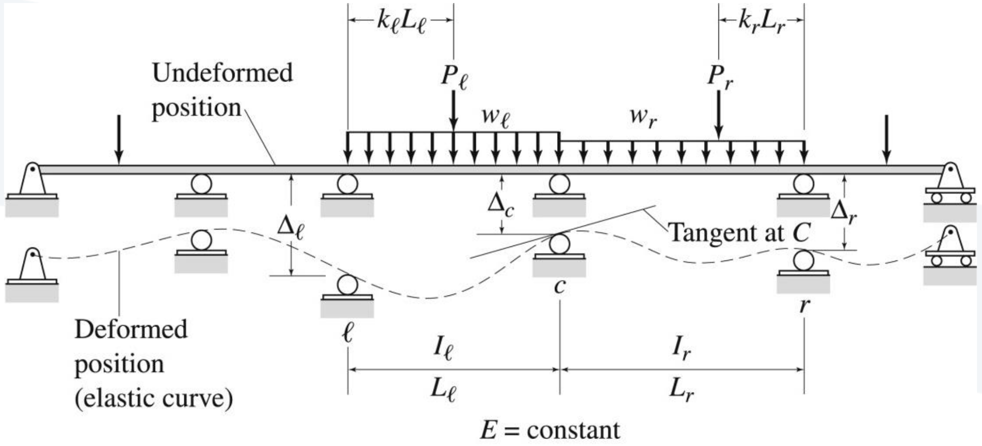
Structural Mechanics (1)

Analysis of Indeterminate Structures - Force Method

- Indeterminate Structures vs. Determinate Structures
- Analysis of Indeterminate Structures.
- Structures with single Degree of Indeterminacy (Beams & Frames)
- Structures with single Degree of Indeterminacy (Trusses: Int. & Ext.)
- Structures with multiple Degrees of Indeterminacy
- Support Settlements
- Three-Moment Equation for Continuous Beams

Derivation of three-Moment Equation

A continuous beam under: ex. loads & sup. settlements.



(a) Continous Beam

Supports: l, c, r. Spans: L_l , L_r. Mo. of In.: I_l , I_r

Con. Loads: P_l at k_lL_l from l & P_r at k_rL_r from r.

Dis. Loads: w_l over L_l and w_r over L_r.

Sup. Settlements: Δ_l at l, Δ_c at c, Δ_r at r.

Redundants are internal moments at supports: …, M_l, M_c, M_r ,.. Compatibility Equations are slope continuities at the supports.

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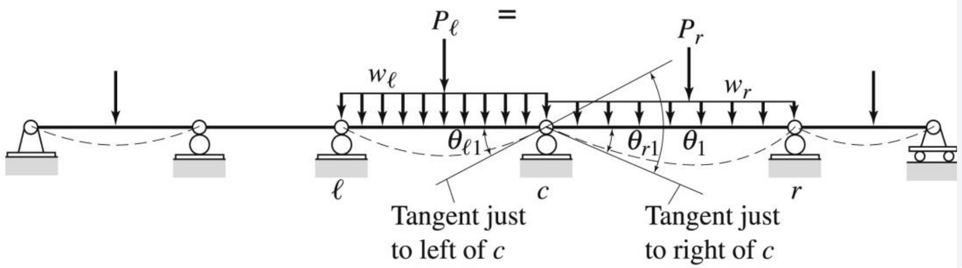

(b) Primary Structure Subjected to External Loading

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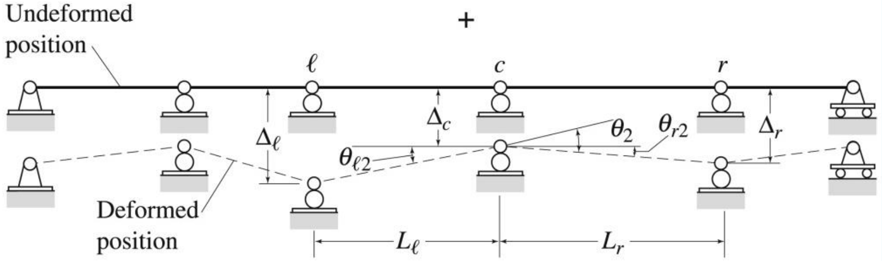

