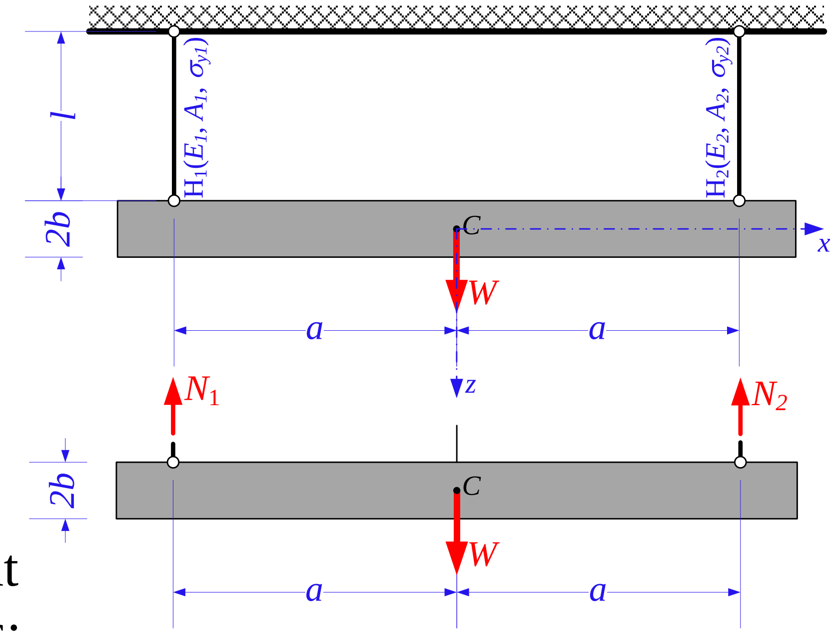
Introductory Problem

A heavy rigid box is suspended from the ceiling by two identical hangers: H_1 & H_2. If the weight of the box W is known, find the tension forces in the two hangers and the vertical displacement w of C.



<u>Solution (Idealized approach):</u>

If the rigid box has to stay horizontal (*the extensions in the two hangers must be equal),* the two equilibrium conditions, are independent of the changing geometry and can be written as:

$$\sum F_z = 0:\ N_1+N_2=W,\ \&\ \sum M_C = 0:\ -aN_1+aN_2=0.\ \Rightarrow\ N_1=N_2=W/2.$$

If the two hangers behave elastically, their extensions δ_1 & δ_2 *are:*

$$\delta_1=l\varepsilon_1=l\sigma_1/E_1=lN_1/A_1E_1=N_1(l/E_1A_1)=N_1 f_1=N_1/k_1,$$

$$\delta_2=l\varepsilon_2=l\sigma_2/E_2=lN_2/A_2E_2=N_2(l/E_2A_2)=N_2 f_2=N_2/k$$

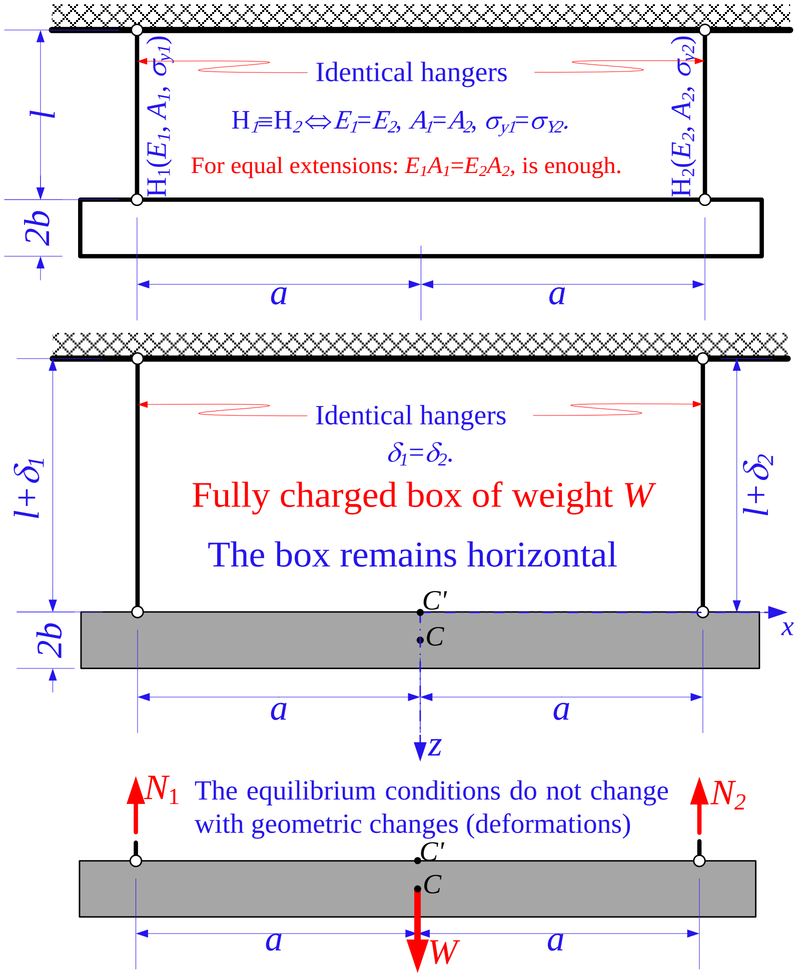
Where k_1, k_2 & f_1, f_2 are respectively stiffness & flexibility coefficients of the hangers. So

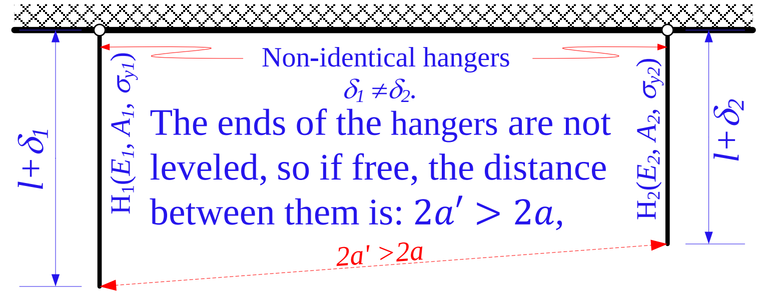
$$k_1=1/f_1=E_1A_1/l\ \&\ k_2=1/f_2=E_2A_2/l$$

Then for identical hangers ($k_1=1/f_1=k_2=1/f_2=k$) the vertical displacement of C, is: $w=W/2k.$

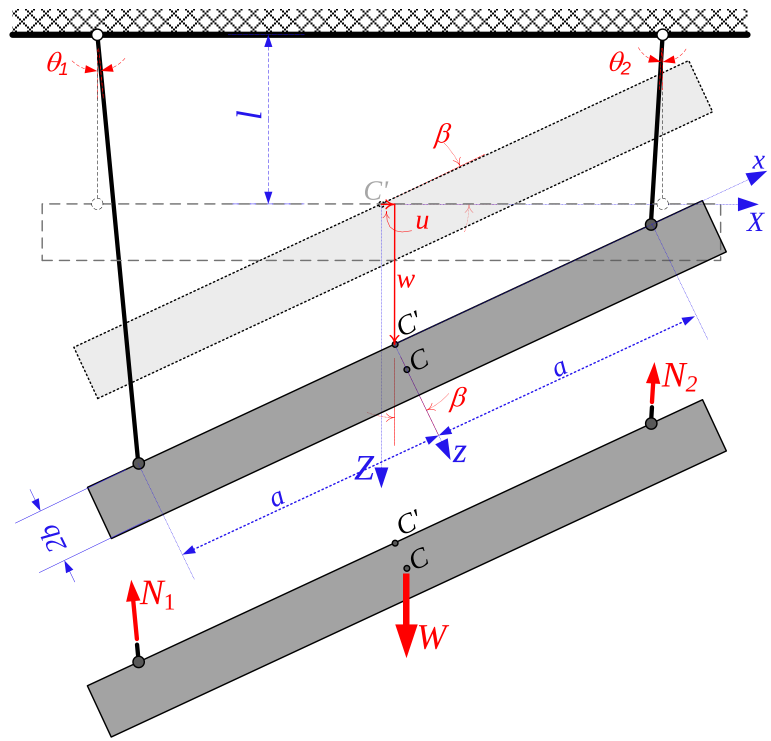
If the hangers are not identical (*more precisely if their extensions are not equal),* this problem becomes very tough (coupled, nonlinear). This situation is illustrated by the following:

Introductory Problem – More realistic approach





but the hangers and the rigid box move to keep it $2a$.



2 independent Eqm. Eqs.:

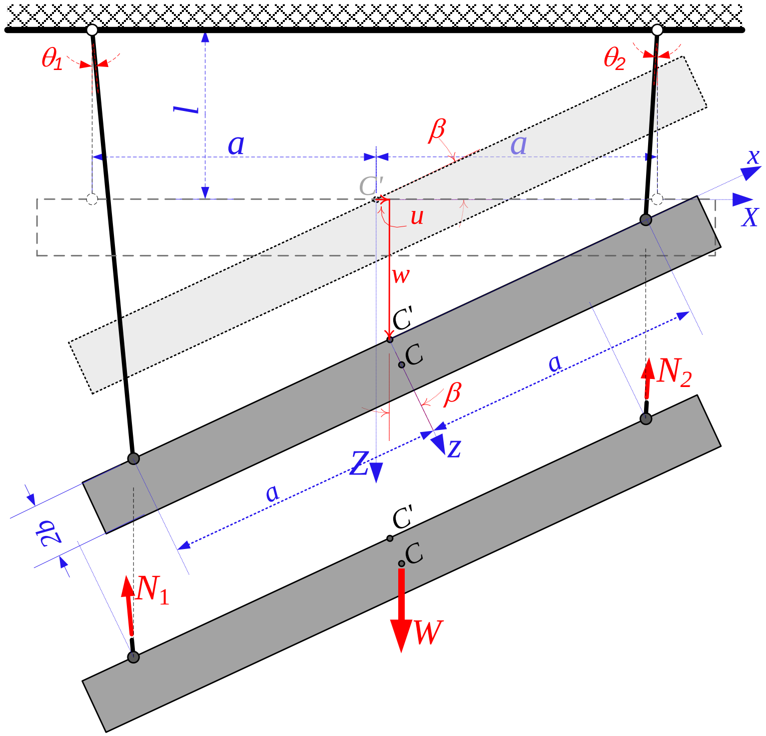
$N_1=N_2=W/2 \Rightarrow \sigma_1=W/2A_1\ \&\ \sigma_2=W/2A_2.$

Then for Elastic Behavior:

$\delta_1=Wl/(2E_1A_1)=\delta_2=Wl/(2E_2A_2)=w.$

3Eqm. Eqs. with 2unknowns: $N_1\ \&\ N_2$, coupled with 5kinematic unknowns: w, u, β, δ_1, δ_2, verifying 2Kin. Eqs. & 2Elastic Beh. Eqs.

Introductory Problem – More realistic approach



3Eqm. Eqs. with 2unknowns: N_1 & N_2 , coupled with 5kinematic unknowns: w, u, β, δ_1, δ_2, verifying 2Kin. Eqs. & 2Elastic Beh. Eqs.

Kinematic Equations (Kin. Eqs.):

The motion of the rigid box is a rotation: β about Y with center C' , composed with two axial translations: w & u on Z and X. Where (X, Y, Z) is a fixed system with origin at the middle of the upper edge of the box.

The lower end of the left hanger moves:
From $(0,-a)$ to $(a\sin\beta, -a\cos\beta)$ by rotation β about Y with center C', and to $(w + a\sin\beta, u - a\cos\beta)$ by the two translations. So the displacement components on Z & X of this end are:

$$w_1 = \text{New } Z - \text{Old } Z = w + a\sin\beta$$
$$u_1 = \text{New } X - \text{Old } X = u - a\cos\beta + a$$

The lower end of the right hanger moves:
From $(0,+a)$ to $(-a\sin\beta, a\cos\beta)$ by rotation β about Y with center C', and to $(w - a\sin\beta, u + a\cos\beta)$ by the two translations. So the displacement components on Z & X of this end are:

$$w_2 = \text{New } Z - \text{Old } Z = w - a\sin\beta$$
$$u_2 = \text{New } X - \text{Old } X = u + a\cos\beta - a$$

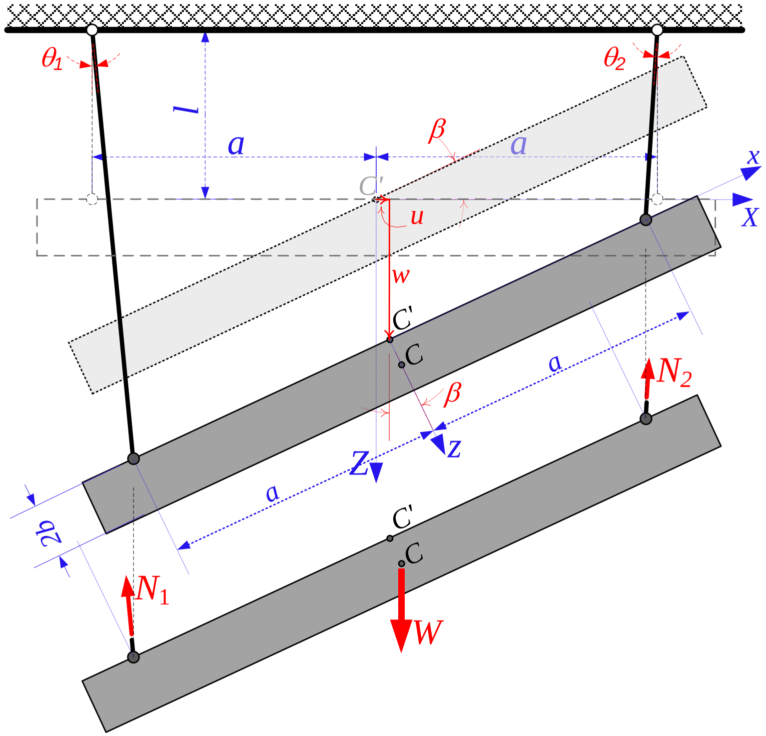
The new lengths of the hangers are

$$l+\delta_1=[(l+w_1)^2+(u_1)^2]^{1/2} \;\&\; l+\delta_2=[(l+w_2)^2+(u_2)^2]^{1/2}$$

Their angles with Z are

$$\theta_1=\sin^{-1}[u_1/(l+\delta_1)]=\cos^{-1}[(l+w_1)/(l+\delta_1)]$$
$$\theta_2=\sin^{-1}[u_2/(l+\delta_2)]=\cos^{-1}[(l+w_2)/(l+\delta_2)]$$

Introductory Problem – More realistic approach



3Eqm. Eqs. with 2unknowns: N_1 & N_2 , coupled with 5kinematic unknowns: w, u, β, δ_1, δ_2, verifying 2Kin. Eqs. & 2Elastic Beh. Eqs.

