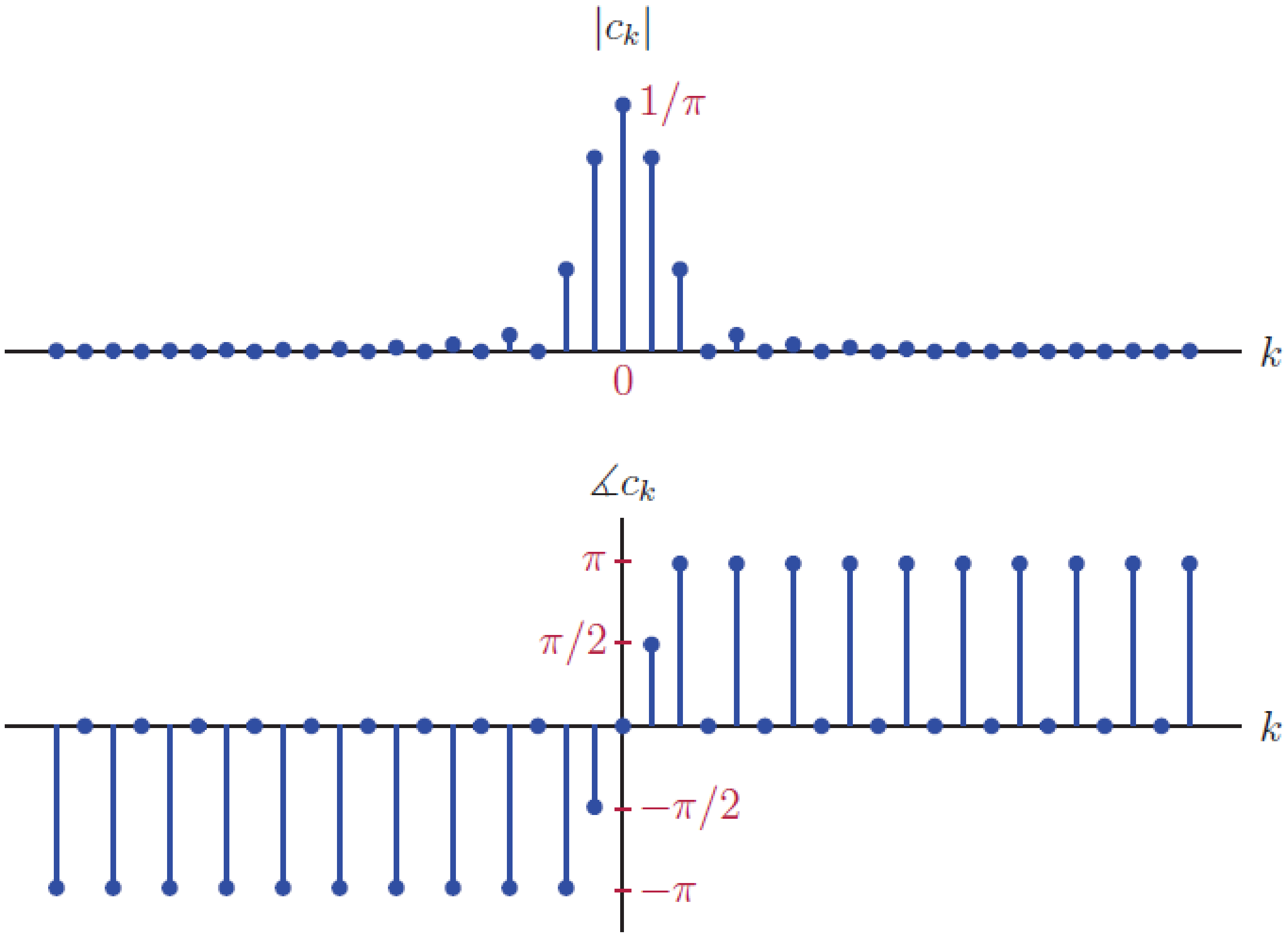
$$= \frac{1}{4\pi(k-1)}\left[e^{-j\pi(k-1)} - 1\right] - \frac{1}{4\pi(k+1)}\left[e^{-j\pi(k+1)} - 1\right]$$

Case 1: k odd and $k \neq \pm 1$ $\quad e^{-j\pi(k-1)} = e^{-j\pi(k+1)} = 1 \Rightarrow c_k = 0$

