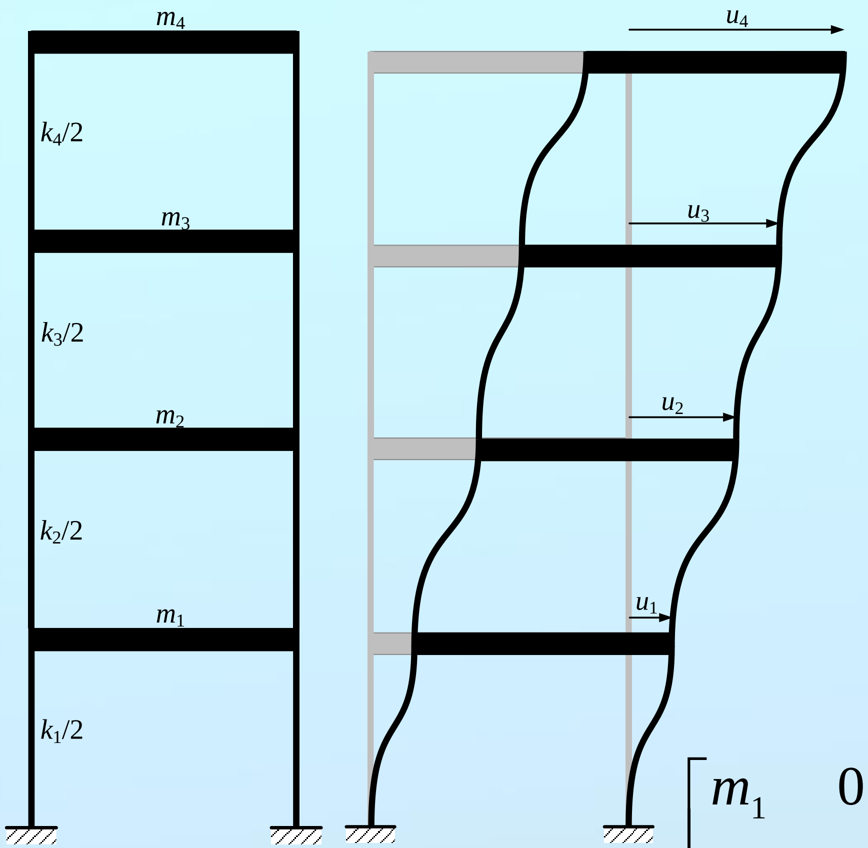
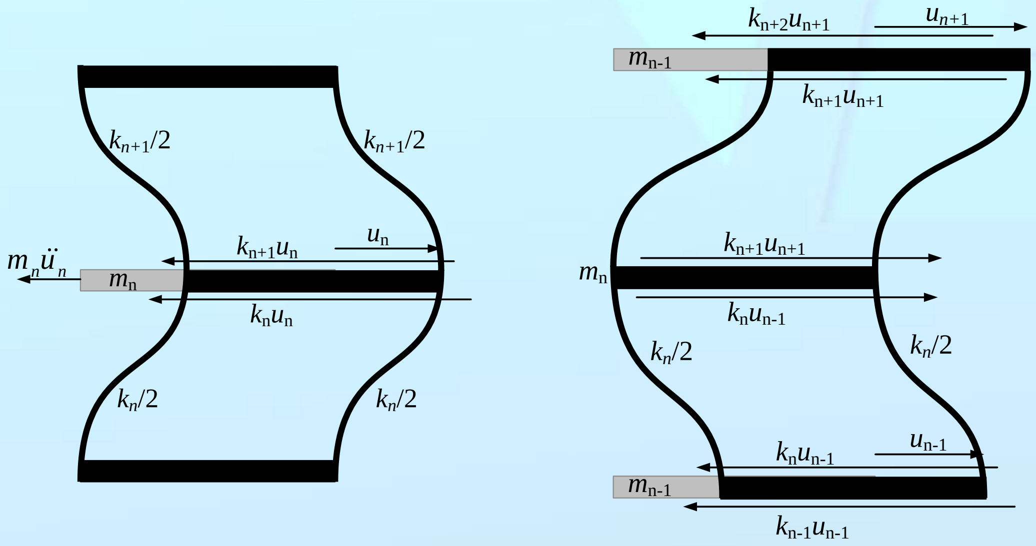
Free Vibration of multi degree of freedom systems:

Equation of Motion of Regular multi Story Building



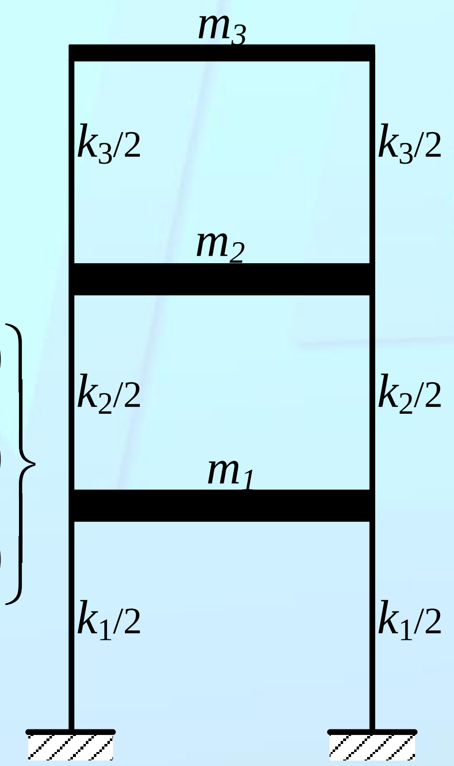
$$m_n\ddot{u}_n + k_n u_n + k_{n+1} u_n - k_n u_{n-1} - k_{n+1} u_{n+1} = 0$$



$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \vdots \\ \ddot{u}_n \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \dots & 0 \\ 0 & -k_3 & k_3+k_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & -k_n \\ 0 & 0 & 0 & -k_n & k_n \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = 0$$

Free Vibration of multi degree of freedom systems

Example: Determine the natural frequencies and natural modes of vibration for the 3-story shear building shown in next figure.



$m_1=m_2=4000$kg, & $m_3=2000$kg
$k_1=3000$kN/m, $k_2=2000$kN/m
& $k_3=1500$kN/m

Solution:

Equation of Motion

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For free vibration

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \cos\omega t \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = -\omega^2 \cos\omega t \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

Sub. Into the equation of motion

$$\left(-\omega^2 \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \right) \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \cos\omega t = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Rewrite this equation by summing the two matrices

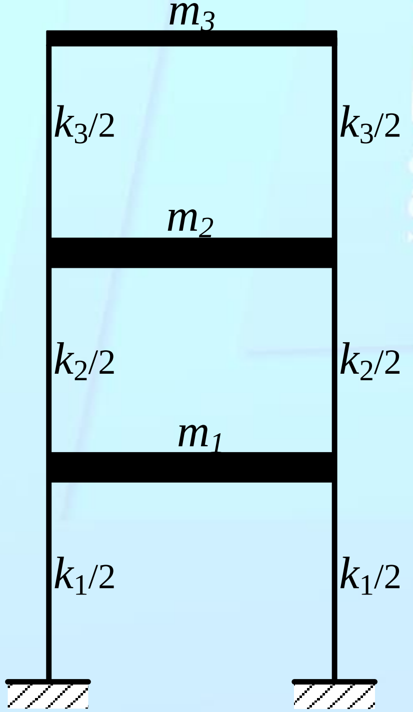
$$\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 & 0 \\ -k_2 & k_2+k_3-\omega^2 m_2 & -k_3 \\ 0 & -k_3 & k_3-\omega^2 m_3 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

If the *determinant* of the matrix is not zero, then $C_1=C_2=C_3=0$. No vibration.

