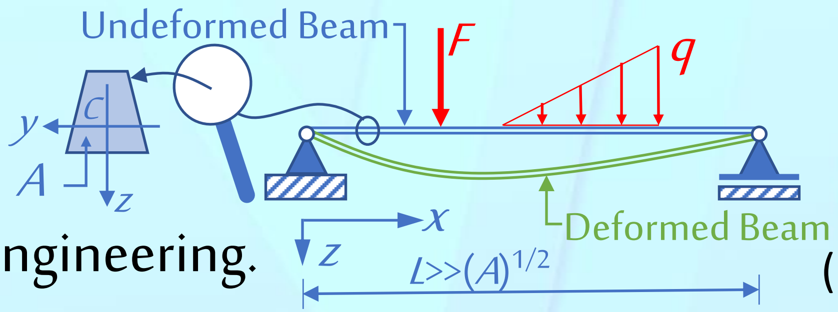
4 Bending of Beams - - إنعطاف الجيزان

4.1 Introduction



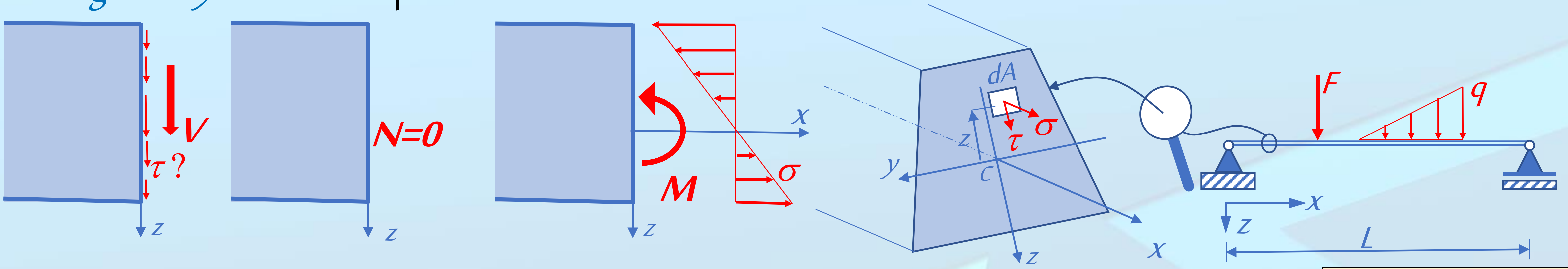
Beams are among the most important elements in structural engineering. (Fig.a).

A beam is straight bar with the dimensions of its cross-sectional area *A* are much smaller than its length *L*.

However, in contrast to the members of a truss it is loaded by forces which are perpendicular to its axis.

Then, the originally straight beam deforms (Fig.a). This is referred to as the ***bending of the beam***.

As a consequence, internal forces (= stresses: σ & τ) are generated in the beam, the resultants of which are the shear force *V* and the bending moment *M* (Mechanics of Materials 1, L6 & L7). It is the aim of the *bending theory* to derive equations that allow the determination of the stresses and the deformations.



$$V = \int_A \tau dA$$

$$N = \int_A \sigma dA = 0$$

$$M = \int_A z\sigma dA$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q(x)$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V(x)$$