Equilibrium Equations (Eqm. Eqs.):

The 3Eqm. Eqs. considering the new geometry are:

$$\sum F_Z = 0:\ N_1\cos\theta_1 + N_2\cos\theta_2 = W, \quad (1)$$

$$\sum F_X = 0:\ N_1\sin\theta_1 + N_2\sin\theta_2 = 0, \quad (2)$$

$$\sum M_{C'} = 0:\ -aN_1\cos(\beta-\theta_1) + aN_2\cos(\beta-\theta_2) = bW\sin\beta. \quad (3)$$

Kinematic Equations (Kin. Eqs.):

$$\theta_1 = \sin^{-1}[u_1/(l+\delta_1)] = \cos^{-1}[(l+w_1)/(l+\delta_1)]$$

$$\theta_2 = \sin^{-1}[u_2/(l+\delta_2)] = \cos^{-1}[(l+w_2)/(l+\delta_2)]$$

$$w_1 = w + a\sin\beta \quad | \quad w_2 = w - a\sin\beta$$

$$u_1 = u - a\cos\beta + a \quad | \quad u_2 = u + a\cos\beta - a$$

$$l+\delta_1 = [(l+w_1)^2 + (u_1)^2]^{½} \quad (4)$$

$$l+\delta_2 = [(l+w_2)^2 + (u_2)^2]^{½} \quad (5)$$

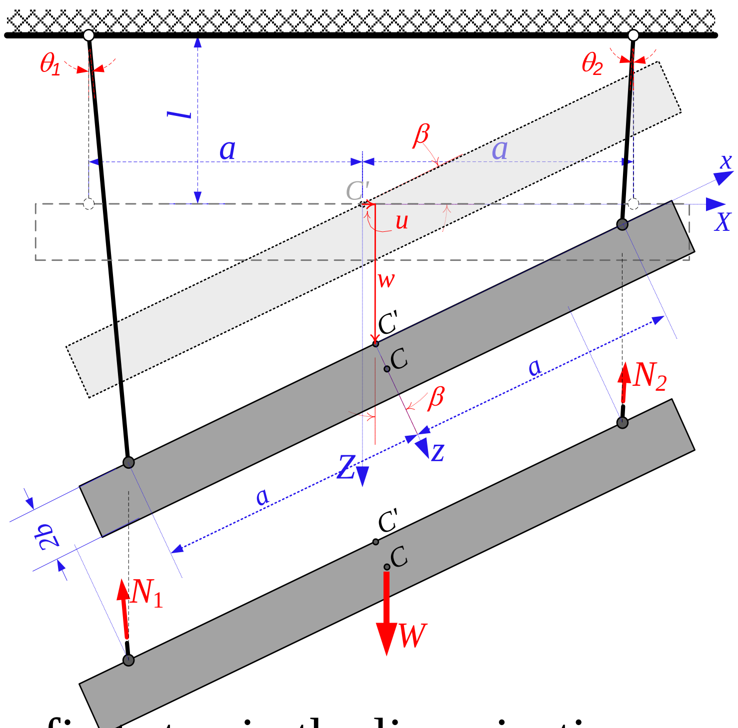
Material Equations (Mat. Eqs):

The material behavior equations relate locally (at every point of the body) the stresses to the strains. For the simple case of axially end-loaded bar with uniform distribution of the normal stress over the bar section, these equations can be reduced to functional relation between the total extension δ_1 or δ_2, in the two hangers to the internal forces by the intermediary of the geometric characteristics l & A, and the material property E, as:

$$\delta_1 = F_1(N_1, l, E_1, A_1) \quad (6)$$

$$\delta_2 = F_2(N_2, l, E_2, A_2) \quad (7)$$

Introductory Problem - Linearization



The first step in the linearization process is to make some simplifying assumptions on the kinematic aspect of the problem.

The main type is the assumption of the *small displacement and deformation variables*.

1. Small rotation: $\beta << 1 \Rightarrow \cos\beta \approx 1 \;\&\; \sin\beta \approx \beta$

So the displacements of the hangers ends, are approximated as:

$w_1 = w + a\sin\beta \approx w + a\beta$ | $w_2 = w - a\sin\beta \approx w - a\beta$

$u_1 = u - a\cos\beta + a \approx u$ | $u_2 = u + a\cos\beta - a \approx u$

And the inclination angles of the hangers defined by:

$$\theta_1 = \sin^{-1}[u_1/(l+\delta_1)] = \cos^{-1}[(l+w_1)/(l+\delta_1)]$$

$$\theta_2 = \sin^{-1}[u_2/(l+\delta_2)] = \cos^{-1}[(l+w_2)/(l+\delta_2)]$$

with the 2d part of the assumption:

2. Small displacements: $u << (a,l) \;\&\; w << (a,l)$

become:

$$\sin\theta_1 = [u/(l+w+a\beta)] \approx u/l$$

$$\cos\theta_1 \approx [(l+w+a\beta)/(l+w+a\beta)] = 1$$

$$\sin\theta_2 = [u/(l+w-a\beta)] \approx u/l$$

$$\cos\theta_2 \approx [(l+w-a\beta)/(l+w-a\beta)] = 1$$

Finally with the 3d part of the assumption:

3. Small deformation: $\delta_1 << l \;\&\; \delta_2 << l$

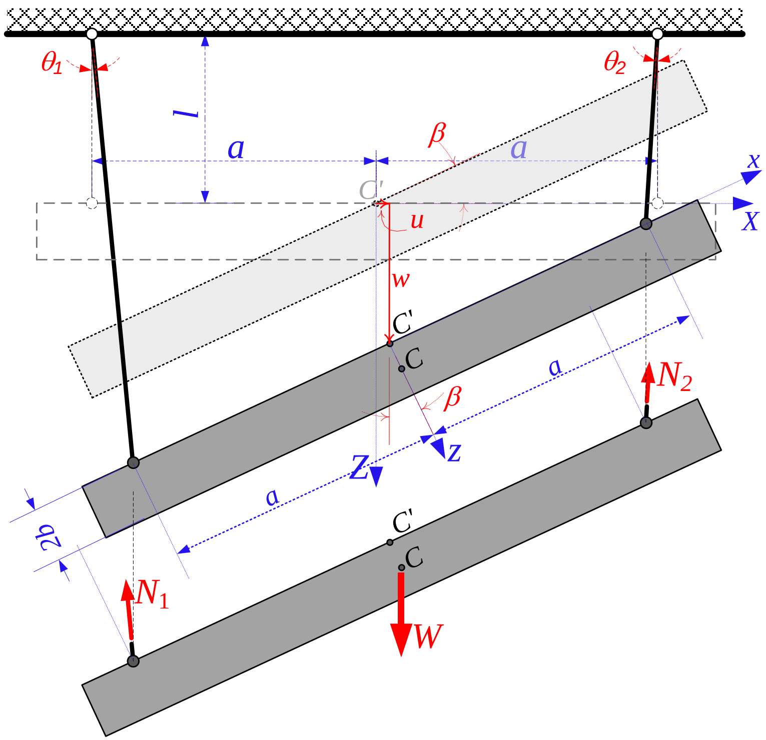
The two kinematic equations defining the deformation:

$$l+\delta_1 = [(l+w_1)^2 + (u_1)^2]^{1/2} \quad (4)$$

$$l+\delta_2 = [(l+w_2)^2 + (u_2)^2]^{1/2} \quad (5)$$

Can be modified and expanded as:

Introductory Problem - Linearization



$l+\delta_1=[l^2+2lw+2\,al\beta+2aw\beta+(w)^2+(a\beta)^2+(u)^2]^{1/2}$ (4')

$l+\delta_2=[l^2+2lw-2\,al\beta-2aw\beta+(w)^2+(a\beta)^2+(u)^2]^{1/2}$ (5')

Then approximated by neglecting 2d order terms, as:

$l+\delta_1\approx[l^2+2lw+2\,al\beta\,]^{1/2}$ (4'')

$l+\delta_2\approx[l^2+2lw-2\,al\beta\,]^{1/2}$ (5'')

and re-approximated once again to:

$l+\delta_1\approx l+w+a\beta$ (4''')

$l+\delta_2\approx l+w-a\beta$ (5''')

Resuming and grouping the consequences of the assumption of *small displacements and deformation*, on the kinematic variables and equations:

displacements of the hangers

$w_1=w+a\beta$ | $w_2=w-a\beta$

$u_1=u$ | $u_2=u$

inclination angles of the hangers

$\sin\theta_1=[u/(l+w+a\beta)]\approx u/l$

$\cos\theta_1\approx[(l+w+a\beta)/(l+w+a\beta)]=1$

$\sin\theta_2=[u/(l+w-a\beta)]\approx u/l$

$\cos\theta_2\approx[(l+w-a\beta)/(l+w-a\beta)]=1$

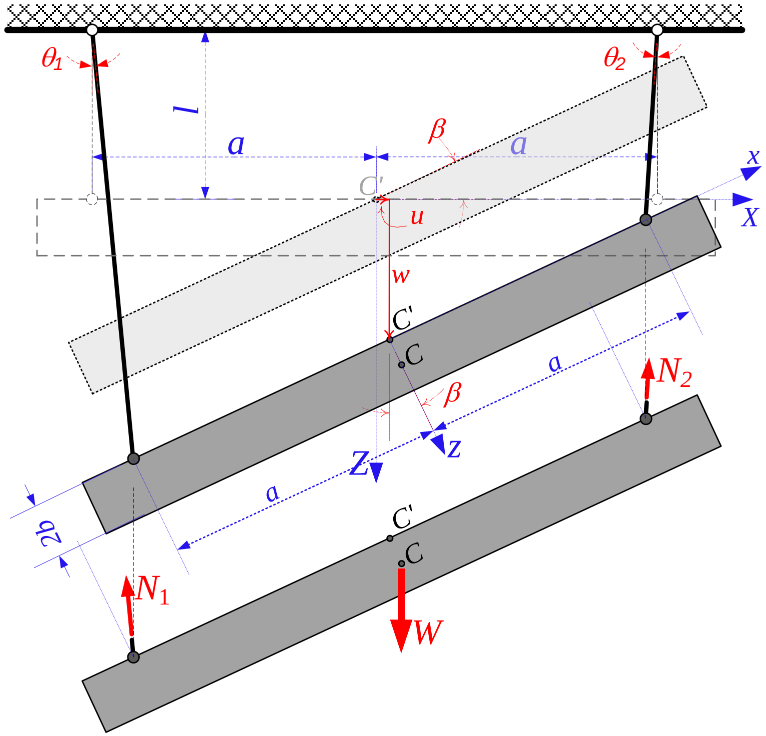
The extensions (deformations) as linearized function of the displacements:

$\delta_1\approx w+a\beta$ (4'''')

$\delta_2\approx w-a\beta$ (5'''')

The 2d step in the linearization process is to modify the equilibrium equations in the deformed geometry according to the approximated displacements:

Introductory Problem - Linearization



$w_1 = w + a\beta$ | $w_2 = w - a\beta$

$u_1 = u$ | $u_2 = u$

$\sin\theta_1 = [u/(l+w+a\beta)] \approx u/l$

$\cos\theta_1 \approx [(l+w+a\beta)/(l+w+a\beta)] = 1$

$\sin\theta_2 = [u/(l+w-a\beta)] \approx u/l$

$\cos\theta_2 \approx [(l+w-a\beta)/(l+w-a\beta)] = 1$

$$\begin{aligned}\delta_1 &\approx w + a\beta \quad (4'''') \\ \delta_2 &\approx w - a\beta \quad (5'''')\end{aligned} \Rightarrow \begin{aligned} w &= (\delta_1+\delta_2)/2 \\ \beta &= (\delta_1-\delta_2)/2a\end{aligned}$$