(c) Primary Structure Subjected to Support Settlements

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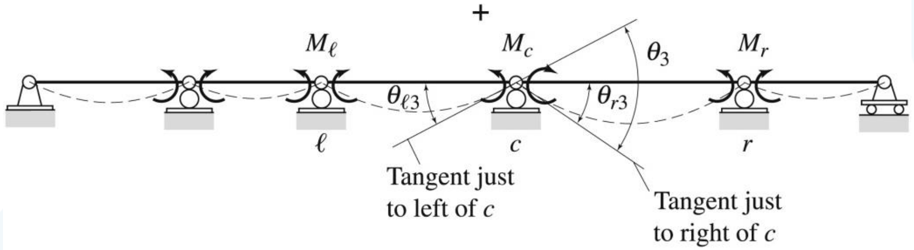


(d) Primary Structure Loaded with Redundant Bending Moments

From ex. Loads: $\theta_1=\theta_{l1}+\theta_{r1}$

From sup. settlements: $\theta_2=\theta_{l2}+\theta_{r2}$

From sup. bending moments: $\theta_3=\theta_{l3}+\theta_{r3}$

Comp. Equation at support c is: $\theta_1+\theta_2+\theta_3=0$

Or in detail: $\theta_{l1}+\theta_{r1}+ \theta_{l2}+\theta_{r2} + \theta_{l3}+\theta_{r3}=0$

Derivation of three-Moment Equation

$$\theta_{l1}=\sum\frac{P_l L_l^2 k_l(1-k_l^2)}{6EI_l}+\frac{w_l L_l^3}{24EI_l} \qquad \theta_{l2}=\frac{\Delta_l-\Delta_c}{L_l}$$

$$\theta_{r1}=\sum\frac{P_r L_r^2 k_r(1-k_r^2)}{6EI_r}+\frac{w_r L_r^3}{24EI_r} \qquad \theta_{r2}=\frac{\Delta_r-\Delta_c}{L_r}$$

$$\theta_{l3}=\frac{M_l L_l}{6EI_l}+\frac{M_c L_l}{3EI_l}; \qquad \theta_{r3}=\frac{M_c L_r}{3EI_r}+\frac{M_r L_r}{6EI_r}$$

Substituting in to the comp. equation at support c:

$\theta_{l1}+\theta_{r1}+\theta_{l2}+\theta_{r2}+\theta_{l3}+\theta_{r3}=0$ we get