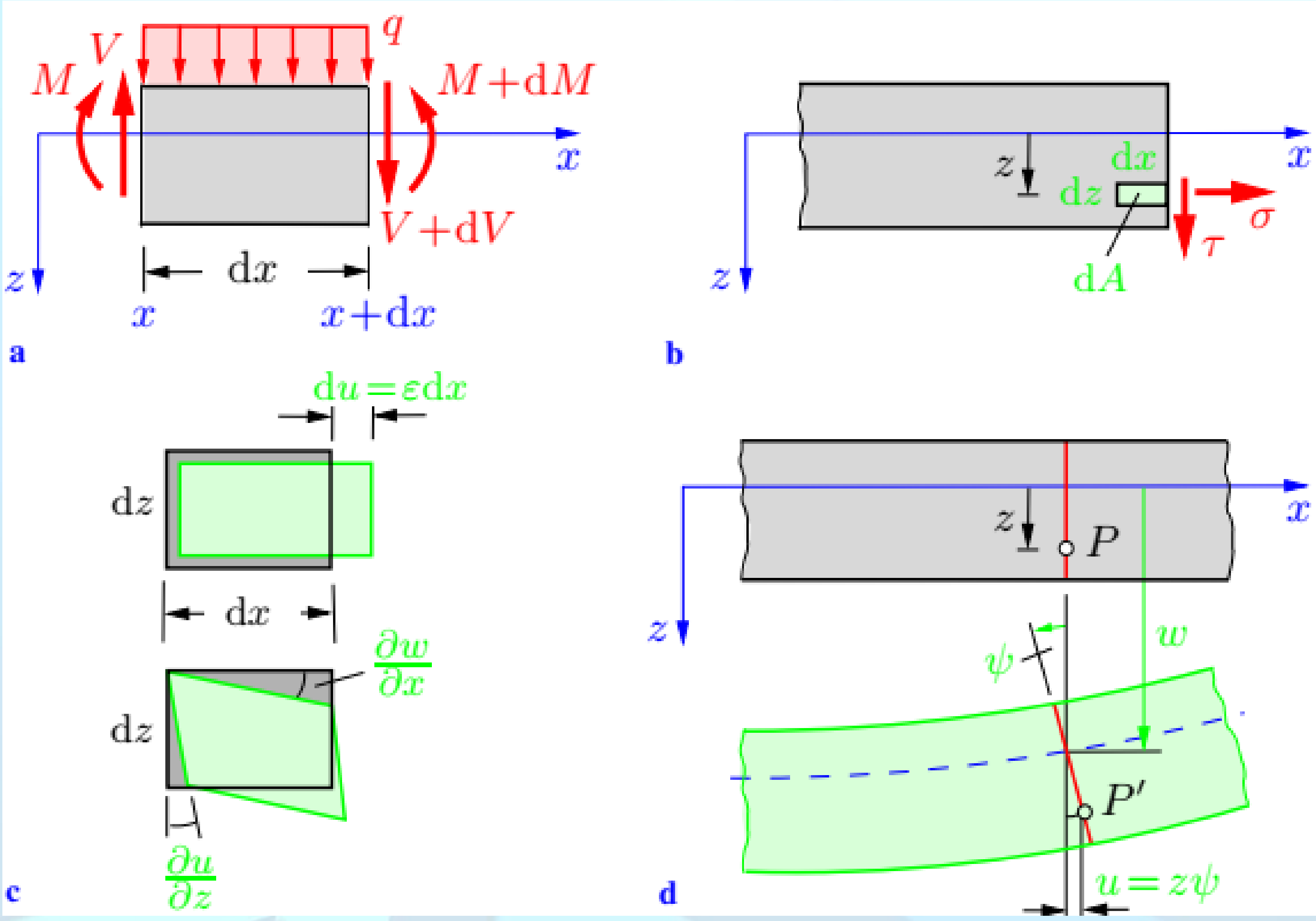
$$\frac{d^2M}{dx^2} = -q(x)$$

4.2 Basic Equations of Ordinary Bending Theory (Simple Beam Theory)

Equations enabling the determination of the stresses and deformations due to the bending of a beam, will now be derived . In the following we restrict ourselves to *ordinary (uniaxial) bending*, i.e., we assume that the axis *z* is an axis of symmetry of the cross section & the loads act in the *z-x* plane.

$$V = \int_A \tau dA \qquad N = \int_A \sigma dA = 0 \qquad M = \int_A z\sigma dA$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q(x) \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V(x) \qquad \frac{d^2M}{dx^2} = -q(x)$$



In addition to the previous statics equations, Hooke's law and the geometrical (kinematic) relations will be used.

Assuming that the normal stresses σ_y & σ_z in the beam are neglected compared with σ_x. Then Hooke's law is given by

$$\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon \;\;\&\;\; \tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$$

$$V = \int_A \tau dA \qquad N = \int_A \sigma dA = 0 \qquad M = \int_A z\sigma dA \qquad \frac{dV}{dx} = -q(x) \qquad \frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = -q(x)$$

$$\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon \;\;\&\;\; \tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$$

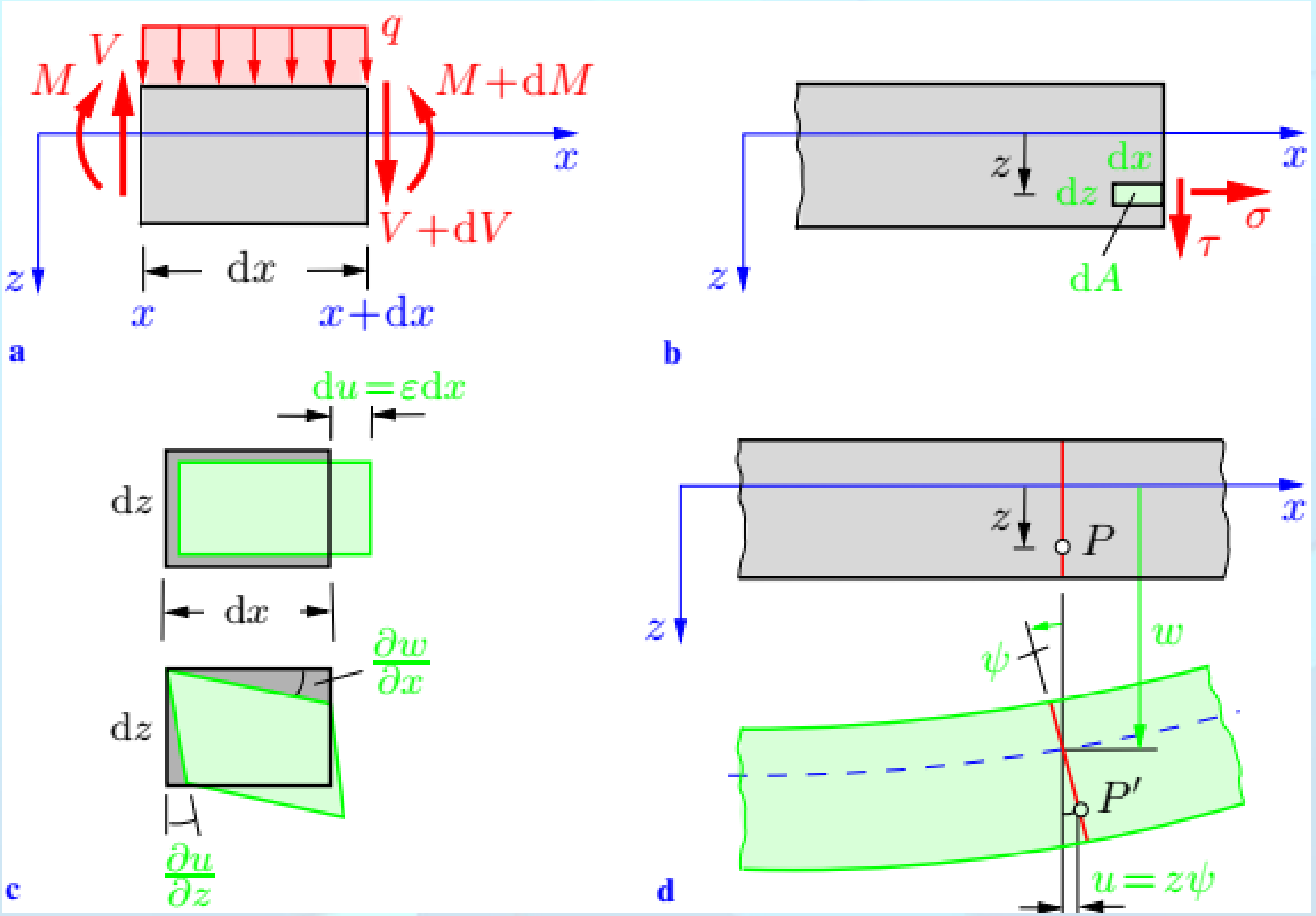
additional assumptions

a) The displacement w is independent of z:

$$w = w(x)$$

This implies that the height of the beam does not change due to bending: $\varepsilon_z = \partial w/\partial z = 0$.

b) Plane cross sections of the beam remain plane during the bending. In addition to the displacement w, a cross section undergoes a rotation. The angle of rotation $\psi = \psi(x)$ is a *small angle*; it is counted as positive if the rotation is counterclockwise. Thus,



The displacement u of a point P which is located at a distance z from the x-axis is given by $u(x,z) = \psi(x)\,z$.