The 3Eqm. Eqs. considering the new geometry are:

$$\sum F_Z = 0: \; N_1\cos\theta_1 + N_2\cos\theta_2 = W, \quad (1)$$

$$\sum F_X = 0: \; N_1\sin\theta_1 + N_2\sin\theta_2 = 0, \quad (2)$$

$$\sum M_{C'} = 0: -aN_1\cos(\beta-\theta_1) + aN_2\cos(\beta-\theta_2) = bW\sin\beta. \quad (3)$$

With the approximated cosins of the hangers inclination angles θ_1, θ_2, the first equilibrium equations becomes:

$$\sum F_Z = 0: \; N_1(1) + N_2(1) = W \Rightarrow N_1 + N_2 = W \quad (1')$$

With the approximated sins of these angles the second equilibrium equations gives:

$$\sum F_X = 0: \; N_1(u/l) + N_2(u/l) = 0$$

$$\Rightarrow (N_1+N_2)(u/l) = 0 \Rightarrow u = 0 \quad (2')$$

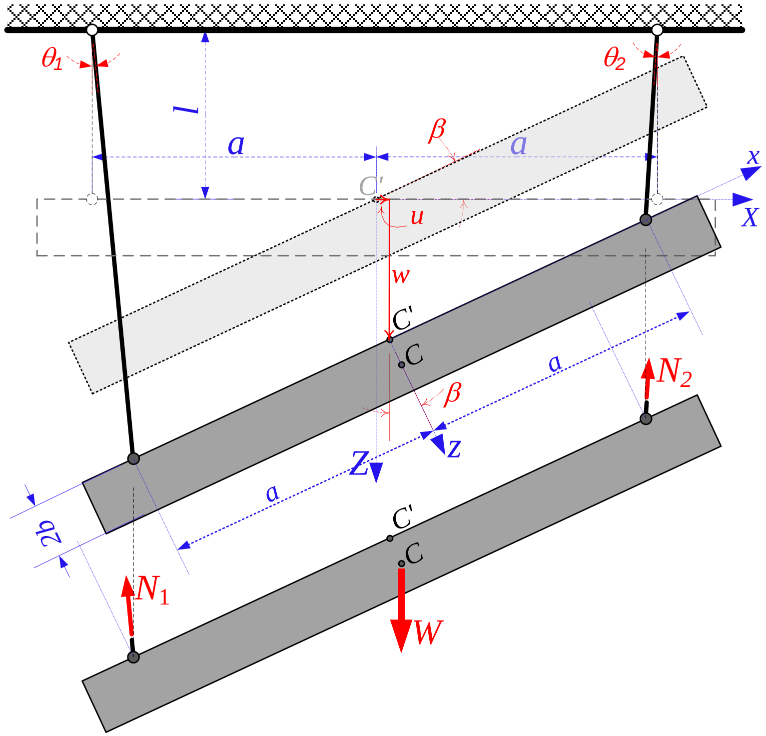
With the approximated sins and cosines of these angles in addition to the angle of rotation b, the third equilibrium equations becomes:

$$\sum M_{C'} = 0: -aN_1(1) + aN_2(1) = bW(\beta).$$

$$\Rightarrow -N_1 + N_2 = bW(\beta)/a \approx 0 \Rightarrow N_1 = N_2 \quad (3')$$

Final Result: the assumption of the *small displacement and deformation* , permits to write the equilibrium equations in the *undeformed geometry*.

Introductory Problem - Linearization



The 3d step in the linearization process is to consider that the material behavior is linear elastic. This means to replace the functional relations:

$$\delta_1 = F_1(N_1, l, E_1, A_1) \quad (6)$$
$$\delta_2 = F_2(N_2, l, E_2, A_2) \quad (7)$$

By the two simplified linear equations:

$$\delta_1 = N_1/k_1 \text{ where } k_1 = E_1A_1/l \quad (6')$$
$$\delta_2 = N_2/k_2 \text{ where } k_2 = E_2A_2/l \quad (7')$$

3Eqm. Eqs. with 2unknowns: N_1 & N_2 , coupled with 5kinematic unknowns: $w, u, \beta, \delta_1, \delta_2$, verifying 2Kin. Eqs. & 2Elastic Beh. Eqs.

$$\sum F_Z = 0: \; N_1\cos\theta_1 + N_2\cos\theta_2 = W, \quad (1)$$
$$\sum F_X = 0: \; N_1\sin\theta_1 + N_2\sin\theta_2 = 0, \quad (2)$$
$$\sum M_{C'} = 0: \; -aN_1\cos(\beta-\theta_1) + aN_2\cos(\beta-\theta_2) = bW\sin\beta. \quad (3)$$
$$\theta_1 = \sin^{-1}[u_1/(l+\delta_1)] = \cos^{-1}[(l+w_1)/(l+\delta_1)]$$
$$\theta_2 = \sin^{-1}[u_2/(l+\delta_2)] = \cos^{-1}[(l+w_2)/(l+\delta_2)]$$
$$w_1 = w + a\sin\beta \qquad w_2 = w - a\sin\beta$$
$$u_1 = u - a\cos\beta + a \qquad u_2 = u + a\cos\beta - a$$
$$l+\delta_1 = [(l+w_1)^2 + (u_1)^2]^{½} \quad (4) \qquad \delta_1 = f_1(N_1, l, E_1, A_1) \quad (6)$$
$$l+\delta_2 = [(l+w_2)^2 + (u_2)^2]^{½} \quad (5) \qquad \delta_1 = f_2(N_2, l, E_2, A_2) \quad (7)$$

2Eqm. Eqs. with 2unknowns: N_1 & N_2 , uncoupled with 4kinematic unknowns: $w, \beta, \delta_1, \delta_2$, verifying 2Kin. Eqs. & 2Elastic Beh. Eqs. Final results:

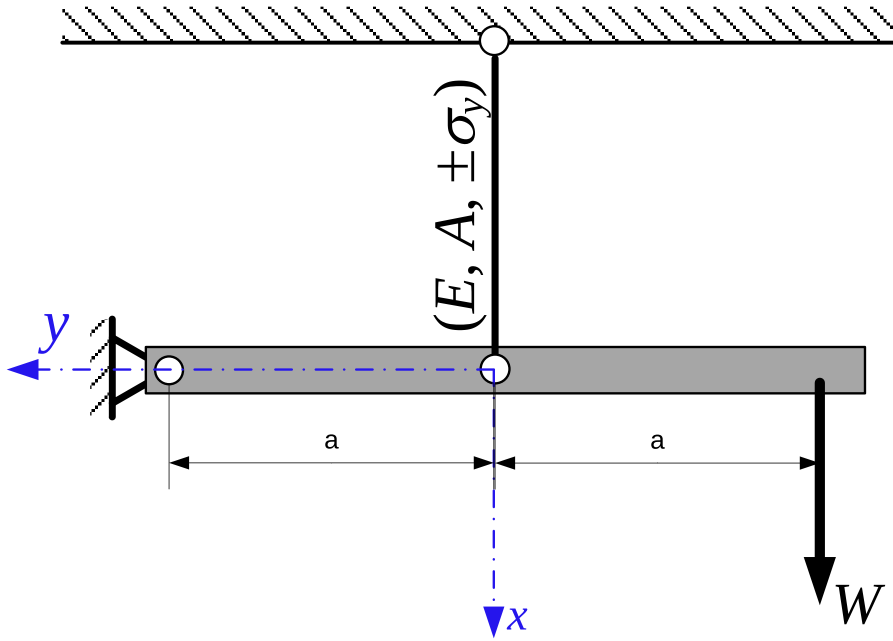
2Eqm. Eqs.
$$\sum F_Z = 0: \; N_1 + N_2 = W \quad (1)$$
$$\sum M_{C'} = 0: \; N_1 = N_2 \quad (2)$$
$$\Rightarrow \; N_1 = N_2 = W/2$$

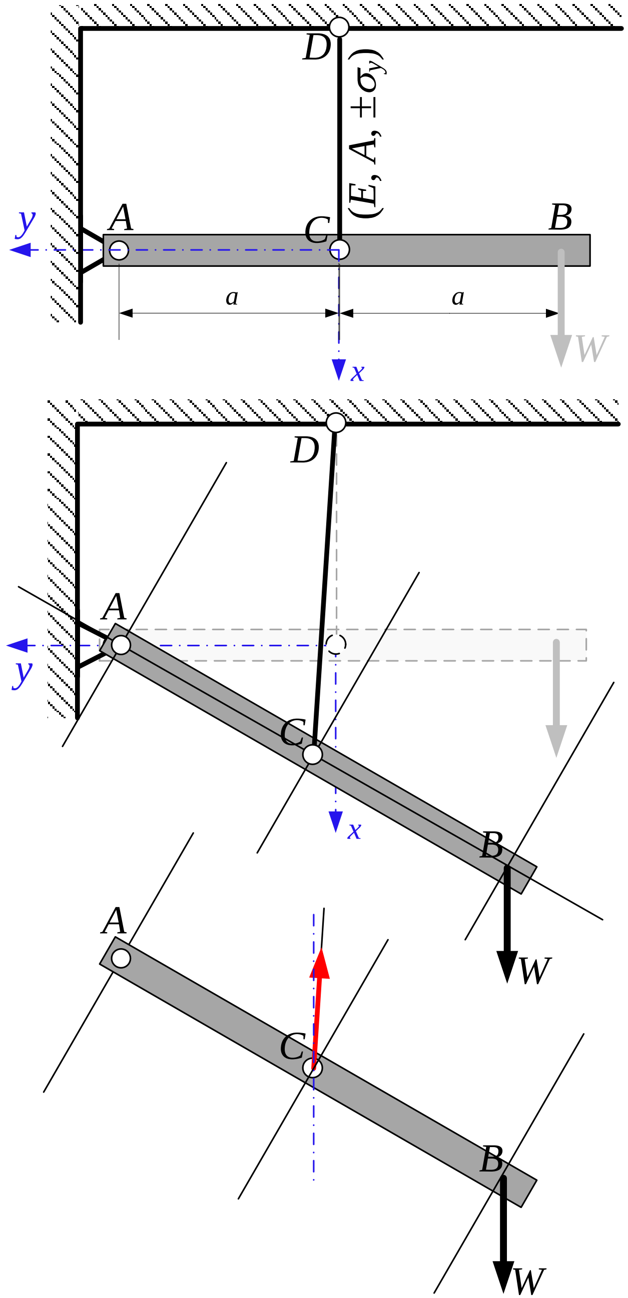
2Kin. Eqs.
$$w = (\delta_1 + \delta_2)/2 \quad (3)$$
$$\beta = (\delta_1 - \delta_2)/2a \quad (4)$$

& 2Beh. Eqs.
$$\delta_1 = N_1/k_1 \quad (5)$$
$$\delta_2 = N_2/k_2 \quad (6)$$
$$\Rightarrow$$

$$\delta_1 = W/2k_1$$
$$\delta_2 = W/2k_2$$

$$w = W(k_1 + k_2)/(2k_1k_2)$$
$$\beta = W(k_2 - k_1)/(4k_1k_2)$$


(E, A, ±σy)
y
a
a
x
W


D
(E, A, ±σy)
A
C
B
y
a
a
W
x
D
A
y
C
x
B
W
A
C
B
W

Plastic Analysis of Structures

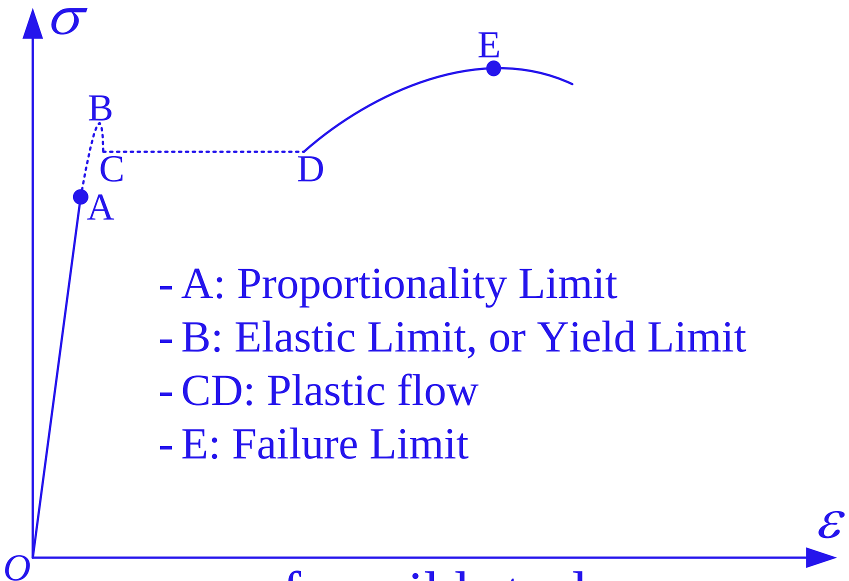
Modeling the behavior of solids is a complex and wide ranging subject: Elasticity, Plasticity, Viscosity… and many combinations.

In modern Structural Engineering students should at least study Elasticity and Plasticity, because most of the *limit state design principles* are based on these two theories.