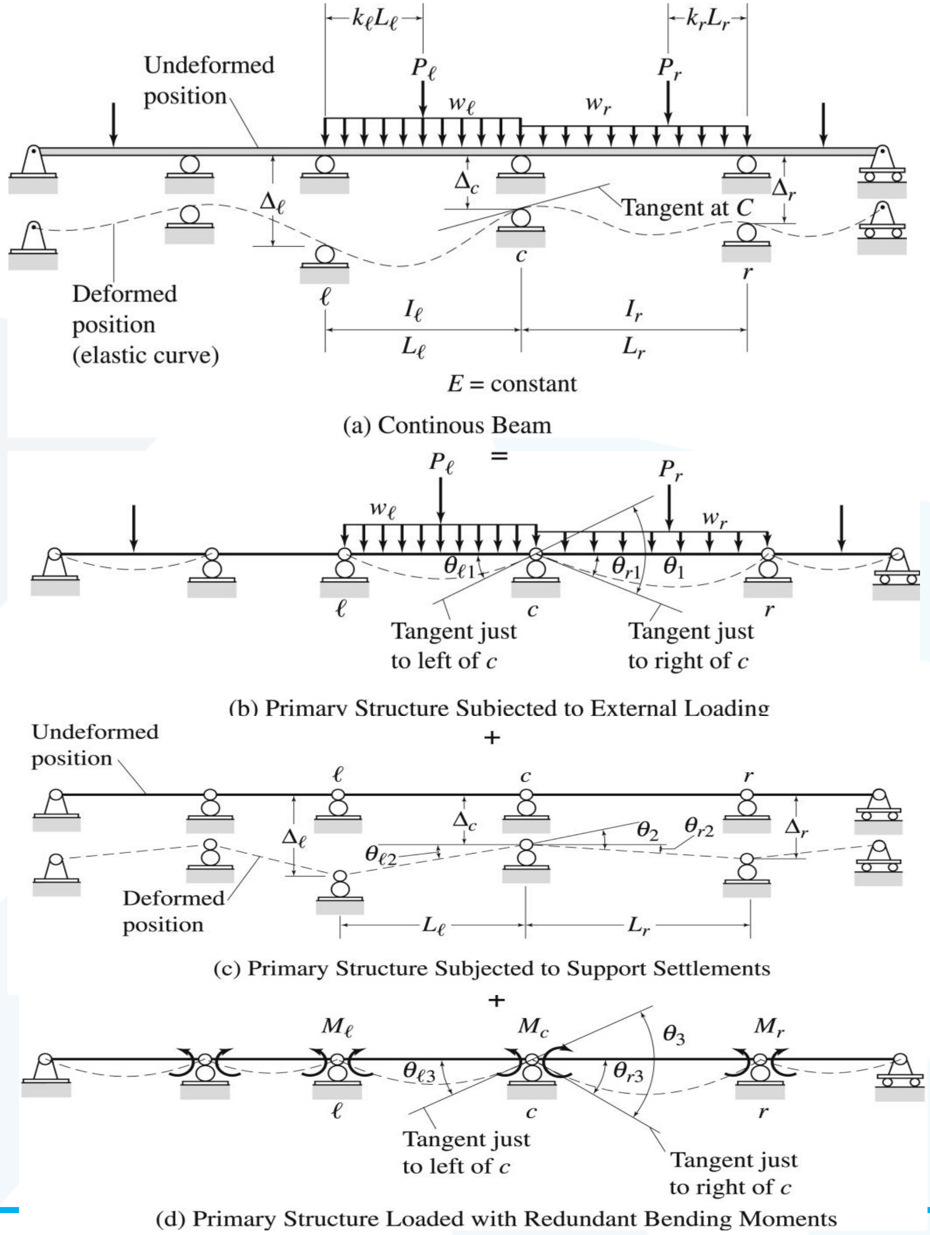
$$\sum\frac{P_l L_l^2 k_l(1-k_l^2)}{6EI_l}+\frac{w_l L_l^3}{24EI_l}+\sum\frac{P_r L_r^2 k_r(1-k_r^2)}{6EI_r}+\frac{w_r L_r^3}{24EI_r}$$

$$+\frac{\Delta_l-\Delta_c}{L_l}+\frac{\Delta_r-\Delta_c}{L_r}+\frac{M_l L_l}{6EI_l}+\frac{M_c L_l}{3EI_l}+\frac{M_c L_r}{3EI_r}+\frac{M_r L_r}{6EI_r}=0$$

Keeping the unknowns in LS and simplifying we get the three-moment equation

$$\frac{M_l L_l}{I_l}+2M_c\left(\frac{L_l}{I_l}+\frac{L_r}{I_r}\right)+\frac{M_r L_r}{I_r}=-\sum\frac{P_l L_l^2 k_l(1-k_l^2)}{I_l}$$

$$-\sum\frac{P_r L_r^2 k_r(1-k_r^2)}{I_r}-\frac{w_l L_l^3}{4I_l}-\frac{w_r L_r^3}{4I_r}-6E\left(\frac{\Delta_l-\Delta_c}{L_l}+\frac{\Delta_r-\Delta_c}{L_r}\right)$$



Derivation of three-Moment Equation

$$\frac{M_l L_l}{I_l} + 2M_c\left(\frac{L_l}{I_l} + \frac{L_r}{I_r}\right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1-k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1-k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r} - 6E\left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r}\right)$$

If the continuous beam has a constant section ($I_l = I_r = I$) the three-moment equation simplifies to

$$M_l L_l + 2M_c (L_l + L_r) + M_r L_r = -\sum P_l L_l^2 k_l (1-k_l^2) - \sum P_r L_r^2 k_r (1-k_r^2) - \frac{1}{4}(w_l L_l^3 + w_r L_r^3) - 6EI\left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r}\right)$$