$V = \int_A \tau dA$ | $N = \int_A \sigma dA = 0$ | $M = \int_A z\sigma dA$ | $\frac{dV}{dx} = -q(x)$ | $\frac{dM}{dx} = V(x)$ | $\frac{d^2M}{dx^2} = -q(x)$

$u(x,z) = \psi(x)\, z$ | $w = w(x)$ | $\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon \;\&\; \tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$

Hooke's Law into Kinematic relations

$$\sigma = E\varepsilon = E\frac{\partial u}{\partial x} = E\frac{\partial \psi}{dx} z = E\psi' z$$

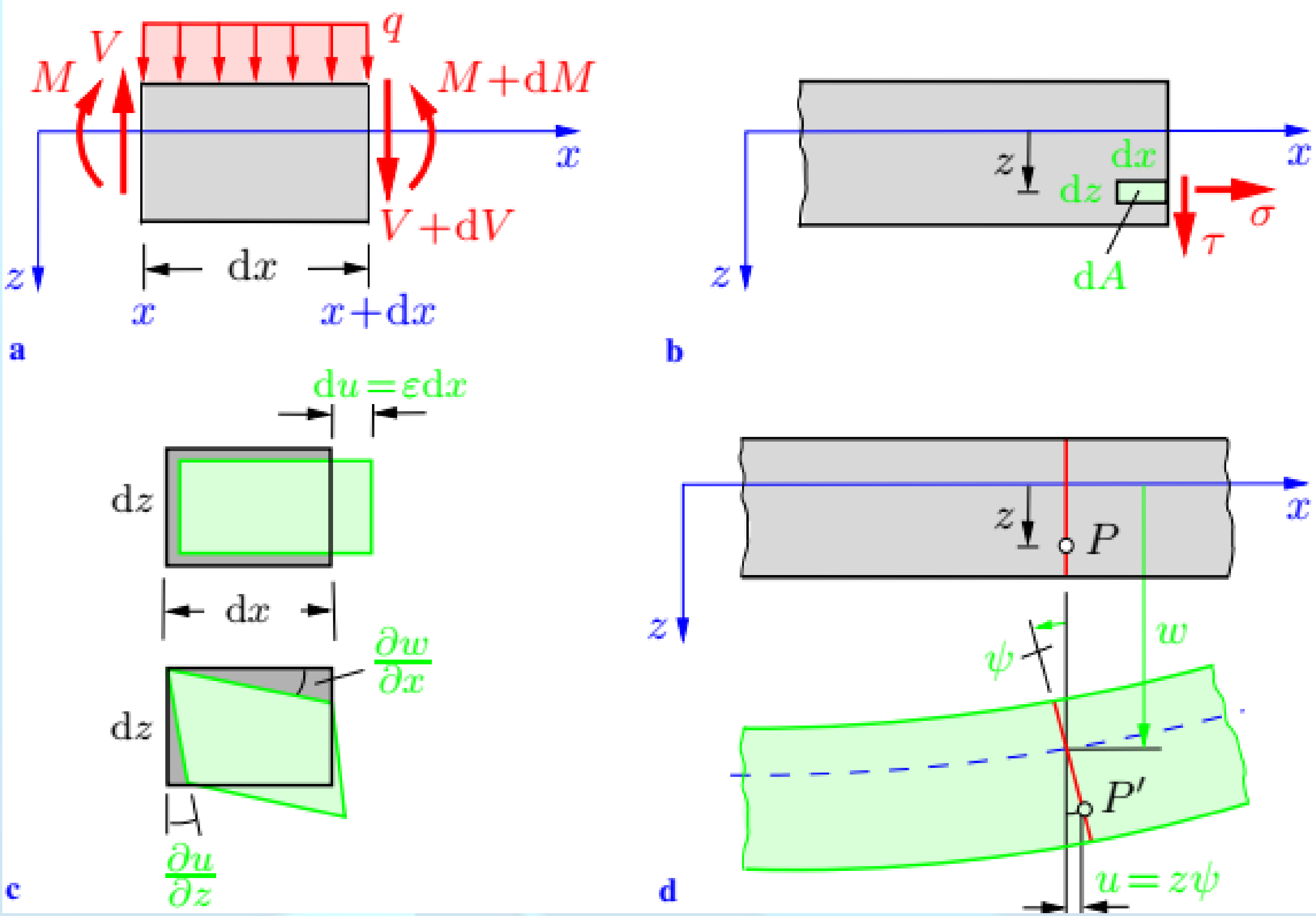
$$\tau = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = G(\psi(x) + w')$$

where $d(\,)/dx = (\,)'$ and w' represents the slope of the deformed axis of the beam.

$$\boxed{N = \int_A \sigma dA = 0 = E\psi' \int_A z dA = 0}$$

which implies that the y-axis has to be a centroidal axis: C is the centroid of the section.

$$\boxed{M = \int_A z\sigma dA = E\psi' \int_A z^2 dA = EI_y\psi'}$$



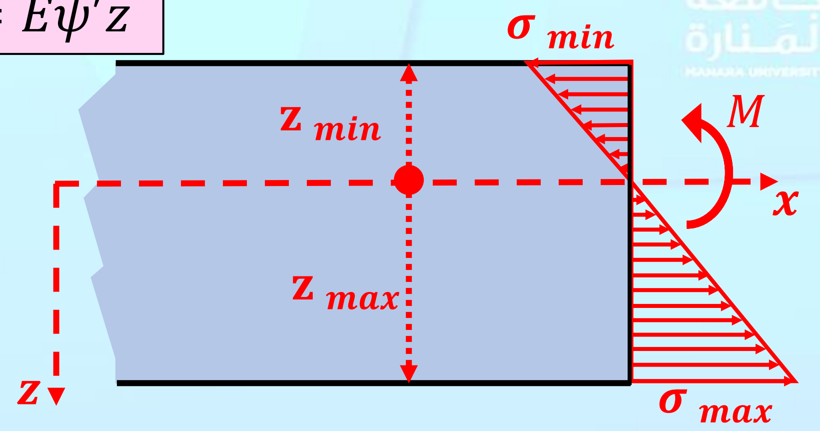
Where $I_y = \int_A z^2 dA$ is the second moment of area about y.