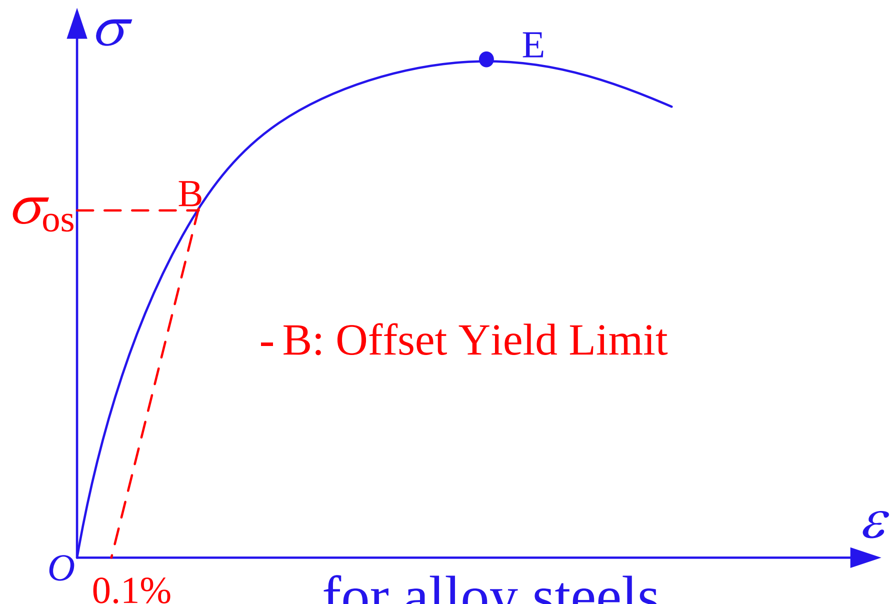
For the purpose of structural design of skeletal structures, its enough to work with uniaxial stress states.

Plastic Behavior in Simple Tension and Compression

Monotonic Loading



for mild steel

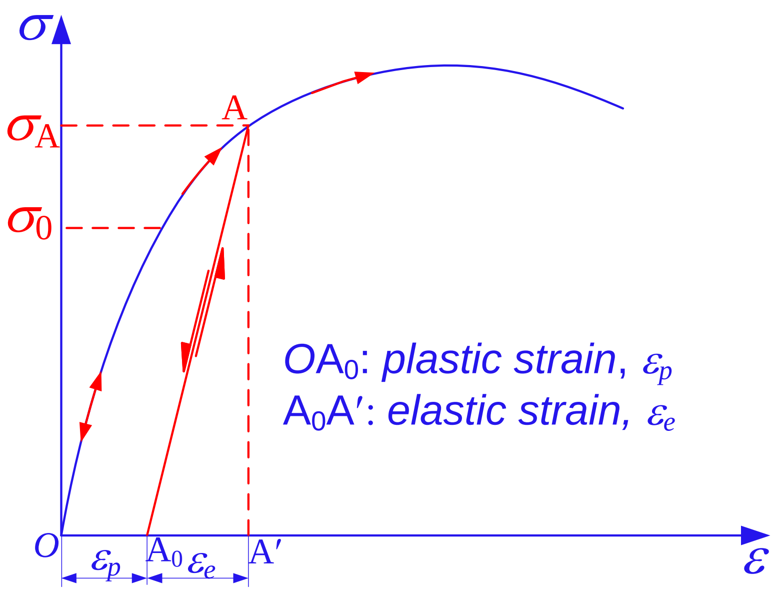


for alloy steels

stress-strain diagram in simple tension. Monotonic loading

Plastic Behavior in Simple Tension and Compression

Unloading and Reloading. Hardening.



Loading, unloading & reloading

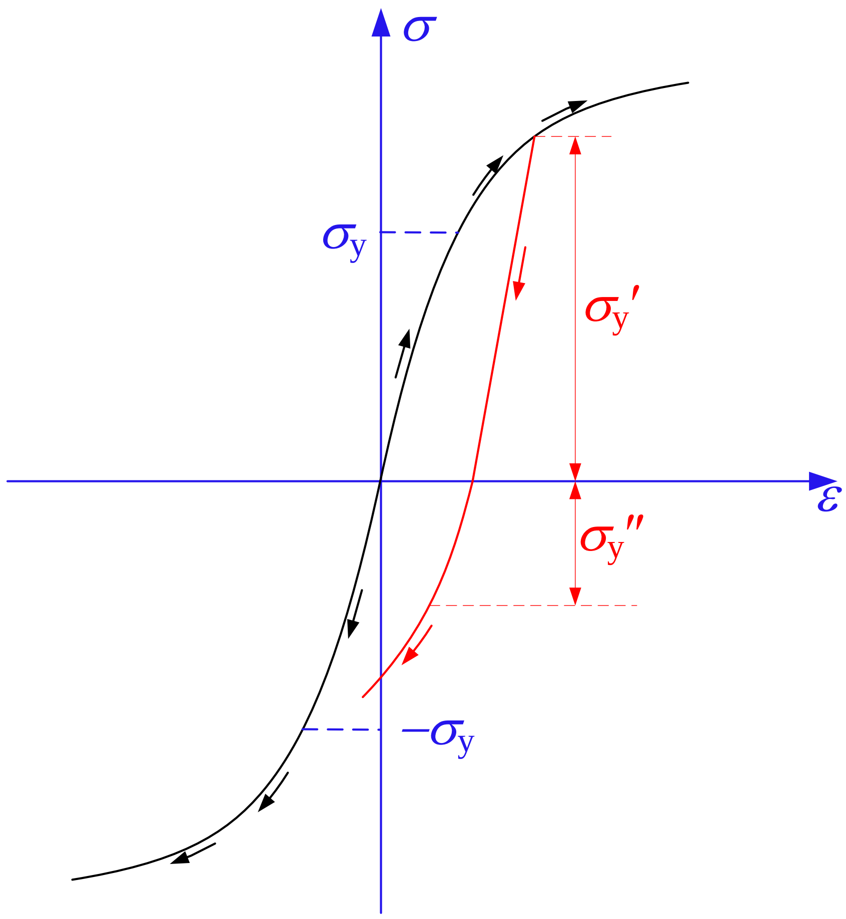
Loading to a value beyond the initial yield, then completely unloading, strain decreases along an almost elastic line AA_0, // to the initially loading curve. Reloading, the σ-ε curve follows the unloading path A_0A.

The material is therefore elastic until the previous maximum stress σ_A is reached. So, the subsequent yield stress increases with further straining. This effect is known as *hardening* .

For brittle materials, such as concrete or rock in compression, there is a region beyond the failure or peak point in which the slope of the curve is negative. Such a behavior is called *strain softening*.

Plastic Behavior in Simple Tension and Compression

Bauschinger Effect



Reversed loading.
Bauschinger Effect

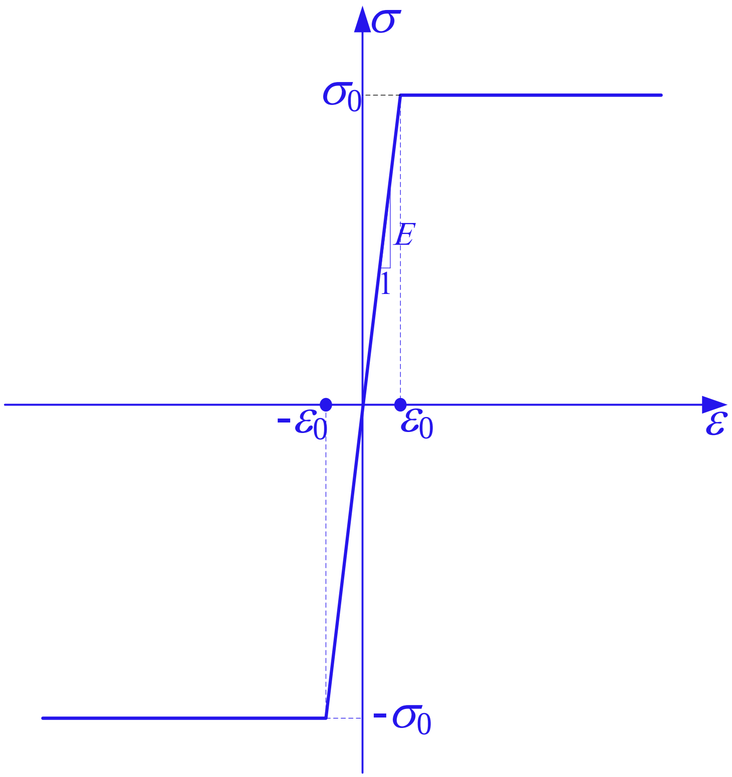
Performing a simple compression test on a metal, an almost identical σ-ε curve as in a simple tension test, is obtained.

After a plastic prestraining in tension, the σ-ε curve in compression differs from the curve obtained on loading in compression.

This phenomenon is known as the *Bauschinger effect* and is present whenever there is a stress reversal.

Modeling of Uniaxial Behavior in Plasticity

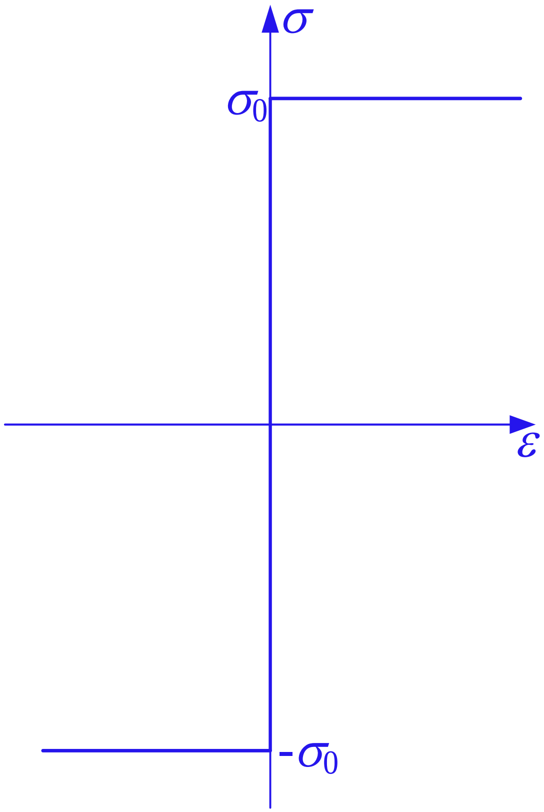
Simplified Uniaxial Stress-Strain Relations



elastic-perfectly plastic

$$\varepsilon = \frac{\sigma}{E} \quad \text{for} \quad \sigma < \sigma_0$$

$$\varepsilon = \frac{\sigma_0}{E} + \lambda \quad \text{for} \quad \sigma = \sigma_0$$

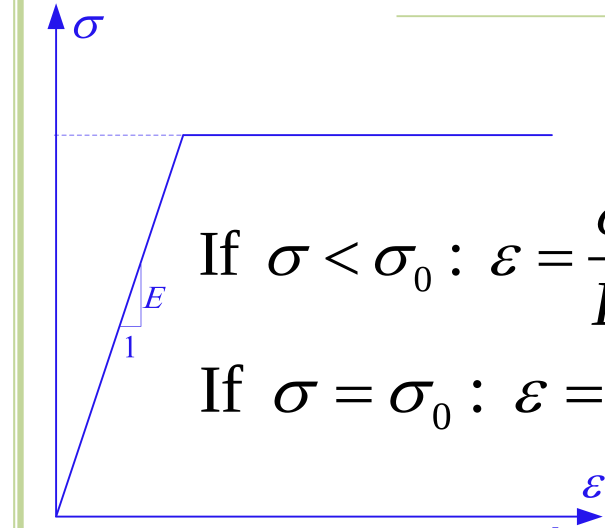


rigid-perfectly plastic

$$\varepsilon = 0 \quad \text{for} \quad \sigma < \sigma_0$$

$$\varepsilon = \lambda \quad \text{for} \quad \sigma = \sigma_0$$

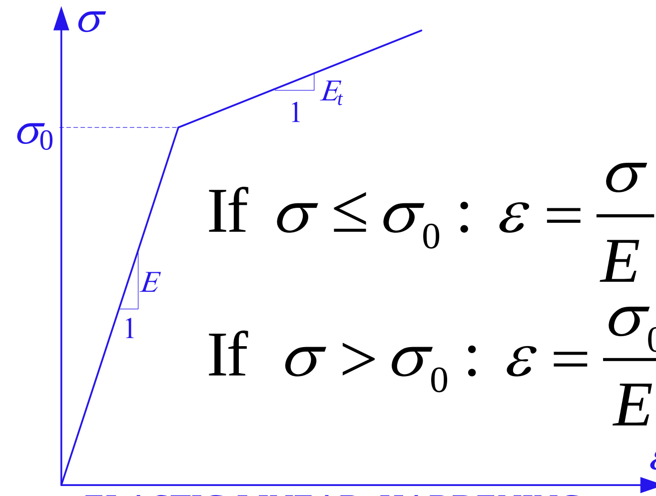
Modeling of Uniaxial Behavior in Plasticity: Hardening Models



$$\text{If } \sigma < \sigma_0 : \varepsilon = \frac{\sigma}{E}$$

$$\text{If } \sigma = \sigma_0 : \varepsilon = \frac{\sigma_0}{E} + \lambda$$

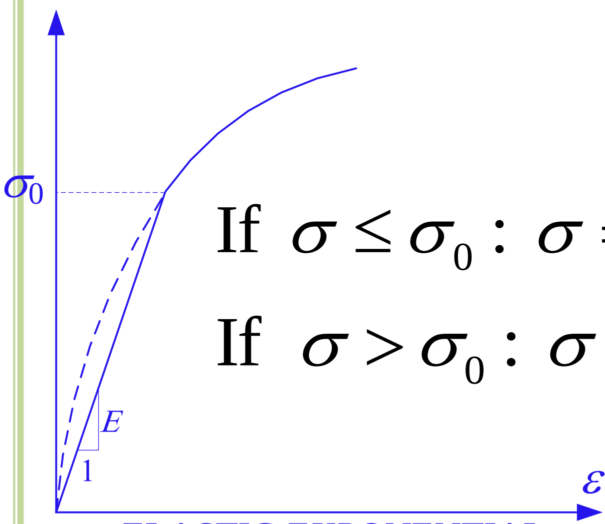
ELASTIC-PERFECTLY PLASTIC
(No Hardening)



$$\text{If } \sigma \le \sigma_0 : \varepsilon = \frac{\sigma}{E}$$

$$\text{If } \sigma > \sigma_0 : \varepsilon = \frac{\sigma_0}{E} + \frac{1}{E_t}(\sigma - \sigma_0)$$