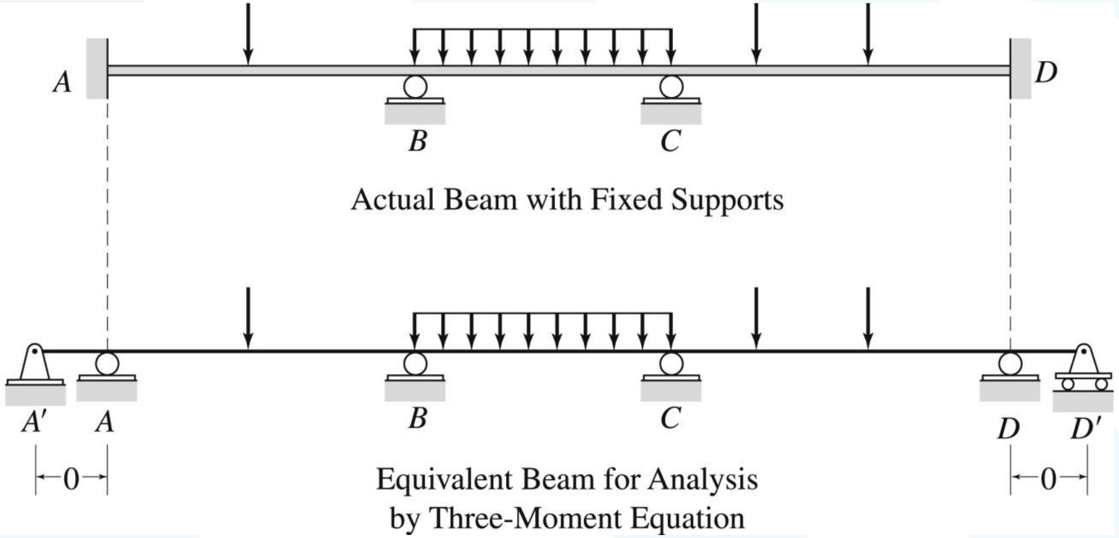
If in addition the two spans are equal ($L_l = L_r = L$) the three-moment equation becomes

$$M_l + 4M_c + M_r = -\sum P_l L k_l (1-k_l^2) - \sum P_r L k_r (1-k_r^2) - \frac{L^2}{4}(w_l + w_r) - \frac{6EI}{L^2}(\Delta_l - 2\Delta_c + \Delta_r)$$

The forgoing forms of the three-moment equation are applicable to any three consecutive supports *l*, *c* & *r* of a continuous beam, provided that there are no discontinuities, such as internal hinges, in the beam between the left support *l* and the right support *r*.

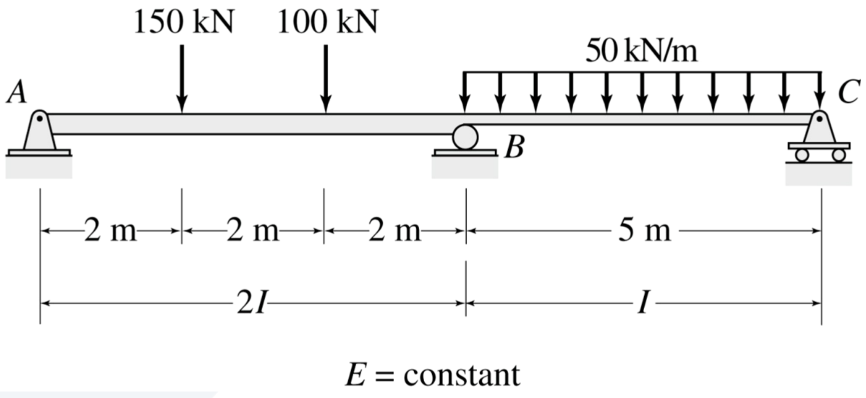
Fixed Supports

The above three-moment equations where derived to satisfy comp. equ. at any interior support. For a comp. equ. of zero slope at a fixed end support, the beam can be imaginary modified by adjoining end spans of zero length simply supported as shown



Three-Moment Equation - Examples

Ex-01. Determine the reactions and draw the shear force & bending moment diagrams for the beam shown in the figure by using the three-moment equation



Solution:

The continuous beam *ABC* has one degree of indeterminacy. It has two different spans of different rigidity. There is no support settlement. So we use the following three-moment equation.