4.3 Normal Stresses in Bending Beams

4.3 الإجهاد الناظمي في انعطاف الجيزان:

$$\boxed{M = \int_A z\sigma dA = E\psi' \int_A z^2 dA = EI_y\psi'} \Rightarrow \boxed{E\psi' = \frac{M}{I_y}}$$ Sub. into $\boxed{\sigma = E\psi' z}$

$\Rightarrow$ Bending formula $\boxed{\sigma = \frac{M}{I_y} z}$ I_y is [L^4] Compare with $\boxed{\sigma = \frac{N}{A}}$



It shows that the normal stresses, which are referred to as the *flexural* or *bending stresses* (إجهاد الانعطاف), are linearly distributed in z-direction as shown in Fig. If the bending moment M is positive, the stresses are positive (tensile stresses) for $z > 0$ and they are negative (compressive stresses) for $z < 0$. For $z = 0$ (i.e., in the x, y-plane) we have $\sigma = 0$. Since $\varepsilon = \sigma/E$, the strain ε is also zero in the x, y plane: the fibers in this plane do not undergo any elongation or contraction. Therefore, this plane is called the *neutral surface* of the beam. The intersection of a cross section of the beam with the neutral surface (i.e., the y-axis) is called the *neutral axis* (المحور السليم) . The bending stresses (tensile or compressive) attain their maximum values at the extreme fibers. With the notation z_{max} for the maximum value of z (often also denoted by c) and : $\sigma_{max} = \frac{M}{I_y} z_{max} = \frac{M}{W}$.

Where $W = \frac{I_y}{z_{max}}$, is [L^3] (often also denoted by S) and called the *section modulus* (معامل المقطع).

If the state of stress in a beam is investigated, it often suffices to determine only the normal stresses since the shear stresses are usually negligibly small (slender beams!). There are several different types of problems arising in this context.

If, for example, the bending moment M, the section modulus W and the allowable stress σ_{allow} are known, one has to verify that the maximum stress $\sigma_{\max}$ satisfies the requirement

$\sigma_{\max} \leq \sigma_{allow} \rightarrow \frac{M}{W} \leq \sigma_{allow}$ this is called stress check. تحقيق الإجهادات

On the other hand, if M and σ_{allow} are given, the required section modulus can be calculated from

$W_{req} = \frac{M}{\sigma_{allow}}$ This is referred to as the *design* of a beam. تصميم الجائز

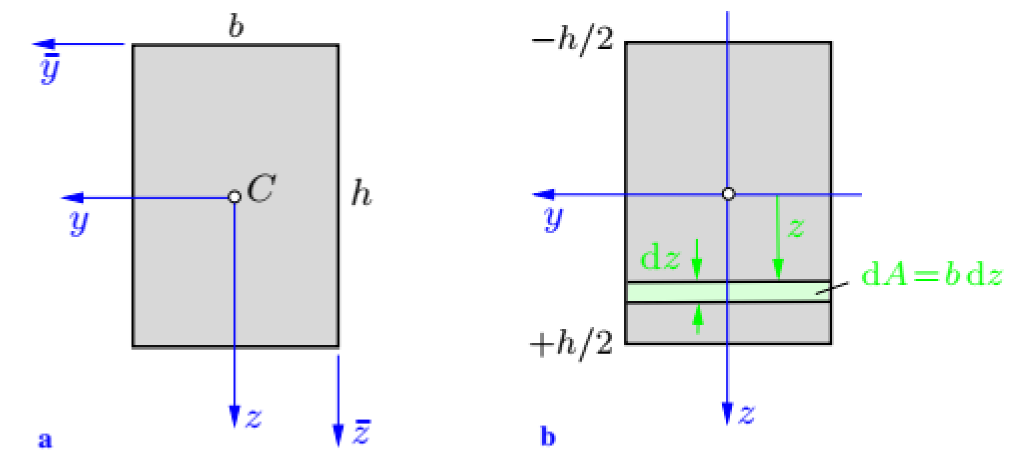
Finally, if W and σ_{allow} are given, the allowable load can be calculated from the condition that the maximum bending moment M_{max} must not exceed the allowable moment $M_{allow} = W\sigma_{allow}$:

$$M_{max} \leq W\,\sigma_{allow}$$

العزم الأعظمي

Ex. 1 As a first example we consider a rectangular area (width b, height h). The coordinate system with the origin at the centroid C is given; (Fig. a). In order to determine I_y, we select an infinitesimal area $dA = bdz$ according to (Fig. b) Then every point of the element has the same distance z from the y-axis. Thus, we obtain

$$I_y = \int z^2 dA = \int_{-h/2}^{+h/2} z^2 (bdz) = \frac{b}{3} [z^3]_{-h/2}^{+h/2} = \frac{bh^3}{12}$$



Ex. 2 In a second example we calculate the moments of inertia of a circular area (radius R)