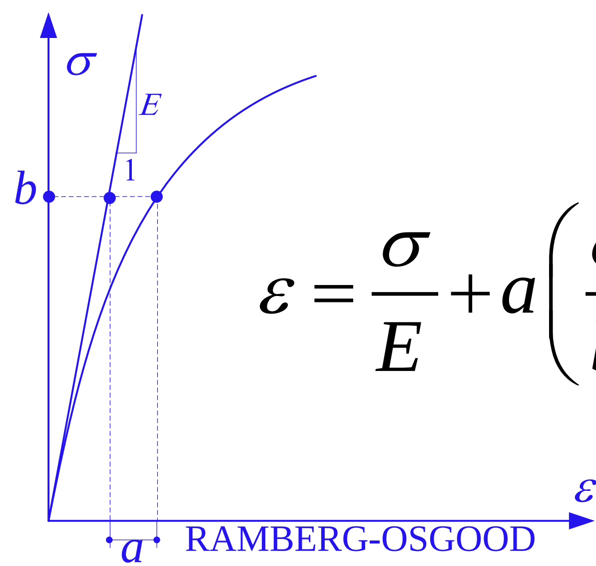
ELASTIC-LINEAR HARDENING.



$$\text{If } \sigma \le \sigma_0 : \sigma = E\,\varepsilon$$

$$\text{If } \sigma > \sigma_0 : \sigma = k\,\varepsilon^n$$

ELASTIC-EXPONENTIAL HARDENING



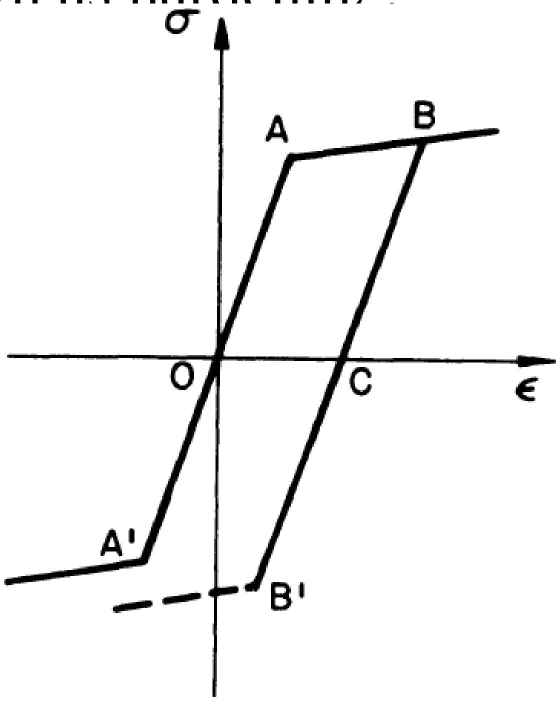
$$\varepsilon = \frac{\sigma}{E} + a\left(\frac{\sigma}{b}\right)^n$$

RAMBERG-OSGOOD HARDENING

Modeling of Uniaxial Behavior in Plasticity: Hardening Rules

As described previously, the phenomenon whereby yield stress increases with further plastic straining is known as hardening.

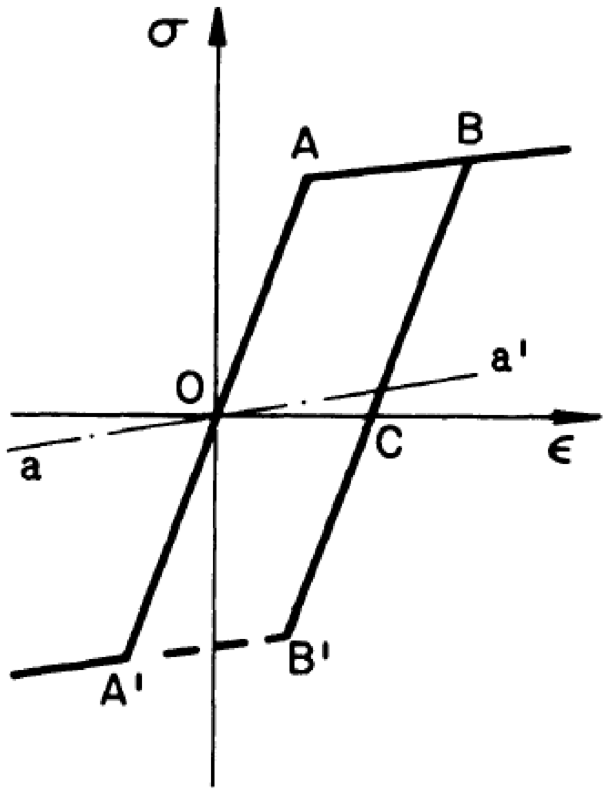
(a) *Isotropic hardening rule:* *The reversed compressive yield stress is assumed* equal to the tensile yield stress. As illustrated in figure where $B'C = BC$, the reversed compressive yield stress $\sigma_{B'}$ is equal to the tensile yielding stress σ_B before load reversal. Thus, the isotropic hardening rule neglects completely the Bauschinger effect, as it assumes that a raised yield point in tension carries over equally in compression. This hardening rule may be expressed mathematically in the form



(a) ISOTROPIC HARDENING

Modeling of Uniaxial Behavior in Plasticity: Hardening Rules

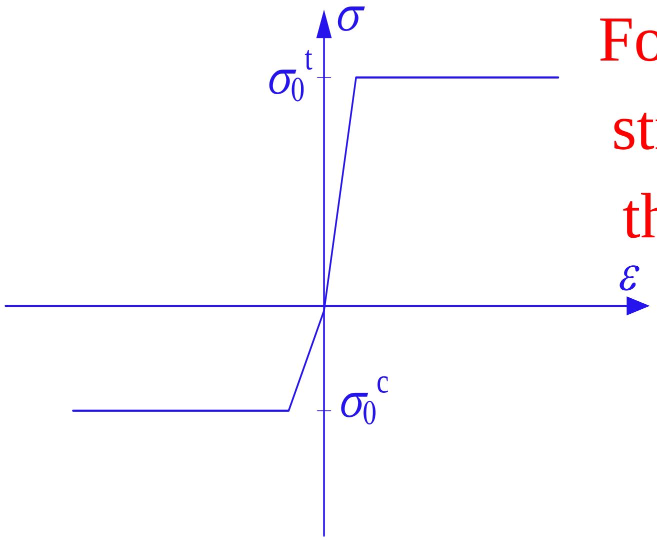
(b) *Kinematic hardening rule:* The elastic range is assumed to be unchanged during hardening. Thus, the kinematic hardening rule considers the Bauschinger effect to its full extent. Kinematic hardening for a linear hardening material is shown in Fig. 1.6b, where BB' = AA'. The center of the elastic region is moved along the straight line aa'.



(b) KINEMATIC HARDENING

Uniaxial Stress-Strain Relations. *Yield Condition*

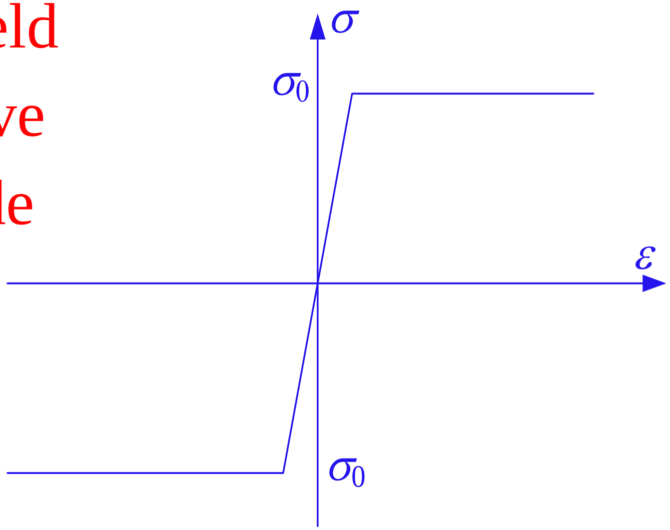
A similar type of idealized description is used for compression. The stress σ cannot exceed the bounds given by the yield stress in tension σ_0^t, and the yield stress in compression σ_0^c.



tension & compression plastic limits

For metals, both yield stresses usually have the same magnitude

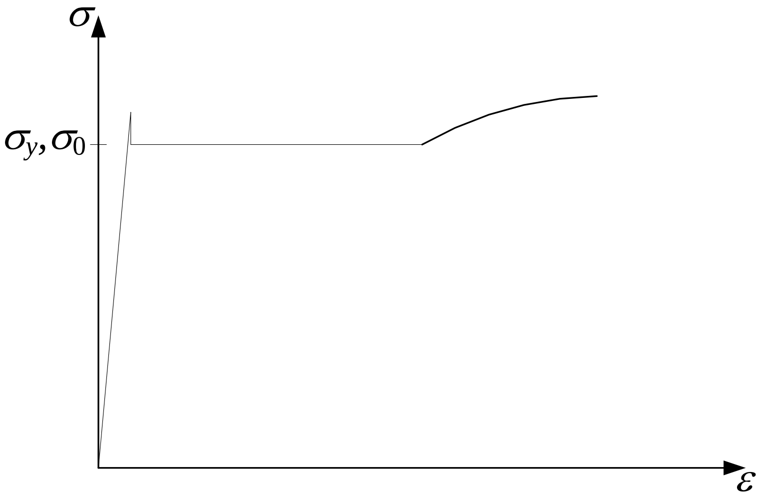
$$\sigma_0^t = \sigma_0^c = \sigma_0$$

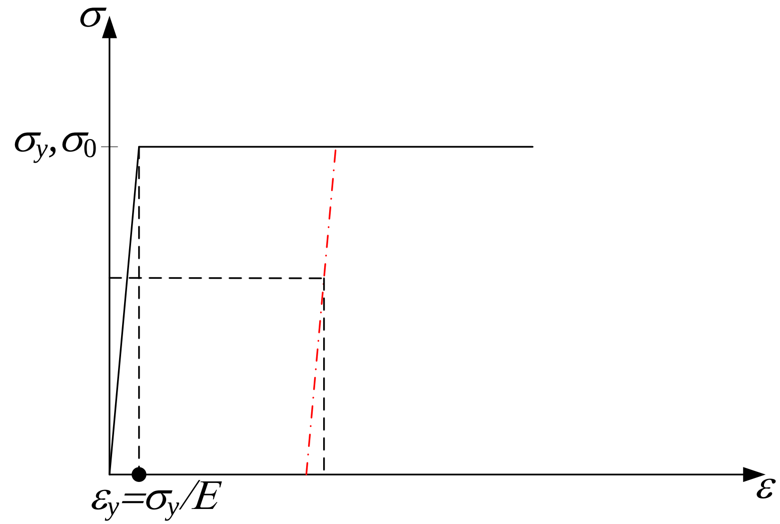


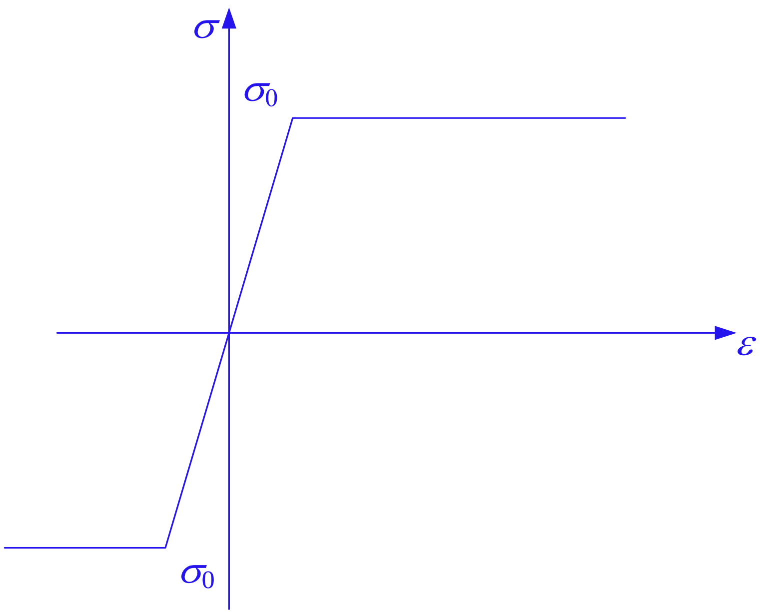
tension & compression plastic limits for metals