$$\frac{M_l L_l}{I_l} + 2M_c\left(\frac{L_l}{I_l} + \frac{L_r}{I_r}\right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1-k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1-k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r}$$

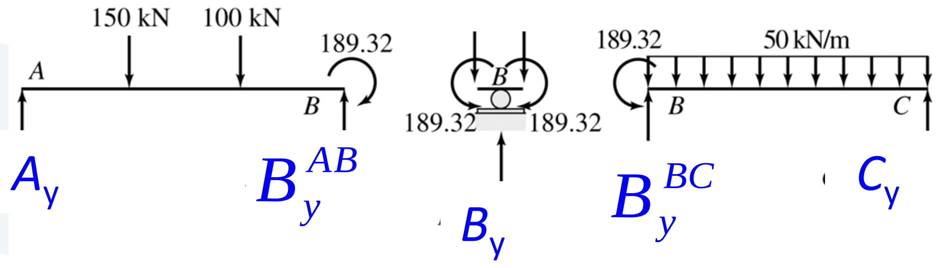
Considering the support *B* as the central one with two null moments at *A* & *B*, we write

$$0 + 2M_B\left(\frac{6}{2I} + \frac{5}{I}\right) + 0 = -\frac{\{(150)(6)^2(2/6)[1-(2/6)^2] + (100)(6)^2(4/6)[1-(4/6)^2]\}}{2I} - \frac{(50)(5)^3}{4I}$$

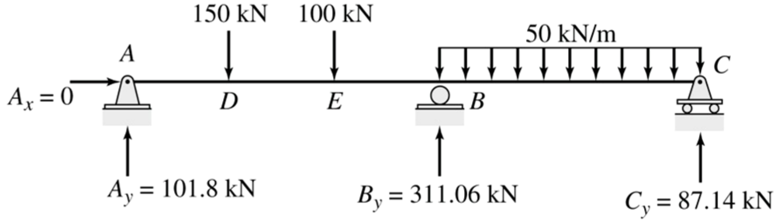
Canceling I in both sides and simplifying we get

$$16M_B = -3029.17 \qquad \Rightarrow \qquad M_B = -189.32\,\text{kN·m}$$

Three-Moment Equation - Examples

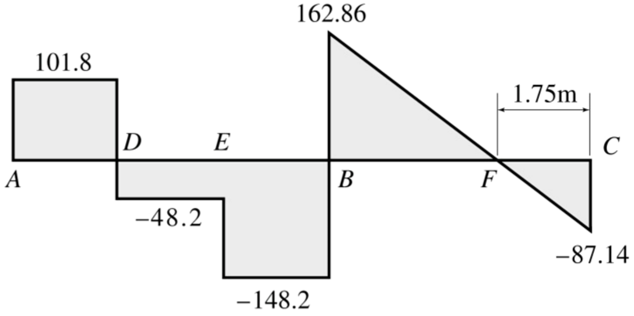


(b) Span End Moments and Shears

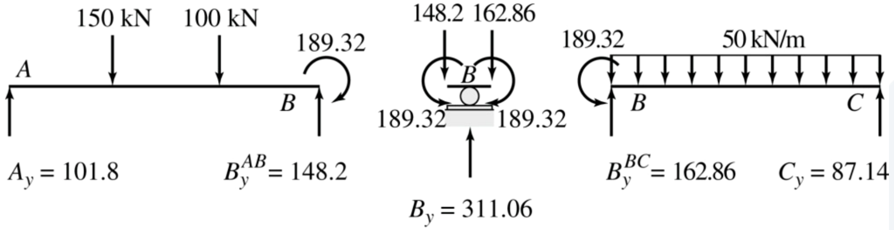


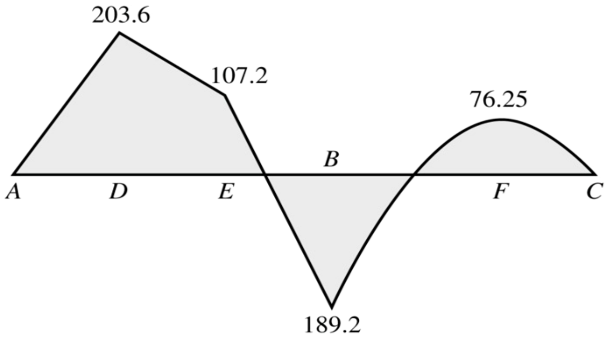
$$+\downarrow\uparrow\Sigma M_B = 0 \Rightarrow -A_y(6)+150(4)+100(2)-189.32 = 0$$

$$\Rightarrow A_y = 101.8 \text{ kN}$$



Shear diagram (k)