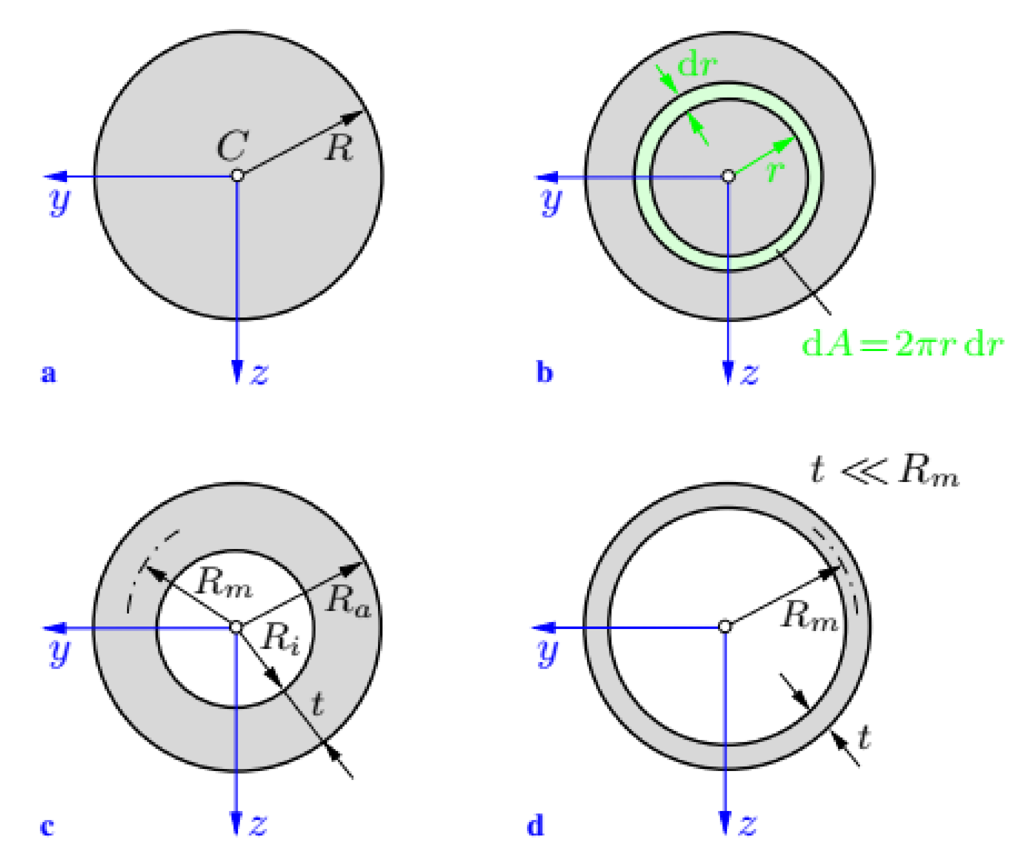
$$I_y = I_z = \frac{1}{2} \int r^2 dA = \frac{1}{2} \int_0^R r^2 (2\pi r dr) = \frac{\pi}{4} R^4$$

Ex. 3 In a third example we calculate the moments of inertia of a ring area (inner radius R_i and outer radius R_a)

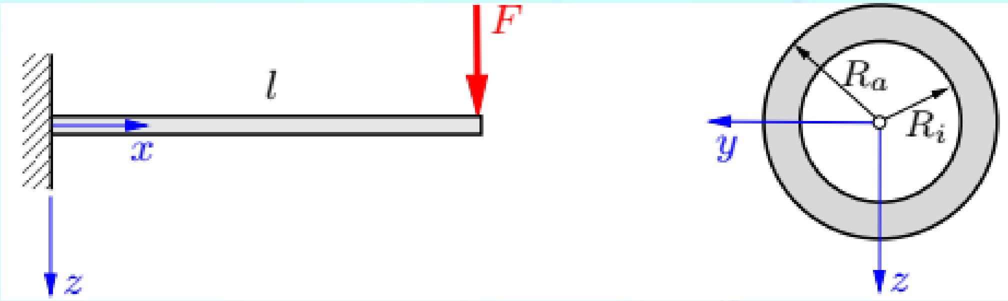
$$I_y = I_z = \frac{\pi}{4} R_a^4 - \frac{\pi}{4} R_i^4 = \pi t R_m (R_m^2 + \tfrac{1}{4} t^2)$$

For the thin ring: $t \ll R_m$

$$I_y = I_z = \pi t R_m^3$$



Example 1 The cross section of a cantilever beam ($l = 3\ m$) consists of a circular ring ($R_i = 4\ cm, Ra = 5\ cm$) The allowable stress is given by $\sigma_{allow} = 150\ MPa$. Determine the allowable value of the load F.



Solution:

$$W = \frac{I_y}{z_{max}} = \frac{\frac{\pi}{4}(R_a^4 - R_i^4)}{R_a} = \frac{\pi(5^4 - 4^4)}{4(5)} = 57.96\ cm^3 = 57960\ mm^3$$

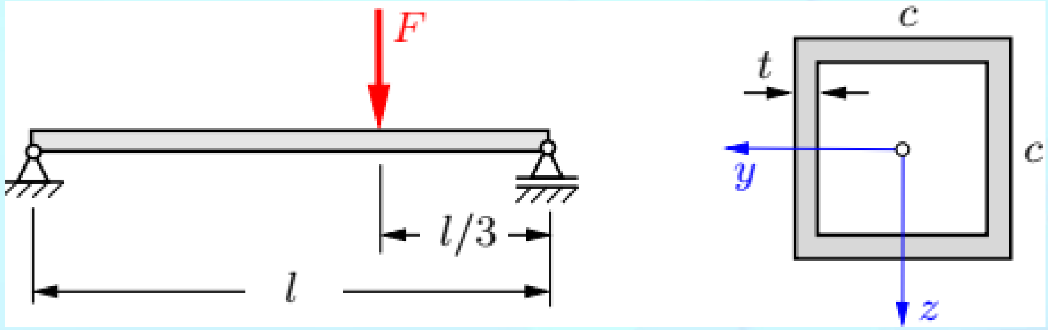
$$M_{allow} = W\sigma_{allow} = 57960 \times 150 = 8694000\ N \cdot mm = 8.694\ kN \cdot m$$

$$M_{max} = Fl \leq M_{allow} = 8.694\ kN \cdot m$$

$$F_{allow}l = M_{allow} = 8.694\ kN \cdot m$$

$$F_{allow} = \frac{M_{allow}}{3} = 2.9\ kN$$

Example 2 The simply supported beam (length $l = 10\ m$) carries the force $F = 200\ kN$. Find the required side length c of the thin-walled quadratic cross section such that the allowable stress $\sigma_{allo} = 200\ MPa$ is not exceeded. The thickness $t = 15\ mm$ of the profile is given



Solution: From the bending moment diagram: $M_{max} = \frac{\left(\frac{2l}{3}\right)\left(\frac{l}{3}\right)}{l} F = \left(\frac{2l}{9}\right) F = 444.4\ kN \cdot m = 444.4 \times 10^6 N \cdot mm$

The value of required section modulus is: $W_{req} = \frac{M_{max}}{\sigma_{allo}} = \frac{444.4 \times 10^6}{200} = 2.222 \times 10^6\ mm^3$

From the shape given in the figure, the section modulus as function of c is: $W = \frac{I_y}{c/2} = \frac{2I_y}{c}$ But for the hollow square section