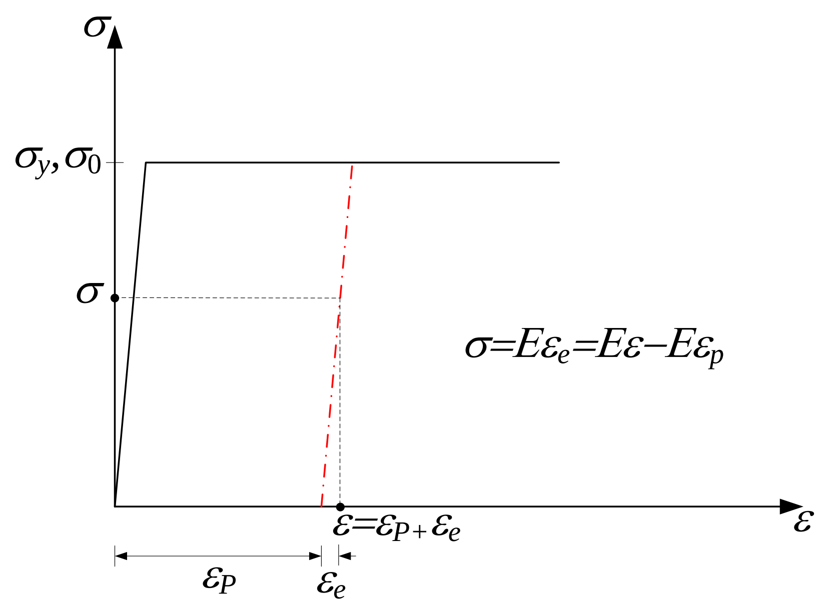
$$-\sigma_0^c \le \sigma \le \sigma_0^t$$ *condition of plastic admissibility* $$-\sigma_0 \le \sigma \le \sigma_0$$

Uniaxial Stress-Strain Relation for Elasto-plastic Material



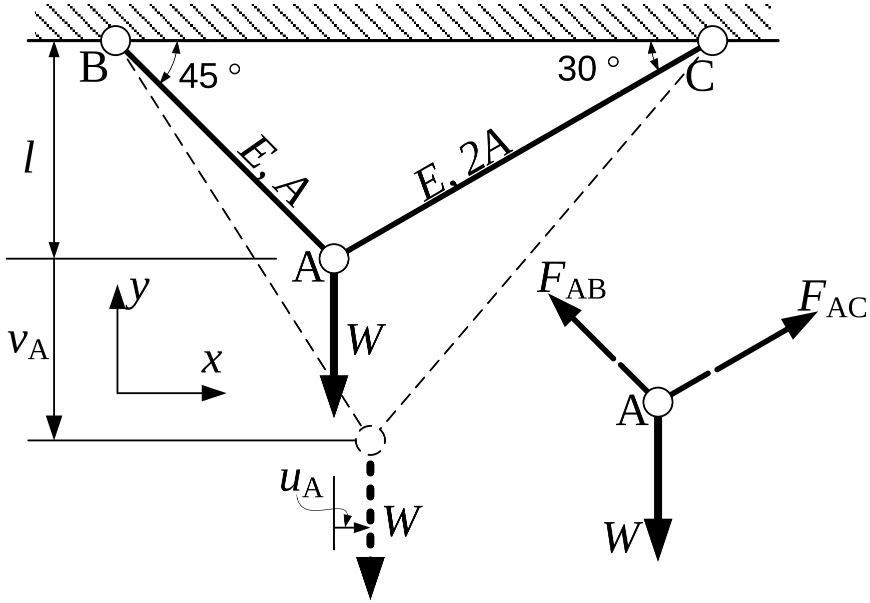






Behavior of Elasto-plastic Structures under Varying Load

1. Statically Determinate Tension Truss



Eq. Eqs.

$$-F_{AB}\cos 45^0 + F_{AC}\cos 30^0 = 0$$
$$F_{AB}\sin 45^0 + F_{AC}\sin 30^0 = W$$

Solving to get internal forces & deformations

$F_{AB} = 0.897W \Rightarrow \sigma_{AB} = F_{AB}/A = 0.897(W/A)$
$\Rightarrow \varepsilon_{AB} = (\sigma_{AB}/E) = 0.897(W/EA) \Rightarrow$
$\delta_{AB} = l_{AB}\varepsilon_{AB} = l\sqrt{2}[0.897(W/EA)] = 1.268(Wl/EA)$

$F_{AC} = 0.732W \Rightarrow \sigma_{AC} = F_{AC}/2A = 0.366(W/A)$
$\Rightarrow \varepsilon_{AC} = (\sigma_{AC}/E) = 0.366(W/EA) \Rightarrow$
$\delta_{AC} = l_{AC}\varepsilon_{AC} = 2l[0.366(W/EA)] = 0.732(Wl/EA)$

Subst. into Kin. Eqs. to get displacements

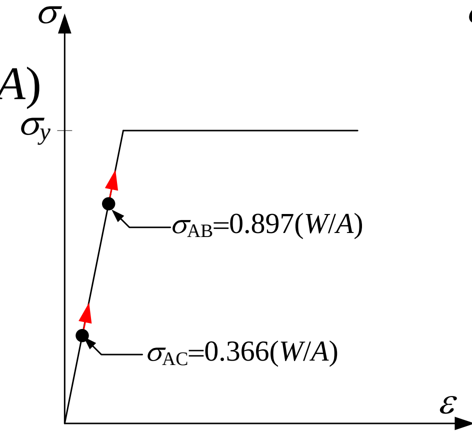
$$\delta_{AB} = \cos 135°(0 - u_A) + \sin 135°(0 - v_A) = 1.268(Wl/EA)$$

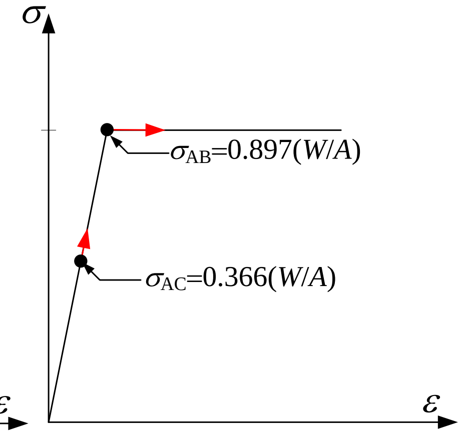
$$\delta_{AC} = \cos 30°(0 - u_A) + \sin 30°(0 - v_A) = 0.732(Wl/EA)$$

$$u_A = 0.121(Wl/EA) \quad \& \quad v_A = -1.673(Wl/EA)$$

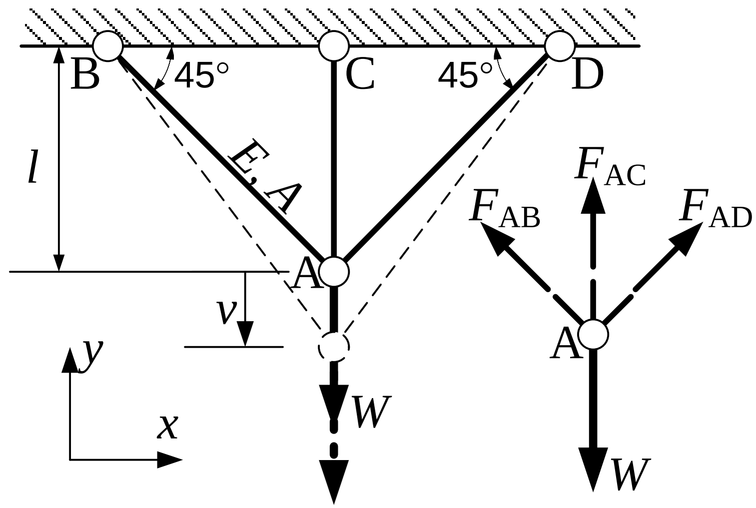
Collapse load W_C:

$$\sigma_{AB} = \sigma_y = 0.897(W_C/A) \Rightarrow W_C = 1.12A\sigma_y$$





2. Statically Indeterminate Tension Truss



Two Eq. Eqs. with three unknowns

$$-F_{AB}\cos 45^0 + F_{AD}\cos 45^0 = 0$$
$$F_{AB}\sin 45^0 + F_{AC} + F_{AD}\sin 45^0 = W$$
$$\Rightarrow \quad F_{AB} = F_{AD}, \quad 2F_{AB}\sin 45^0 + F_{AC} = W$$

To solve we need Kin. Eqs.

$$\delta_{AB} = v\sin 45^0 \qquad \delta_{AC} = v \qquad \delta_{AD} = v\sin 45^0$$

Sub. into behavior Eqs.

$$F_{AB} = E(\delta_{AB}/l_{AB})A = (EA/l)v\sin^2 45^0 = (EA/\,2l)v$$
$$F_{AC} = E(\delta_{AC}/l_{AC})A = (EA/l)v \;\Rightarrow\; F_{AC} = 2F_{AB}$$

Then into Eq. Eqs. $F_{AC} = 2F_{AB} = W\sqrt{2}/(1+\sqrt{2}) \Rightarrow F_{AB} = 0.293W \quad \& \quad F_{AC} = 0.585W$

Back into Eq. Eqs. to get displacement

$$[(EA/\,l)\sin 45^0]v + (EA/l)v = W \qquad v = Wl/[EA(1+\sin 45^0)]$$

First the highly loaded bar plastifies when

$$F_{AC} = 0.585W = \sigma_y A \Rightarrow W_1 = 1.707\sigma_y A \qquad v_1 = l\sigma_y/E$$

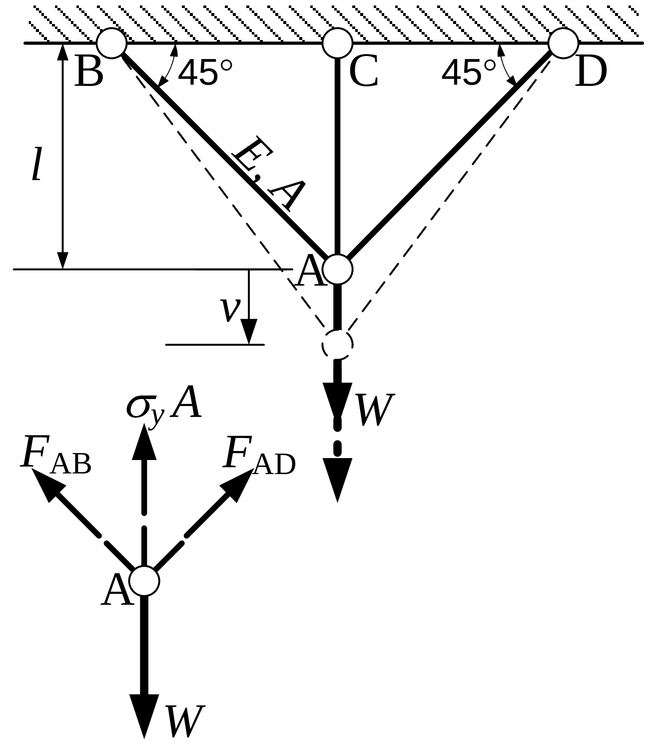
After this value the force in AC remains constant and forces in AB and AD increase

2. Statically Indeterminate Tension Truss (*Cont.*)

$F_{AB}=F_{AD}$
$2F_{AB}\sin 45^0+\sigma_y A=W$

Internal forces are

$F_{AB}=F_{AD}=0.707(W-\sigma_y A)$
$F_{AC}=\sigma_y A$



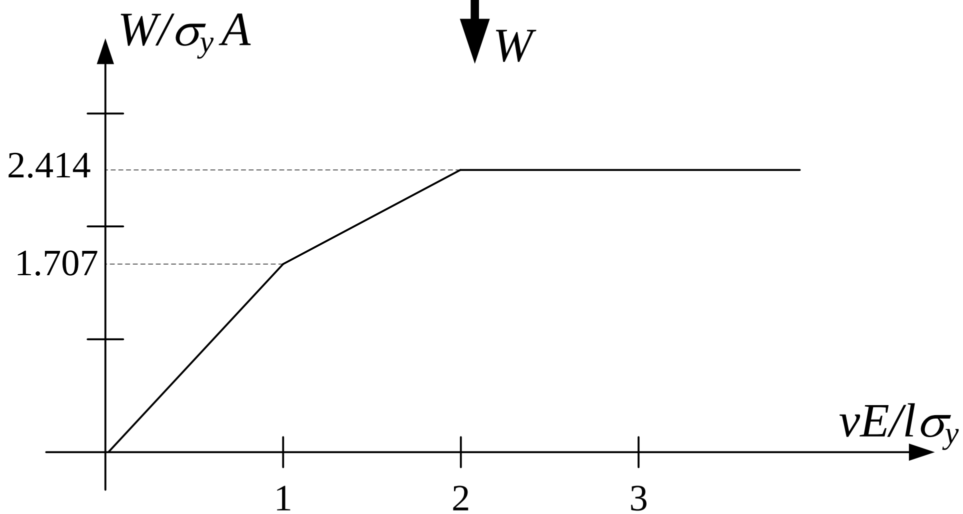
Displacement is $v=2F_{AB}l/EA=1.414(W-\sigma_y A)l/EA$