So the natural vibration occurs when the matrix *determinant* is zero:

$$\det \begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 & 0 \\ -k_2 & k_2+k_3-\omega^2 m_2 & -k_3 \\ 0 & -k_3 & k_3-\omega^2 m_3 \end{bmatrix} = 0$$

With the given data in right units and by dividing all terms over 1000.



$m_1=m_2=4000$kg, & $m_3=2000$kg
$k_1=3000$kN/m, $k_2=2000$kN/m
& $k_3=1500$kN/m

$$\det \begin{bmatrix} 5000-4\omega^2 & -2000 & 0 \\ -2000 & 3500-4\omega^2 & -1500 \\ 0 & -1500 & 1500-2\omega^2 \end{bmatrix} = 0$$

$$(5000-4\omega^2)[(3500-4\omega^2)(1500-2\omega^2)-(1500)^2]+2000[-2000(1500-2\omega^2)]=0$$

$$(5000-4\omega^2)(3500-4\omega^2)(1500-2\omega^2)-(1500)^2(5000-4\omega^2)-(2000)^2(1500-2\omega^2)=0$$

$$(5000-4\omega^2)(5250000-13000\omega^2+8\omega^4)-(2250000)(5000-4\omega^2)-(4000000)(1500-2\omega^2)=0$$

$$\omega^6(-32)+\omega^4(40000+52000)+\omega^2(-65000000-21000000+9000000+8000000)$$

$$+(26250000000-11250000000-6000000000)=0 \quad \Rightarrow \quad -32\left(\omega^2\right)^3+92000\left(\omega^2\right)^2-69000000\left(\omega^2\right)+9000000000=0$$

The roots of this equation: 164, 1711 & 1000, are the squared frequencies: $(\omega_i)^2$

To determine the vibration modes C_i, go back to:

$$\begin{bmatrix} 5000-4\omega^2 & -2000 & 0 \\ -2000 & 3500-4\omega^2 & -1500 \\ 0 & -1500 & 1500-2\omega^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = 0$$

For: $\omega_1^2 = 164 \Rightarrow \omega_1 = 12.81 rad/\sec$

$$\begin{bmatrix} 5000-4(164) & -2000 & 0 \\ -2000 & 3500-4(164) & -1500 \\ 0 & -1500 & 1500-2(164) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4344 & -2000 & 0 \\ -2000 & 2844 & -1500 \\ 0 & -1500 & 1172 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$C_2 = 1 \Rightarrow C_1 = 0.460 \,\&\, C_3 = 1.28$

For: $\omega_2^2 = 1000 \Rightarrow \omega_2 = 31.6 rad/\sec$

$$\begin{bmatrix} 5000-4(1000) & -2000 & 0 \\ -2000 & 3500-4(1000) & -1500 \\ 0 & -1500 & 1500-2(1000) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1000 & -2000 & 0 \\ -2000 & -500 & -1500 \\ 0 & -1500 & -500 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