Case 2: $k = 1$ $\quad c_1 = -\frac{j}{4}$

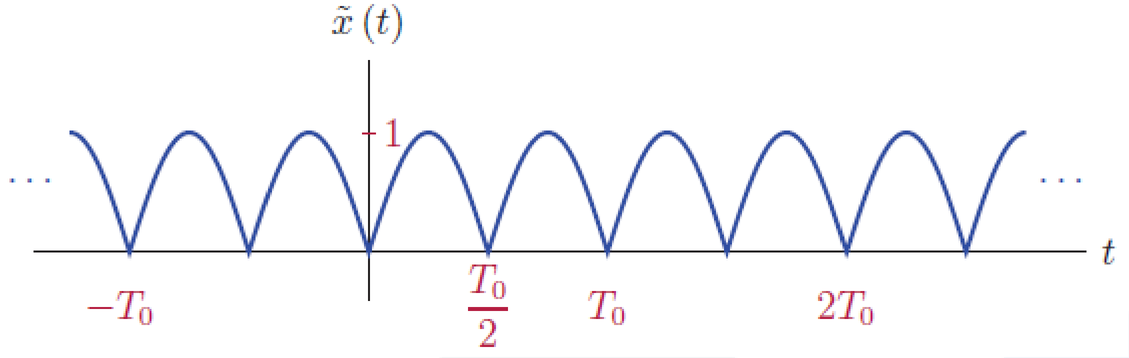
Case 3: $k = -1$ $\quad c_{-1} = \frac{j}{4}$

Case 4: k even $\quad e^{-j\pi(k-1)} - 1 = e^{-j\pi(k+1)} - 1 = -2 \Rightarrow c_k = \frac{-1}{\pi(k^2 - 1)}$



|ck|
1/π
0
k
∡ck
π
π/2
−π/2
−π
k

4. Determine the EFS coefficients of the full-wave rectified signal shown:



$$c_k = \frac{1}{T_0}\int_0^{T_0} \left|\sin(2\pi t/T_0)\right| e^{-j2\pi kt/T_0} dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} \sin(2\pi t/T_0) e^{-j2\pi kt/T_0} dt - \frac{1}{T_0}\int_{T_0/2}^{T_0} \sin(2\pi t/T_0) e^{-j2\pi kt/T_0} dt$$

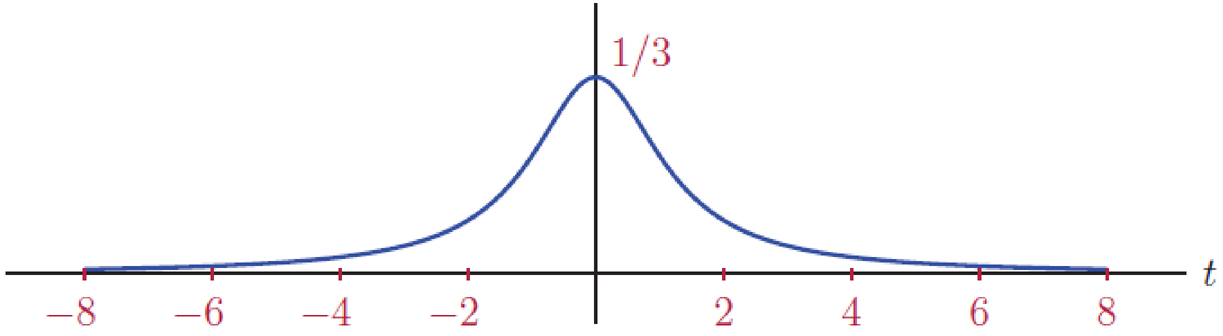
$$c_k = \begin{cases} \dfrac{2}{\pi(1-k^2)}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