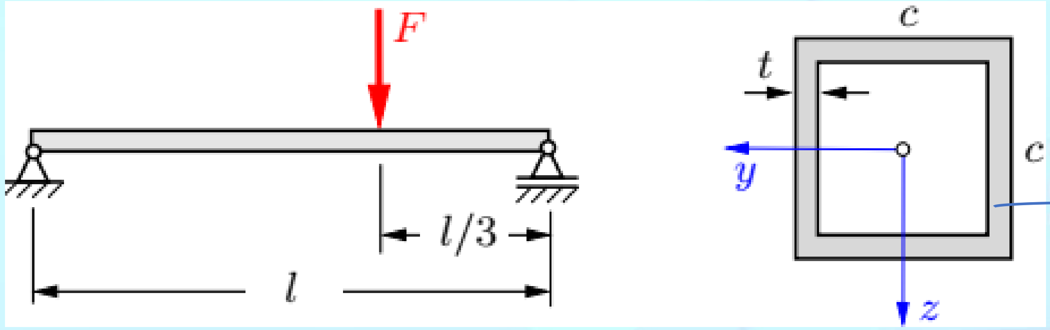
$$I_y = \frac{c^4 - (c-2t)^4}{12} = \frac{[c^2 - (c-2t)^2][c^2 + (c-2t)^2]}{12} = \frac{(2t)(2c-2t)(2c^2 - 4ct + 4t^2)}{12} = \frac{2t(c-t)(c^2 - 2ct + 2t^2)}{3}$$

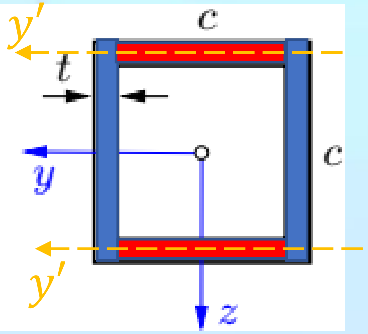
$$I_y = \frac{2t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3} \Rightarrow W = \frac{4t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3c} = \frac{60(c^3 - 45c^2 + 900c - 6750)}{3c} = 2.222 \times 10^6$$

$$\Rightarrow c^3 - 45c^2 + 900c - 6750 = \left(\frac{2.222 \times 10^6}{20}\right) c \Rightarrow c^3 - 45c^2 - 110211c - 6750 = 0$$

$$\Rightarrow c_1 = -310, c_2 = 335, c_3 = -0.061 \qquad \Rightarrow c = 335\ mm$$

Example 2 The simply supported beam (length $l = 10\ m$) carries the force $F = 200\ kN$. Find the required side length c of the thin-walled quadratic cross section such that the allowable stress $\sigma_{allo} = 200\ MPa$ is not exceeded. The thickness $t = 15\ mm$ of the profile is given





Solution:

From the bending moment diagram: $M_{max} = \dfrac{\left(\frac{2l}{3}\right)\left(\frac{l}{3}\right)}{l} F = \left(\dfrac{2l}{9}\right) F 444.4 \times 10^6 N \cdot mm$

The value of required section modulus is: $W_{req} = \dfrac{M_{max}}{\sigma_{allo}} = \dfrac{444.4 \times 10^6}{200} = 2.222 \times 10^6\ mm^3$

From the shape given in the figure, the section modulus as function of c is: $W = \dfrac{I_y}{c/2} = \dfrac{2I_y}{c}$

$$I_y = \frac{2t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3}$$

$$I_y = \frac{2t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3}$$

But the inertia moment for the thin-walled section can be simplified as:

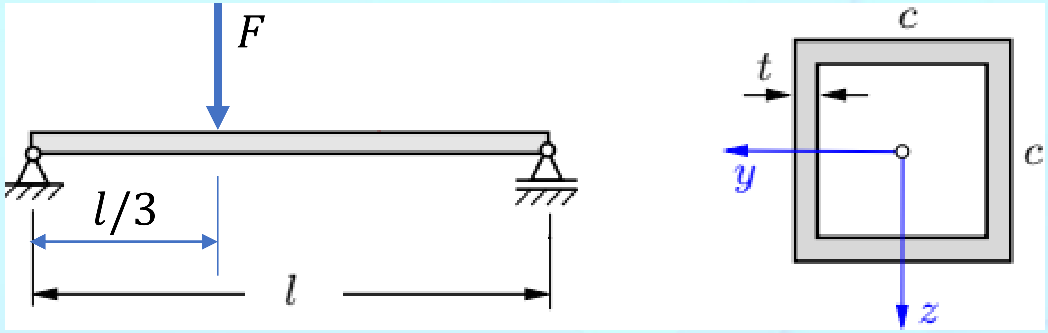
$$I_y = 2\frac{tc^3}{12} + 2tc\left(\frac{c-2t}{2}\right)^2 + 2\frac{(c-2t)t^3}{12} \approx \frac{tc^3}{6} + \frac{tc^3}{2} = \frac{4tc^3}{6} = \frac{2tc^3}{3}$$

$$\Rightarrow W = \frac{2I_y}{c} = \frac{4tc^3}{3c} = \frac{4tc^2}{3} = 2.222 \times 10^6\ mm^3$$

Take $t = 15\ mm$ to get: $20c^2 = 2.222 \times 10^6$

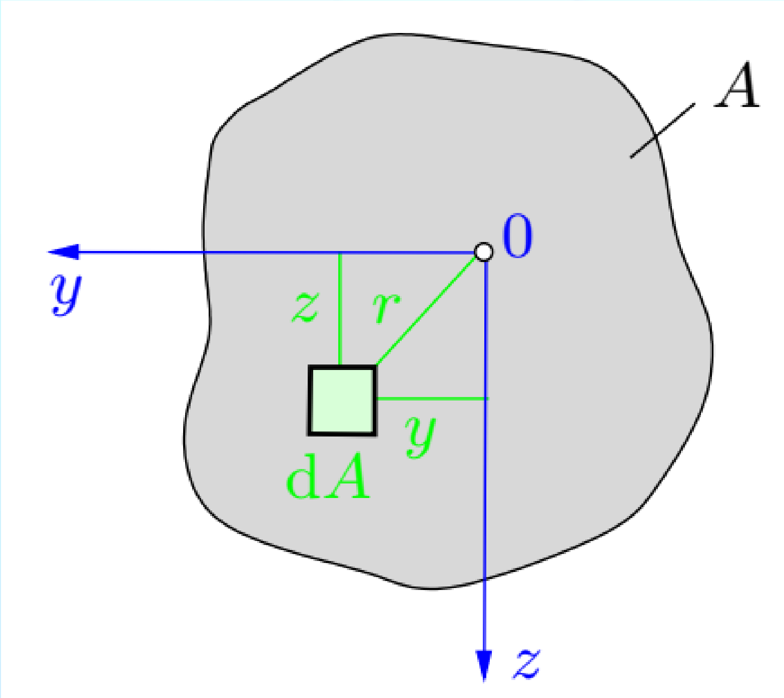
$$\Rightarrow c = \sqrt{0.1111 \times 10^6} = 333\ mm \approx 335\ mm$$