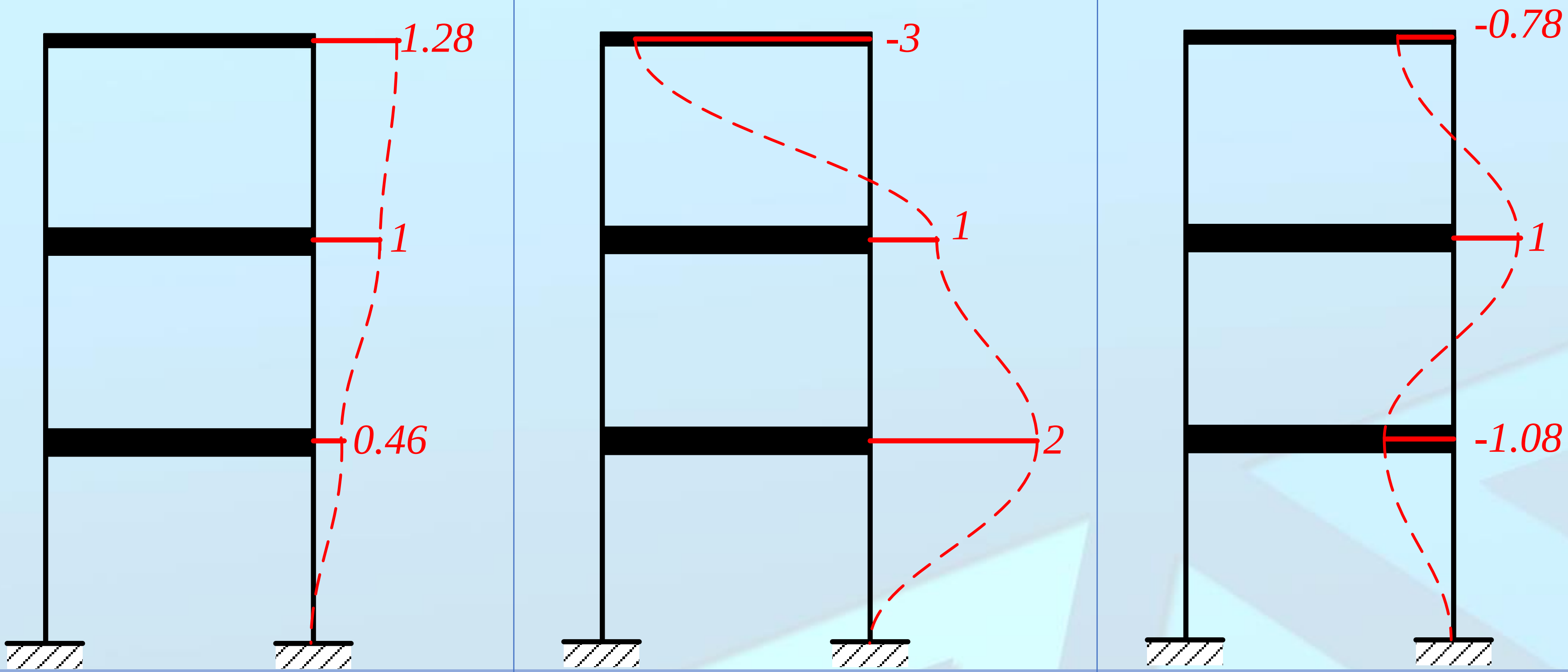
$C_2 = 1 \Rightarrow C_1 = 2 \,\&\, C_3 = -3$

For: $\omega_3^2 = 1711 \Rightarrow \omega_3 = 41.4 rad/\sec$

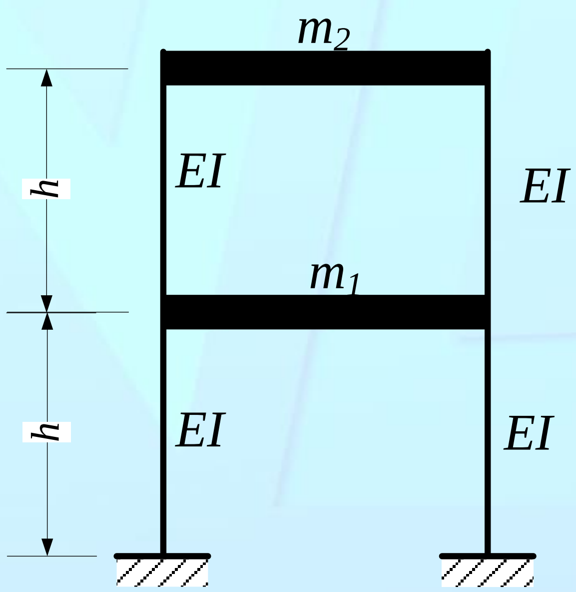
$$\begin{bmatrix} 5000-4(1711) & -2000 & 0 \\ -2000 & 3500-4(1711) & -1500 \\ 0 & -1500 & 1500-2(1711) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1844 & -2000 & 0 \\ -2000 & -3344 & -1500 \\ 0 & -1500 & -1922 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$C_2 = 1 \Rightarrow C_1 = -1.08 \,\&\, C_3 = -0.780$



Ex.1:Determine the natural frequencies and natural modes of vibration for the 2- story shear building shown in next figure.