5. The transform pair

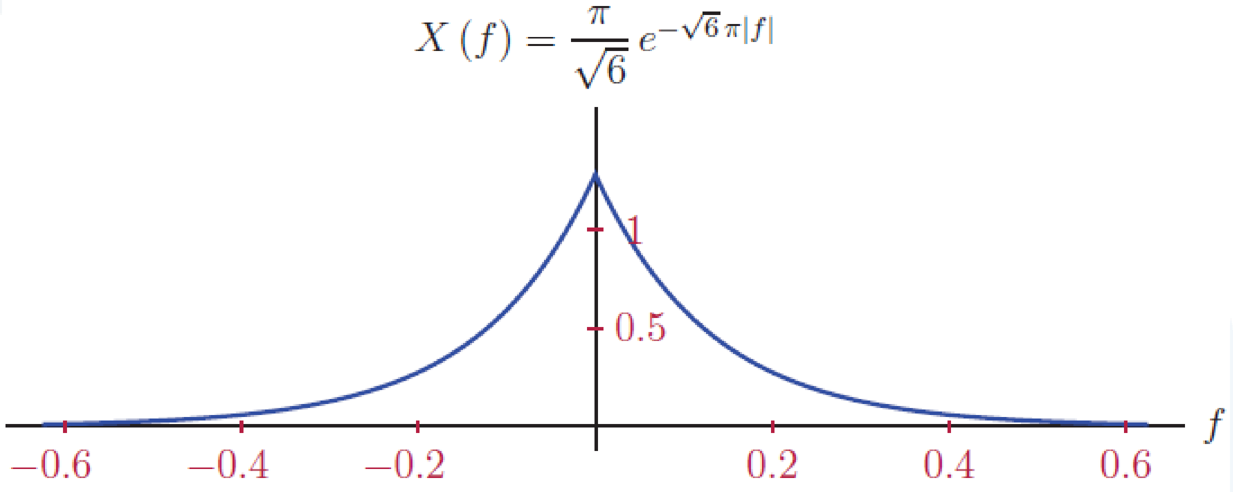
$$e^{-a|t|} \overset{F}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

Using this pair along with the duality property, find the Fourier transform of the signal

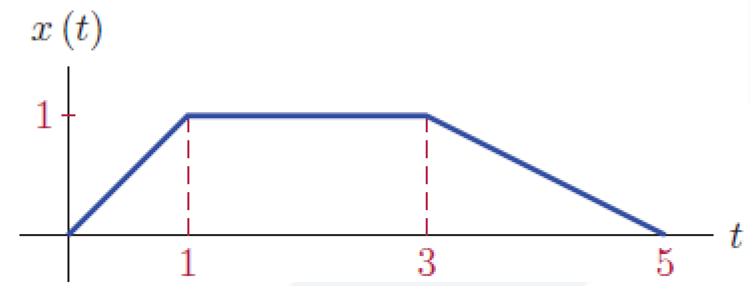
$$x(t) = \frac{1}{3 + 2t^2}$$



$$\frac{1}{3+2t^2} \overset{F}{\longleftrightarrow} \frac{\pi}{\sqrt{6}} e^{-\sqrt{3/2}|\omega|}$$



6. Using the differentiation-in-time property of the Fourier transform, determine the transform of the signal shown.



Let $w(t)$ be the derivative of the signal $x(t)$, that is

$$w(t) = \frac{dx(t)}{dt} = 2\Pi(t-0.5) - 0.5\Pi(\frac{t-4}{2})$$

$$W(f) = \text{sinc}(f)e^{j\pi f} - \text{sinc}(2f)e^{j8\pi f}$$

$$W(f) = j2\pi X(f) \Rightarrow X(f) = \frac{1}{j2\pi}W(f) = \frac{1}{j2\pi}\left[\text{sinc}(f)e^{j\pi f} - \text{sinc}(2f)e^{j8\pi f}\right]$$

7. Determine the Fourier transform of the signal

$$x(t) = \sin(\pi t)\Pi\left(t - \frac{1}{2}\right) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Using the modulation property of the Fourier transform

b. Using the multiplication property of the Fourier transform

$$a.\ \mathcal{F}\{p(t)\sin(2\pi f_0 t)\} = \frac{1}{2}\left[P(f - f_0)e^{-j\pi/2} + P(f + f_0)e^{j\pi/2}\right]$$

$$p(t) = \Pi\left(t - \frac{1}{2}\right) \Rightarrow P(f) = \mathrm{sinc}(f)\mathrm{e}^{-j\pi f}$$

f_0 = 0.5 Hz

$$X(f) = \frac{1}{2}\left[P(f-0.5)e^{-j\pi/2} + P(f+0.5)e^{j\pi/2}\right]$$

$$= \frac{1}{2}\left[\text{sinc}(f-0.5)e^{-j\pi(f-0.5)}e^{-j\pi/2} + \text{sinc}(f+0.5)e^{-j\pi(f+0.5)}e^{j\pi/2}\right]$$