Example 3 The simply supported beam (length $l = 9\ m$) carries the force $F = 210\ kN$. Find the required side length c of the thin-walled quadratic cross section such that the allowable stress $\sigma_{allo} = 200\ MPa$ is not exceeded. The thickness $t = 12\ mm$ of the profile is given



4.2 Second Moments of Area

4.2.1 Definitions: The shown coordinate system is arbitrary



The coordinates of the centroid C of an area may be obtained from:

$$y_C = \frac{1}{A}\int_A ydA\,,\, z_C = \frac{1}{A}\int_A zdA$$

First moments of area (Static moments of area)

$$S_y = \int_A zdA \quad , \quad S_z = \int_A ydA$$

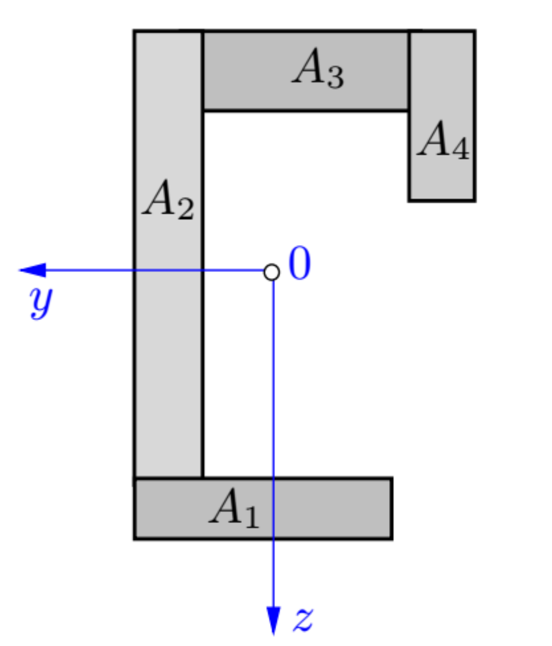
Second moments of area (Inertia moments of area)

$$I_y = \int_A z^2 dA \qquad I_z = \int_A y^2 dA \qquad I_{yz} = I_{zy} = -\int_A yzdA \qquad I_P = \int_A (y^2 + z^2)dA = I_y + I_z$$

Radii of gyration (Radii plural of radius)

$$r_{gy} = \sqrt{\frac{I_y}{A}} \qquad r_{gz} = \sqrt{\frac{I_z}{A}} \qquad r_{gP} = \sqrt{\frac{I_P}{A}}$$

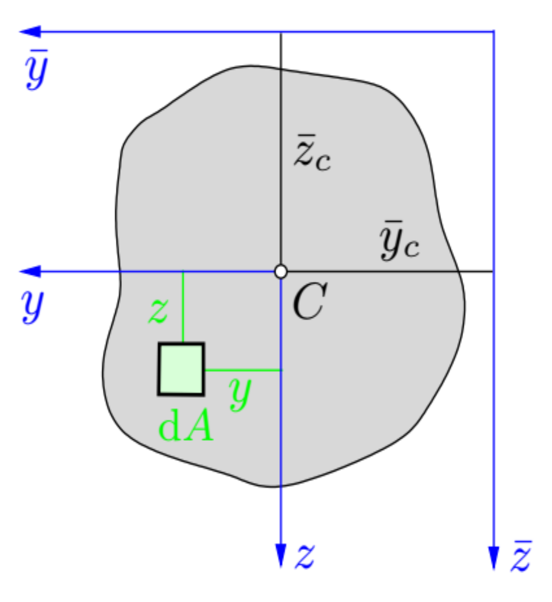
Frequently, an area A is composed of several parts A_i the moments of inertia of which are known (Fig.). In this case, the moment of inertia about the y-axis, for example, is obtained as the sum of the moments of inertia I_{y_i} of the individual parts about the *same axis*:



$$\boxed{I_y = \int_A z^2 dA = \int_{A_1} z^2 dA + \int_{A_2} z^2 dA + \cdot = \sum I_{y_i}} \quad \boxed{I_z = \sum I_{z_i}} \quad \boxed{I_{yz} = \sum I_{yz_i}}$$

4.2.2 Parallel-Axis Theorem

$$\boxed{\bar{y} = y + \bar{y}_C} \quad \boxed{\bar{z} = z + \bar{z}_C}$$

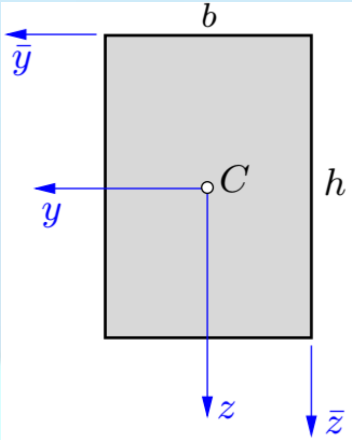


$$I_{\bar{y}} = \int \bar{z}^2 dA = \int (z + \bar{z}_C)^2 dA = \int z^2 dA + 2\bar{z}_C \int z dA + \bar{z}_C^2 \int dA$$