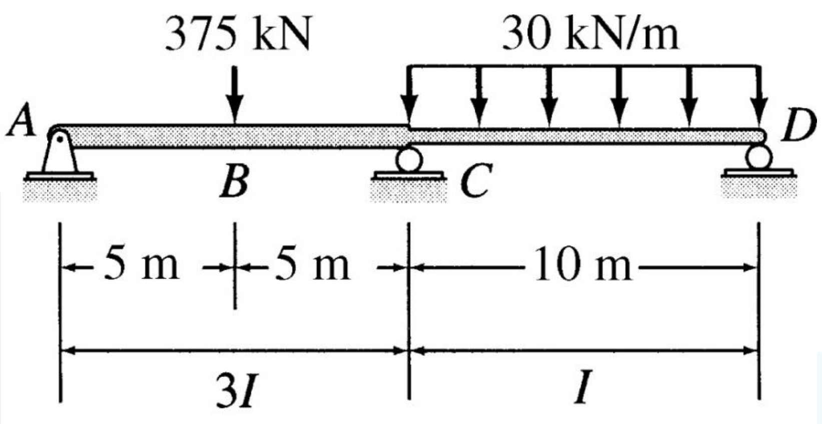




Bending moment diagram (kN-m)

Three-Moment Equation - Examples

Ex. 02. Determine the reaction for the continuous beam ABCD shown in the figure.

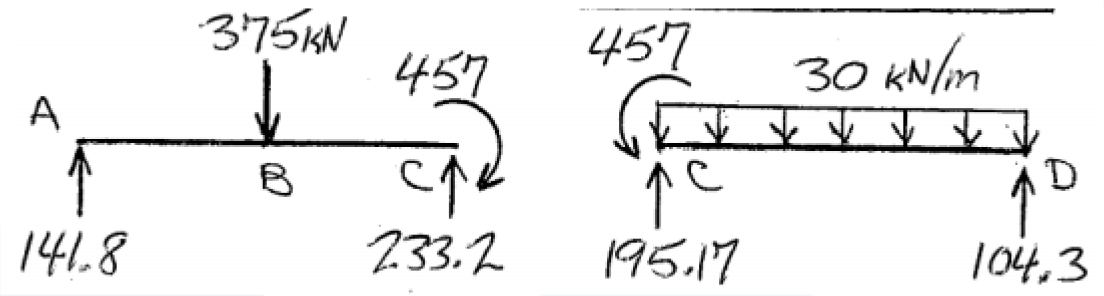


Solution:

The continuous beam *ABCD has one degree of indeterminacy. It has two* spans of different rigidity. There is no support settlement. So we use the following three-moment equation.

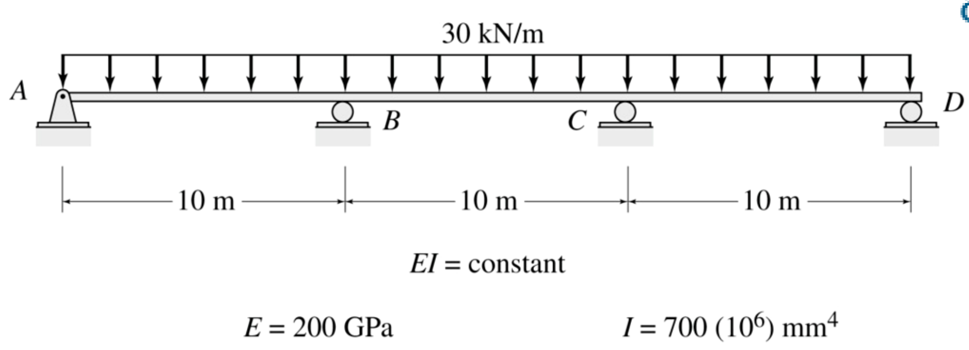
$$\frac{M_l L_l}{I_l} + 2M_c\left(\frac{L_l}{I_l} + \frac{L_r}{I_r}\right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1-k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1-k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r}$$

$$M_C = -457 \text{ kN·m}$$



Three-Moment Equation - Examples

Ex. 03. Determine the reactions for the continuous beam ABCD shown in the figure, due to the uniformly distributed load and due to support settlement of 10 mm at A, 50 mm at B, 20 mm at C and 40 mm at D. Using the three-moment equation