$$= \frac{1}{2}[\text{sinc}(f-0.5) + \text{sinc}(f+0.5)]e^{-j\pi f}$$

b. $\mathcal{F}\{p(t)q(t)\} = P(t) * Q(t)$

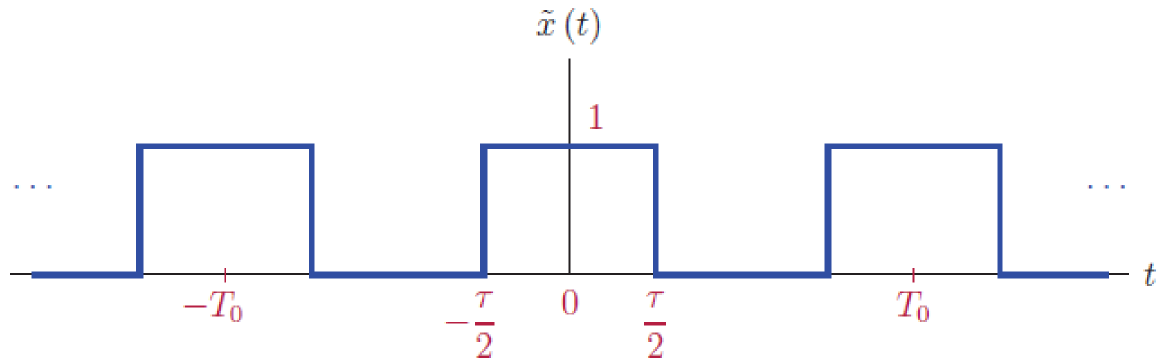
$$p(t) = \Pi\left(t - \frac{1}{2}\right) \Rightarrow P(f) = \text{sinc}(f)\text{e}^{-j\pi f}$$

$$q(t) = \sin(\pi t) \Rightarrow Q(f) = -j\frac{1}{2}\delta(f-0.5) + j\frac{1}{2}\delta(f+0.5)$$

$$X(f) = Q(f) * P(f) = \frac{1}{2}[\text{sinc}(f-0.5) + \text{sinc}(f+0.5)]e^{-j\pi f}$$

8. Consider the pulse train with duty cycle d shown. Its EFS $c_k = d\,\text{sinc}\,(kd)$.

a. Working in the time domain, compute the power of the pulse train as a function of the duty cycle d.



b. Sketch the power spectral density based on the EFS coefficients.

c. Let $d = 0.5$. Suppose this signal is processed through a lowpass system that only retains the first m harmonics and eliminates the others. How many harmonics should be retained if we want to preserve at least 90 percent of the signal power?

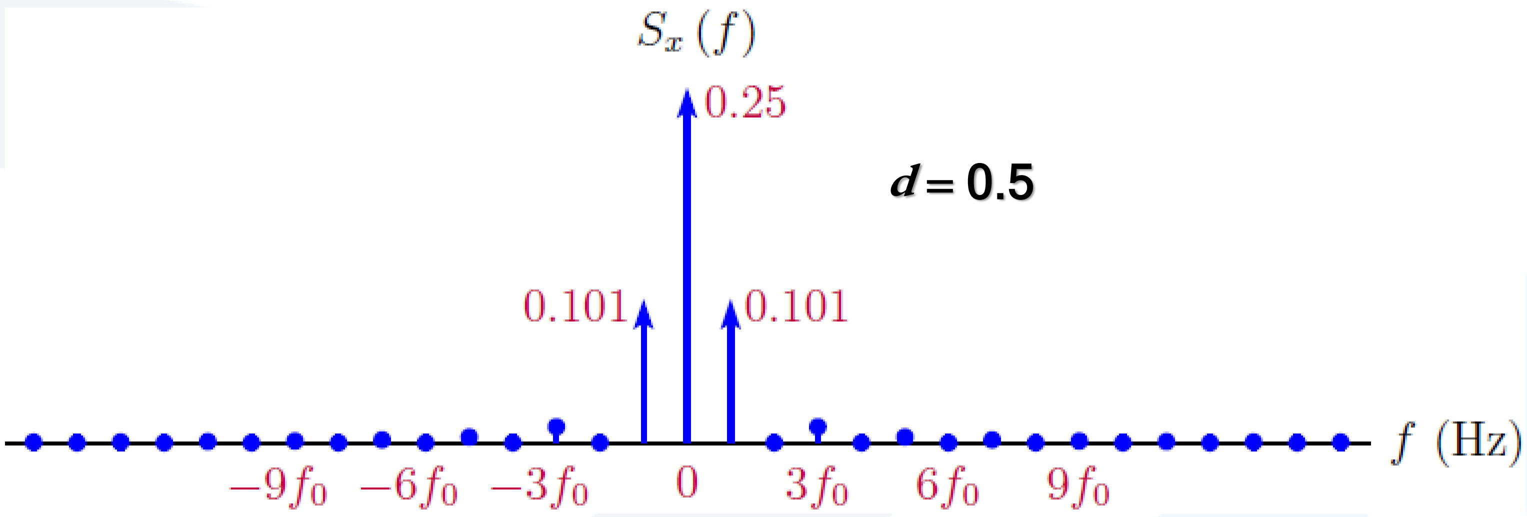
d. How many harmonics should be retained to preserve at least 95 percent of the signal power?