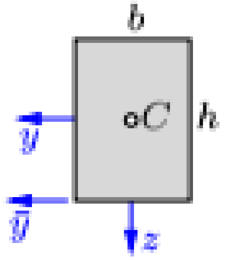
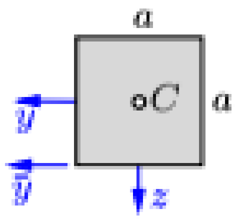
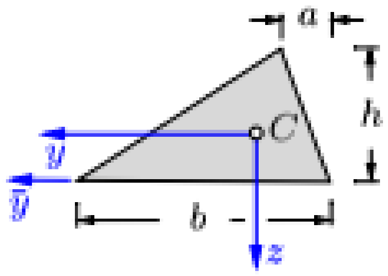
$$I_{\bar{y}} = \int z^2 dA + 2\bar{z}_C(0) + \bar{z}_C^2 A = I_y + \bar{z}_C^2 A$$

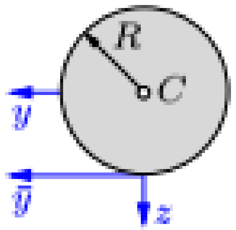
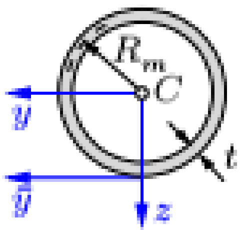
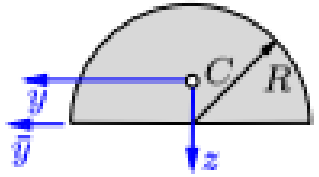
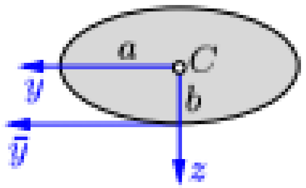
$$\boxed{I_{\bar{y}} = I_y + \bar{z}_C^2 A} \quad \boxed{I_{\bar{z}} = I_z + \bar{y}_C^2 A} \quad \boxed{I_{\bar{y}\bar{z}} = I_{yz} - \bar{y}_C \bar{z}_C A}$$

Ex. Determine the moment of inertia with respect to the $\bar{y}$ axis for the shown rectangle.



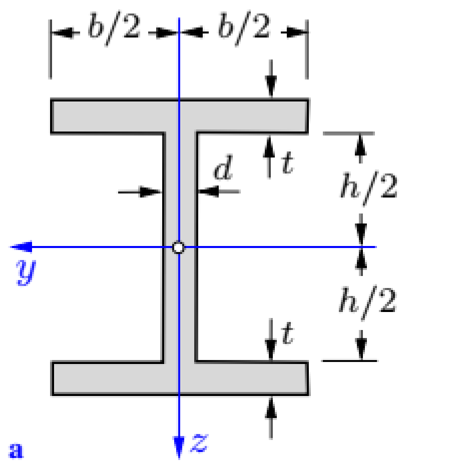
$$I_{\bar{y}} = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 (bh) = \frac{bh^3}{3}$$

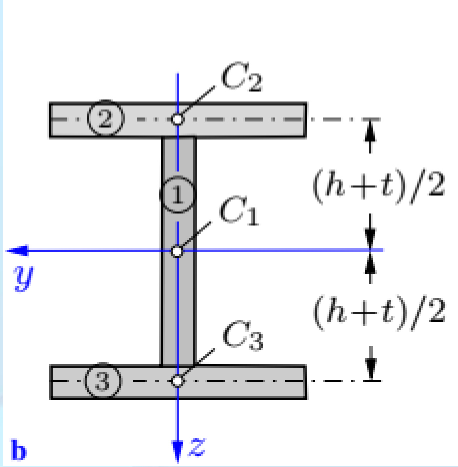
Area	I_y	I_z	I_{yz}	I_p	$I_{\bar{y}}$
Rectangle 	$\frac{b\,h^3}{12}$	$\frac{h\,b^3}{12}$	0	$\frac{b\,h}{12}(h^2+b^2)$	$\frac{b\,h^3}{3}$
Square 	$\frac{a^4}{12}$	$\frac{a^4}{12}$	0	$\frac{a^4}{6}$	$\frac{a^4}{3}$
Triangle 	$\frac{b\,h^3}{36}$	$\frac{b\,h}{36}(b^2-b\,a+a^2)$	$-\frac{b\,h^2}{72}(b-2\,a)$	$\frac{b\,h}{36}(h^2+b^2-b\,a+a^2)$	$\frac{b\,h^3}{12}$

Circle 	$\frac{\pi R^4}{4}$	$\frac{\pi R^4}{4}$	0	$\frac{\pi R^4}{2}$	$\frac{5\pi}{4} R^4$
Thin Circular Ring $t \ll R_m$ 	$\pi R_m^3 t$	$\pi R_m^3 t$	0	$2 \pi R_m^3 t$	$3 \pi R_m^3 t$
Semi-Circle 	$\frac{R^4}{72 \pi}(9 \pi^2 - 64)$	$\frac{\pi R^4}{8}$	0	$\frac{R^4}{36 \pi}(9 \pi^2 - 32)$	$\frac{\pi R^4}{8}$
Ellipse 	$\frac{\pi}{4} a b^3$	$\frac{\pi}{4} b a^3$	0	$\frac{\pi a b}{4}(a^2 + b^2)$	$\frac{5\pi}{4} a b^3$

Ex. 2 Determine the moments of inertia for the I-profile shown in Fig. a. Simplify the results for $d, t \ll b, h$.

Solution We consider the area to be composed of three rectangles (Fig. b).





$$I_y = \frac{dh^3}{12} + 2\left[\frac{bt^3}{12} + \left(\frac{t}{2} + \frac{h}{2}\right)^2 bt\right] = \frac{dh^3}{12} + \frac{bt^3}{6} + \frac{bt^3}{2} + bht^2 + \frac{h^2bt}{2}$$

$$I_y = \frac{dh^3}{12} + \frac{2bt^3}{3} + bht^2 + \frac{h^2bt}{2} \approx \frac{dh^3}{12} + \frac{h^2bt}{2} = \frac{dh^3}{12} + 2\left[\left(\frac{h}{2}\right)^2 bt\right]$$

$$I_z = \frac{ht^3}{12} + 2\frac{tb^3}{12} \approx \frac{tb^3}{6}$$

Ex. 3. A cantilever beam with the depicted cross section (constant wall thickness $t, t \ll a$, is subjected to a concentrated force F at one end. Determine the maximum stress in the cross section at the support.