$m_1 = m_2 = 20000$kg

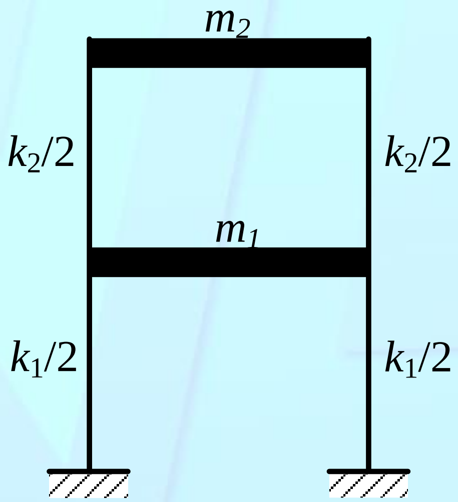
$EI/h^3 = 7.5 \times 10^5$ N/m

Ex.2. The shear frame shown in figure is constructed of rigid girders and flexible columns.

1.Write the general matrix equation of lateral free vibration of the frame.

2.Find the natural frequencies of the frame.

3.Find the vibration modes, then draw a small sketch for every mode.



$m_1=4(10^6)$kg and $m_2=2(10^6)$kg

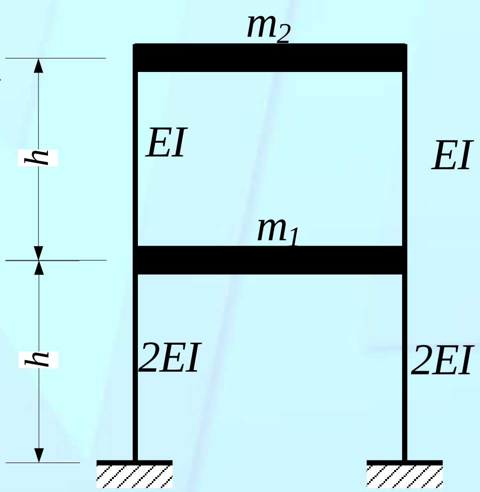
$k_1=6(10^4)$kN/m and $k_2=4(10^4)$kN/m

Ex.3. The shear frame shown in figure is constructed of rigid girders and flexible columns.

1.Write the general matrix equation of lateral free vibration of the frame.

2.Find the natural frequencies of the frame.

3.Find the vibration modes, then draw a small sketch for every mode.



$m_1 = m_2 = 18000$ kg

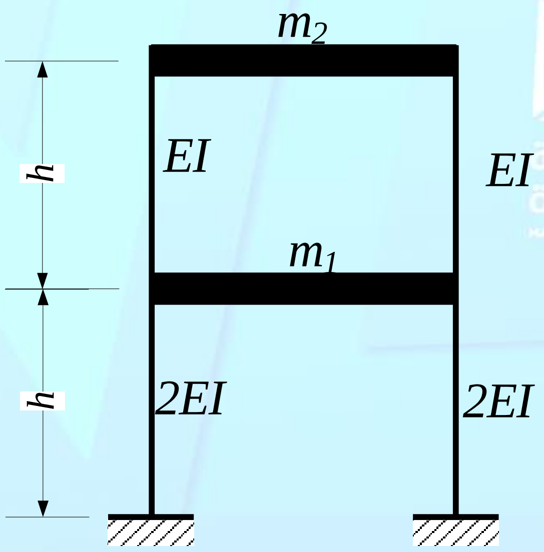
$EI/h^3 = 150$ kN/m

Ex.4. The shear frame shown in figure is constructed of rigid girders and flexible columns.

1. Write the general matrix equation of lateral free vibration of the frame.

2. Find the natural frequencies of the frame.

3. Find the vibration modes, then draw a small sketch for every mode.



$m_1 = m_2 = 20000$ kg

$EI/h^3 = 200$ kN/m