e. How many harmonics should be retained to preserve at least 99 percent of the signal power?

$$a.\ P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} dt = d \quad (\tau = dT_0)$$

$$b.\ c_k = d\,\text{sinc}(kd) \quad k = -\infty, \cdots, \infty$$

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(k - f_0) = \sum_{k=-\infty}^{\infty} d^2 \text{sinc}^2(kd)\delta(k - f_0)$$



c. If EFS terms up to and including the M-*th* harmonic are retained, the normalized average power of the signal would be

$$P_x^{(M)} = \sum_{k=-M}^{M} |c_k|^2 = \sum_{k=-M}^{M} d^2 \text{sinc}^2(kd)$$

and the percentage of this to the total average power in the signal $x(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^{M} d\,\mathrm{sinc}^2(kd)$$

It can be shown that, with $d = 0.5$ and $M = 1$ we get $\sum_{k=-1}^{1} 0.5\,\mathrm{sinc}^2(0.5d) = 0.9053$

d. Frequencies up to the third harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-3}^{3} 0.5\,\mathrm{sinc}^2(0.5d) = 0.9503$$

e. Frequencies up to the 21-st harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-21}^{21} 0.5\,\mathrm{sinc}^2(0.5d) = 0.9908$$

9. Repeat parts (c)-(e) of Problem 8 using $d = 0.2$. Does it take fewer or more harmonics to preserve the same percentage of power when the duty cycle is reduced?

c. If EFS terms up to and including the M-th harmonic are retained, the percentage of this to the total average power in the signal $x(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^{M} d\,\text{sinc}^2(kd)$$

It can be shown that, with $d = 0.2$ and $M = 4$ we get

$$\sum_{k=-4}^{4} 0.2\,\text{sinc}^2(0.2d) = 0.9029$$

d. Frequencies up to the 11-*st* harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-11}^{11} 0.2\,\mathrm{sinc}^2(0.2d) = 0.9528$$

e. Frequencies up to the 51-*st* harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-51}^{51} 0.2\,\mathrm{sinc}^2(0.2d) = 0.9900$$

10. Determine and sketch the power spectral density of the following signals:

a. $x(t) = 3\cos(20\pi t)$

b. $x(t) = 2\cos(20\pi t) + 3\cos(30\pi t)$

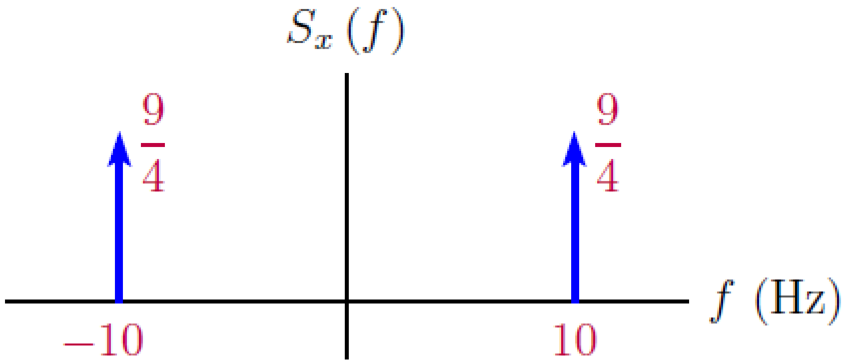
c. $x(t) = 5\cos(200\pi t) + 5\cos(200\pi t)\cos(30\pi t)$

a. For the signal $x(t)$ the fundamental frequency is f_0 = 10 Hz, and the EFS coefficients are

$$c_k = \begin{cases} \frac{3}{2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