Collapse occurs when the bars AB and AD plastify

$F_{AB}=0.707(W-\sigma_y A)=\sigma_y A \Rightarrow W_c=2.414\sigma_y A$ $v_c=2l\sigma_y/E$

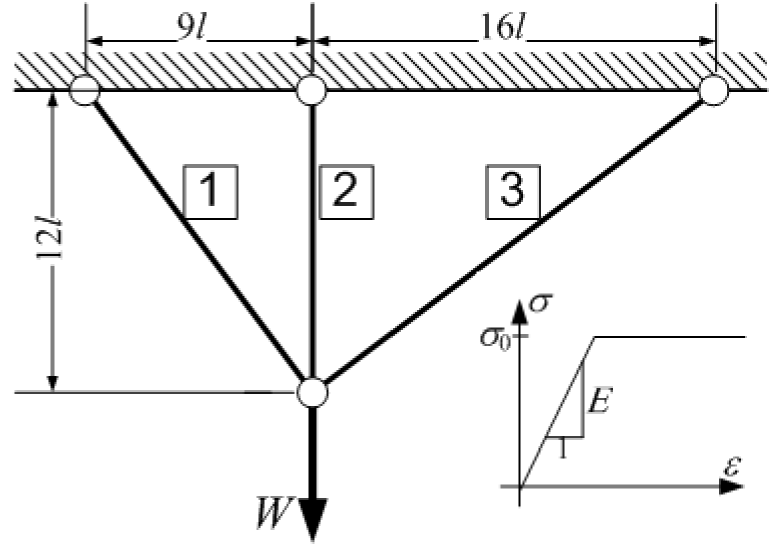
$W_1=1.707\sigma_y A,\ v_1=l\sigma_y/E$

$W_c=2.414\sigma_y A,\ v_c=2l\sigma_y/E$

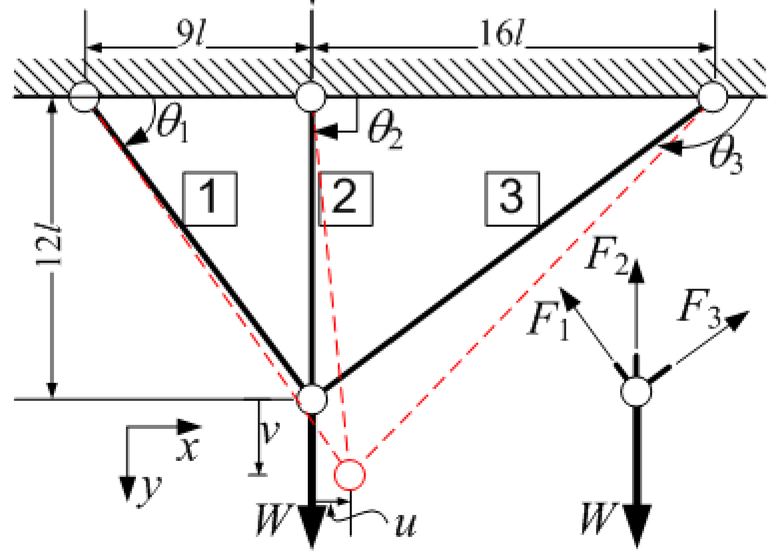


This diagram shows the behavior of the truss under the increasing load

Ex1. Consider a simple truss consisting of three pin-ended bars connected to a common suspended joint. All bars have the same cross sectional area *A*, and the same material properties (E,σ_0). The vertical Load *W*, applied at the suspended joint, increases in magnitude. Determine the evolution of *W* to the total collapse.



Solution: 8Unknowns: $F_1, F_2, F_3, u, v, \delta_1, \delta_2, \delta_3$.



Geom. Relations:

$l_1=15l.\ \cos\theta_1=0.6.\ \sin\theta_1=0.8$

$l_2=12l.\ \cos\theta_2=0.\ \sin\theta_2=1.$

$l_3=20l.\ \cos\theta_3=-0.8.\ \sin\theta_3=0.6$

Eqm. Eqs.

$\sum F_x=-0.6F_1+0.8F_3=0$

$\sum F_y=0.8F_1+F_2+0.6F_3=W$

Statically Indeterminate

Kin. Eqs.: $\delta_{IJ}=\vec{e}_{IJ}\cdot(\vec{u}_J-\vec{u}_I)$

$\delta_1=0.6u+0.8v$

$\delta_2=v$

$\delta_3=-0.8u+0.6v$

Beh. Eqs.: full Elastic

$F_1=(EA/l_1)\delta_1=(EA/15l)\delta_1$

$F_2=(EA/l_2)\delta_2=(EA/12l)\delta_2$

$F_3=(EA/l_3)\delta_3=(EA/20l)\delta_3$

Or Beh. Eqs.: full Elastic

$\delta_1=(l_1/EA)F_1=(15l/EA)F_1$

$\delta_2=(l_1/EA)F_2=(12l/EA)F_2$

$\delta_3=(l_1/EA)F_3=(20l/EA)F_3$

8Unknowns: $F_1, F_2, F_3, u, v, \delta_1, \delta_2, \delta_3$.

Eqm. Eqs.

$-0.6F_1+0.8F_3=0$

$0.8F_1+F_2+0.6F_3=W$

Kin. Eqs.:

$\delta_1=0.6u+0.8v$

$\delta_2=v$

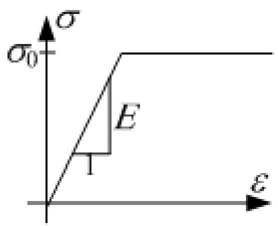
$\delta_3=-0.8u+0.6v$

Beh. Eqs.: full Elastic

$F_1=(EA/l_1)\delta_1=(EA/15l)\delta_1$

$F_2=(EA/l_2)\delta_2=(EA/12l)\delta_2$

$F_3=(EA/l_3)\delta_3=(EA/20l)\delta_3$



1- a- Solution in the full Elastic Phase: *Displacement Method*. Sub. The Kin. Eqs. into the Beh. Eqs. to get internal forces in terms of displacements u & v.

Internal forces in terms of Displacements

$F_1=(EA/15l)\delta_1=(EA/15l)(0.6u+0.8v)$

$F_2=(EA/l_2)\delta_2=(EA/12l)(v)$

$F_3=(EA/l_3)\delta_3=(EA/20l)(-0.8u+0.6v)$

Eqm. Eqs. in terms of Displacements

$-0.6[(EA/15l)(0.6u+0.8v)]+0.8[(EA/20l)(-0.8u+0.6v)]=0$

$0.8[(EA/15l)(0.6u+0.8v)]+(EA/12l)(v)+0.6[(EA/20l)(-0.8u+0.6v)]=W$

Simplifying:

$3.36u+0.48v=0$

$0.48u+8.64v=(60Wl/EA)$

Solve to get: $u=-Wl/EA.$ & $v=7Wl/EA.$

Sub. into internal force expressions in terms of u&v, to get:

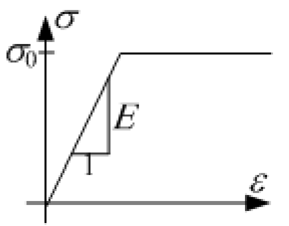
$F_1=W/3,\ F_2=7W/12,\ F_3=W/4.$

Then sub. into the Kin. Eqs. to get:

$\delta_1=5Wl/EA,\ \delta_2=7Wl/EA,\ \delta_3=3.4\ Wl/EA.$

8Unknowns: F_1, F_2, F_3, u, v, δ_1, δ_2, δ_3.

Eqm. Eqs.	Kin. Eqs.:	Or Beh. Eqs.: full Elastic
$-0.6F_1+0.8F_3=0$	$\delta_1=0.6u+0.8v$	$\delta_1=(l_1/EA)F_1=(15l/EA)F_1$
$0.8F_1+F_2+0.6F_3=W$	$\delta_2=v$	$\delta_2=(l_1/EA)F_2=(12l/EA)F_2$
$F_1-F_2+F_3=0$.	$\delta_3=-0.8u+0.6v$	$\delta_3=(l_1/EA)F_3=(20l/EA)F_3$



1- b- Solution in the full Elastic Phase: Compatibility Method. Eliminate u &v, from the Kin. Eqs., to get a **Comp. Eq.** between the 3 extensions: δ_1, δ_2, δ_3.

Comp. Eq. in terms of: δ_1, δ_2, δ_3: $0.8\delta_1+0.6\delta_3=v=\delta_2 \Rightarrow 0.8\delta_1-\delta_2+0.6\delta_3=0$

Comp. Eq. in terms of: F_m: $\delta_m=(l_m/EA)F_m$, m=1, 2, 3.

$0.8\ (l_1/EA)F_1-(l_2/EA)F_2+0.6\ (l_3/EA)F_3=0 \Rightarrow F_1-F_2+F_3=0$. Adding it to the two Eqm. Eqs., we get the same solution: $F_1=W/3$, $F_2=7W/12$, $F_3=W/4$. and…

2- End of the full Elastic Phase or first yielding:

The three bars behave elastically until the maximum stress reaches the yield stress σ_0. As the three bars have the same section area A, this is equivalent to:

$$\text{Max}(F_1, F_2, F_3)=A\sigma_0. \Rightarrow F_2=A\sigma_0.$$

This occurs when the load W reaches the value W_1, given as:

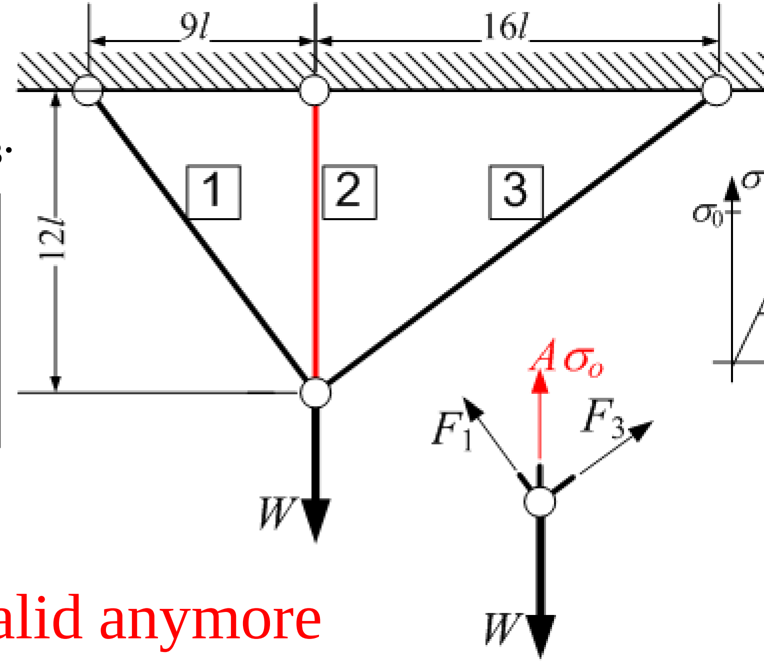
$$F_2=7W_1/12=A\sigma_0. \Rightarrow W_1=(12/7)A\sigma_0.$$

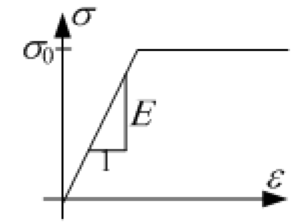
The others unknowns for this load W_1 still can be computed from the previous relations. Especially the vertical displacement v reaches a value v_1, given as:

$$v_1=7W_1 l/EA. \Rightarrow v_1=7[(12/7)A\sigma_0]l/EA. \Rightarrow v_1=12\sigma_0 l/E$$

3- the Elasto-plastic Phase:

7Unknowns: F_1, $F_2=A\sigma_0$, F_3, u, v, δ_1, δ_2, δ_3.





Eqm. Eqs.

$-0.6F_1+0.8F_3=0$

$0.8F_1+A\sigma_0+0.6F_3=W$

with $W>W_1$.

Kin. Eqs.:

$\delta_1=0.6u+0.8v$

$\delta_2=v$

$\delta_3=-0.8u+0.6v$

Or Beh. Eqs.: full Elastic

$\delta_1=(l_1/EA)F_1=(15l/EA)F_1$

~~$\delta_2=(l_1/EA)F_2=(12l/EA)F_2$~~ Not valid anymore

$\delta_3=(l_1/EA)F_3=(20l/EA)F_3$

During this phase the problem is statically determinate. The two forces F_1, & F_3, can be computed from the two Eqm. Eqs. rewritten in the form:

Eqm. Eqs.

$-0.6F_1+0.8F_3=0$

$0.8F_1+0.6F_3=W-A\sigma_0$

with $W>W_1$.

Solve to get: $F_1=0.8(W-A\sigma_0)$. & $F_3=0.6(W-A\sigma_0)$.