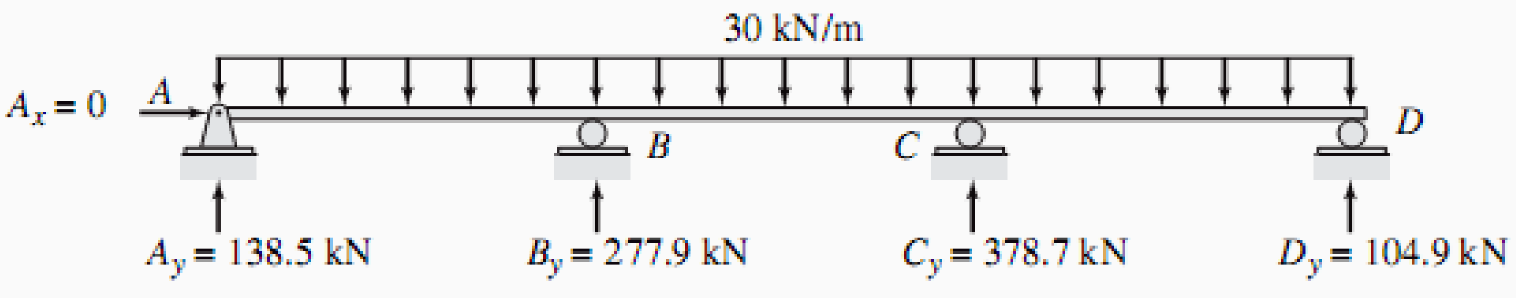


Solution:

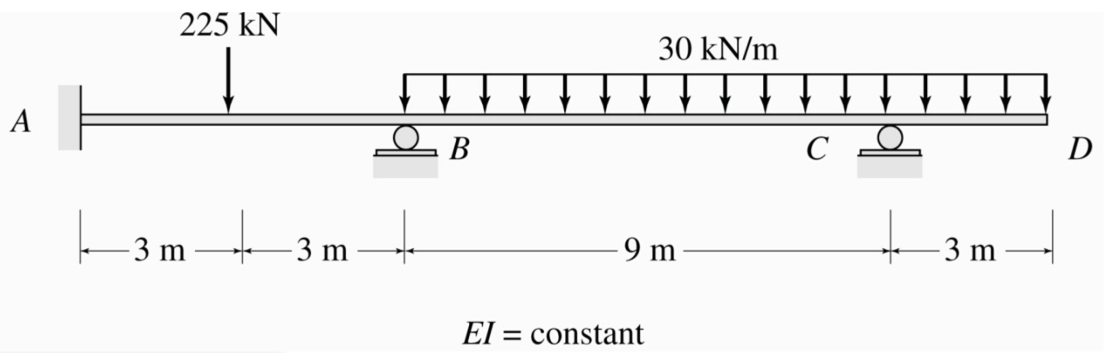
$$M_l + 4M_c + M_r = -\sum P_l L k_l (1-k_l^2) - \sum P_r L k_r (1-k_r^2) - \frac{L^2}{4}(w_l + w_r) - \frac{6EI}{L^2}(\Delta_l - 2\Delta_c + \Delta_r)$$

$$M_B = -115.2 \text{ kN-m}$$

$$M_C = -451.2 \text{ kN-m}$$



Ex. 04. Determine the reaction for the continuous beam ABCD shown in the figure,



Fixed Support & Extended Cantilever

$$M_l L_l + 2M_c(L_l + L_r) + M_r L_r = -\sum P_l L_l^2 k_l (1-k_l^2) - \sum P_r L_r^2 k_r (1-k_r^2) - \frac{1}{4}(w_l L_l^3 + w_r L_r^3) - 6EI\left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r}\right)$$

$$M_A = -146.25 \text{ kN-m}$$

$$M_B = -213.75 \text{ kN-m}$$