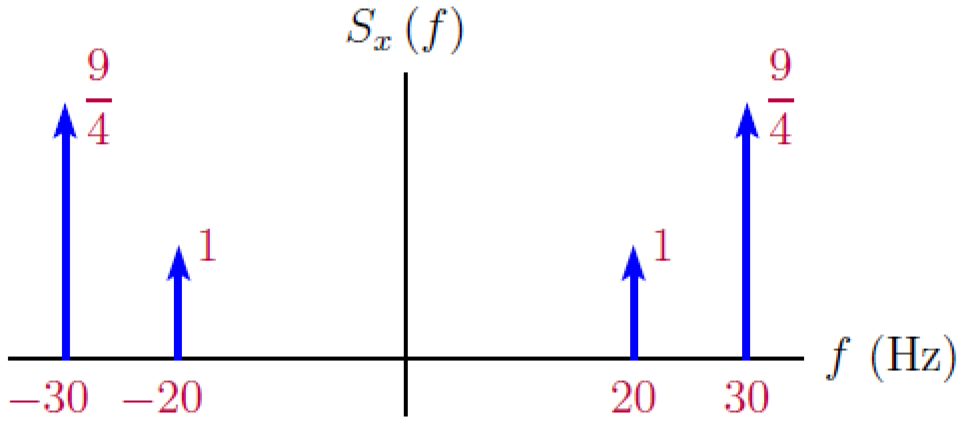
The power spectral density is

$$S_x(f) = \frac{9}{4}\delta(f+10) + \frac{9}{4}\delta(f-10)$$



b. For the signal $x(t)$ the fundamental frequency is f_0 = 5 Hz, and the EFS coefficients are

$$c_k = \begin{cases} 1, & k = \pm 2 \\ \frac{3}{2}, & k = \pm 3 \\ 0, & \text{otherwise} \end{cases}$$

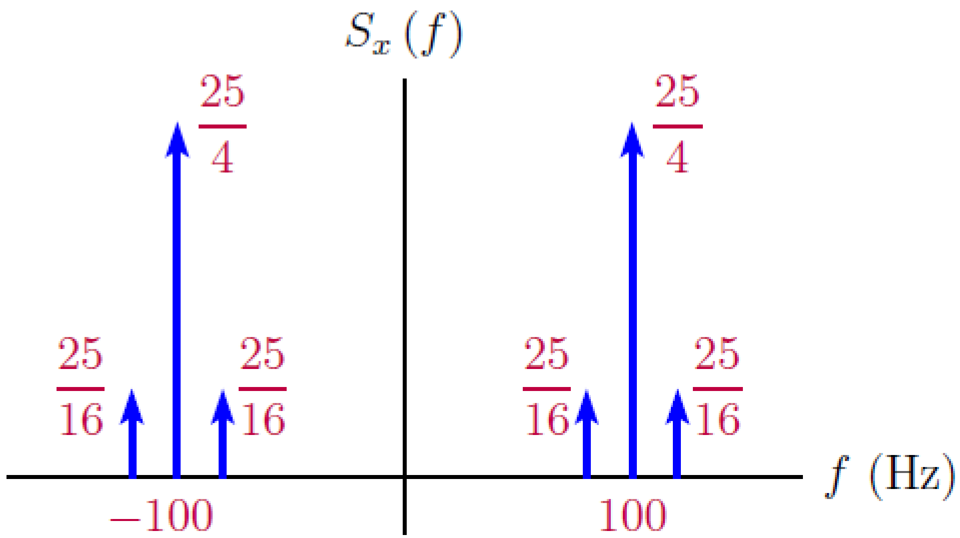


The power spectral density is

$$S_x(f) = \tfrac{9}{4}\,\delta(f+30) + \delta(f+20) + \delta(f-20) + \tfrac{9}{4}\,\delta(f-30)$$

c. For the signal $x(t)$ the fundamental frequency is f_0 = 5 Hz, and the EFS coefficients are

$$c_k = \begin{cases} \frac{5}{4}, & k = \pm 17 \\ \frac{5}{2}, & k = \pm 20 \\ \frac{5}{4}, & k = \pm 23 \\ 0, & \text{otherwise} \end{cases}$$



The power spectral density is

$$S_x(f) = \tfrac{25}{16}\delta(f+230) + \tfrac{25}{4}\delta(f+200) + \tfrac{25}{16}\delta(f+170) + \tfrac{25}{16}\delta(f-170) + \tfrac{25}{4}\delta(f-200) + \tfrac{25}{16}\delta(f-230)$$