Sub. into the two valid Beh. Eqs. to get extensions:

$\delta_1=(15l/EA)[0.8(W-A\sigma_0)]=12l(W-A\sigma_0)/EA.$

$\delta_3=(20l/EA)[0.6(W-A\sigma_0)]=12l(W-A\sigma_0)/EA.$

Sub. These values of δ_1 & δ_3 into the first and the third Kin. Eqs. to get displacements:

$u=-0.2[12l(W-A\sigma_0)/EA]=-2.4l(W-A\sigma_0)/EA.$

$v=1.4[12l(W-A\sigma_0)/EA]=16.8l(W-A\sigma_0)/EA.$

4- End of the Elasto-plastic Phase (total collapse)

The two non yielded bars (1&3) behave elastically until the maximum between F_1, & F_3. reaches the yielding limit $A\sigma_0$. So

$$\text{Max}(F_1, F_3)=A\sigma_0. \Rightarrow F_1= A\sigma_0.$$

This occurs when the load W reaches the value W_c, given as:

$$F_1=0.8(W_c - A\sigma_0)=A\sigma_0. \Rightarrow W_c - A\sigma_0 = 1.25\,A\sigma_0 \Rightarrow W_c=2.25A\sigma_0\ .$$

The others unknowns for this load W_c still can be computed from the previous relations. Especially the vertical displacement v reaches a value v_c, given as:

$$v_c=16.8l(W - A\sigma_0)/EA=21\sigma_0 l\ /E$$

The evolution of W during the two phases: full elastic phase then the elasto-plastic phase can be shown by observing the following diagram reflecting the relation between W and v.

The full elastic phase starts at:

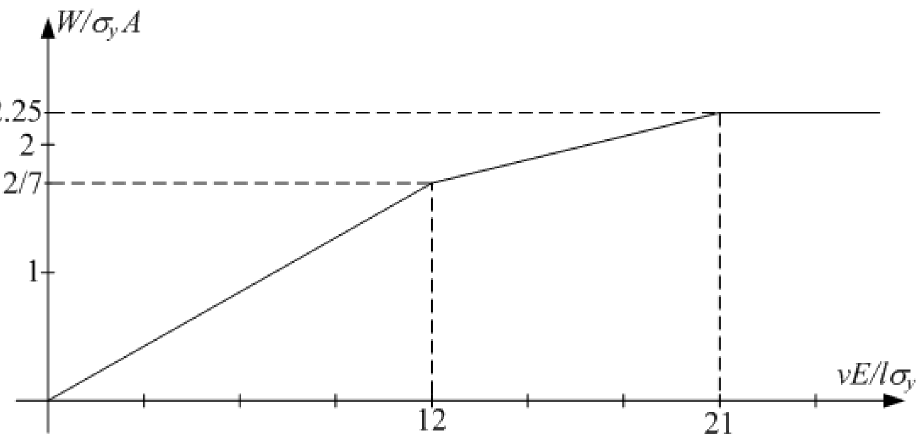
$W=0$ & $v=0$.

It ends at:

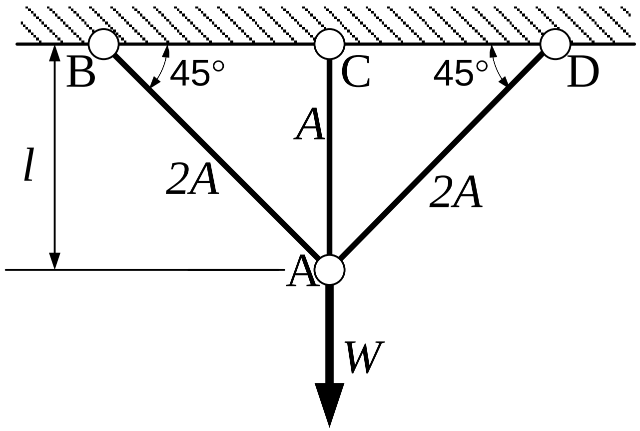
$W_1=(12/7)A\sigma_0$. & $v_1=12\sigma_0 l\ /E$.

Where the elasto-plastic phase starts to end at total collapse where:

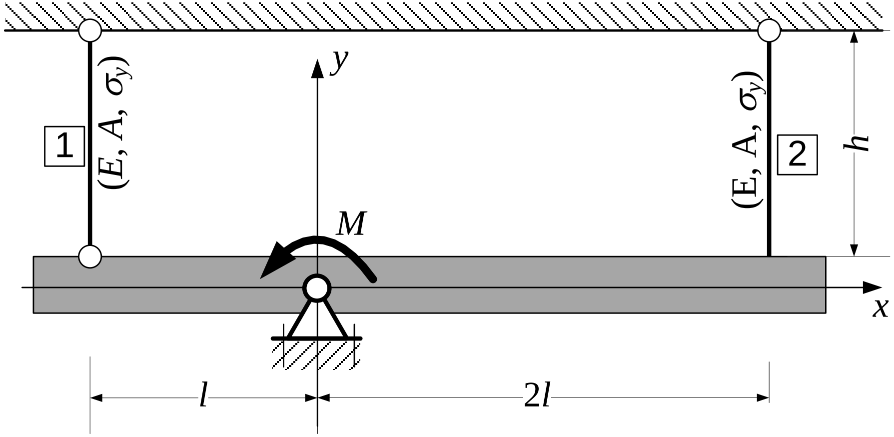
$W_c=2.25A\sigma_0$. & $v_c=21\sigma_0 l\ /E$

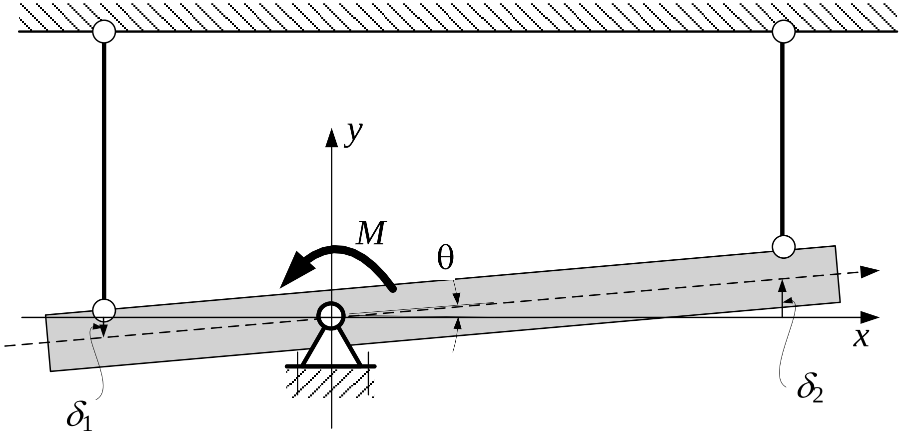


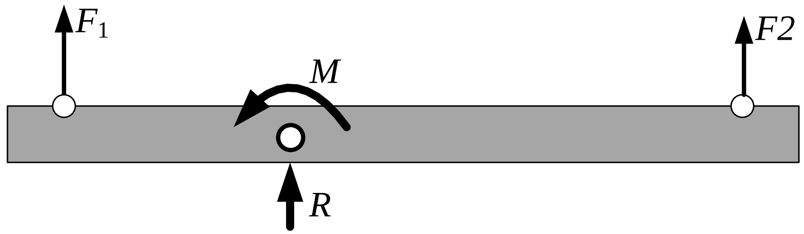
Ex.2. Consider a simple truss consisting of three pin-ended bars connected to a common suspended joint. The bars have the different cross sectional area as indicated on the figure, and the same material properties (E, σ_0). The vertical Load W, applied at the suspended joint, increases in magnitude. Determine the evolution of W to the total collapse.



Ex3. Consider a rigid horizontal beam suspended to the roof by two different bars as indicated on the figure, and rotates about the shown hinge by a small angle θ, under the action of a moment M, applied at the hinge. If the moment M increases in magnitude. Determine the evolution of M to the total collapse.

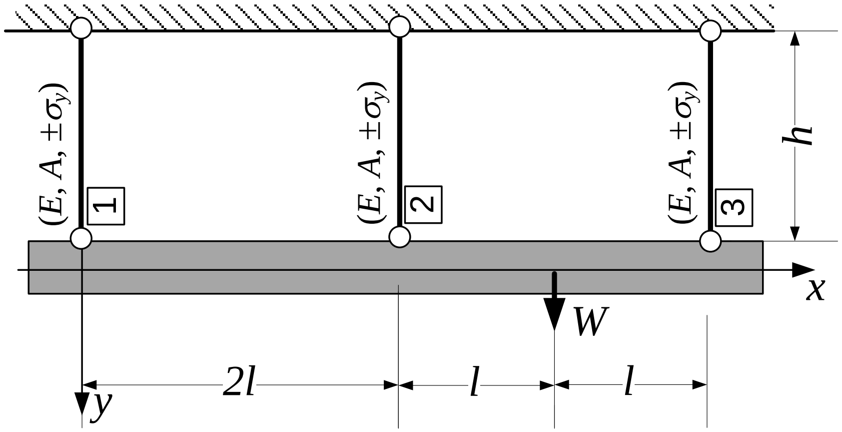




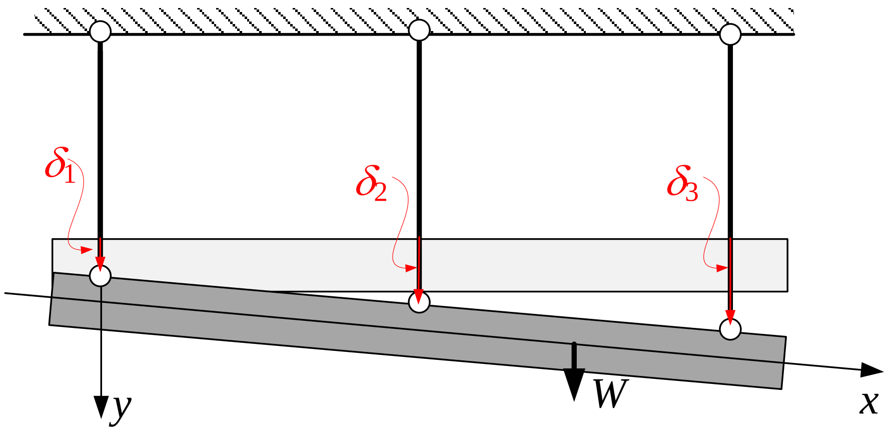


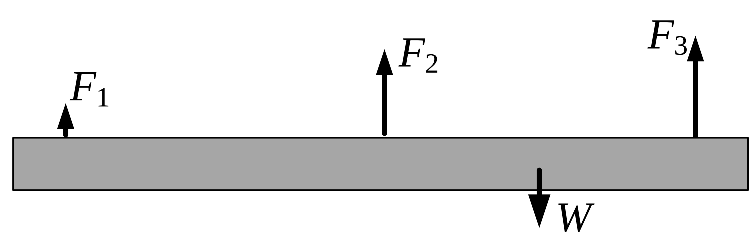
Solution:

Ex4. Consider a rigid horizontal beam suspended to the roof by three identical bars as indicated on the figure. If the load *W* increases in magnitude. Determine the evolution of *W* to the total collapse.

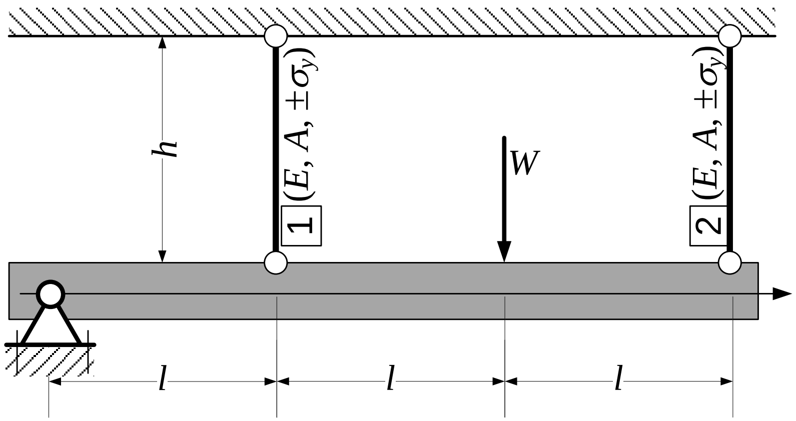


Solution:





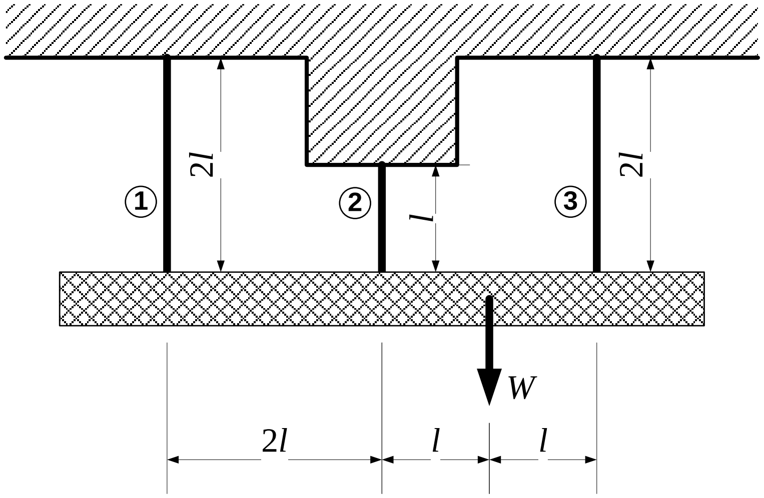
Ex.5. Consider a rigid horizontal beam suspended to the roof by two identical bars and supported by a pin as shown on the figure. The beam rotates about the pin by a small angle θ, under the action of a force W, acting as shown.



1. If the force W increases in magnitude, from zero, determine the evolution of W to the total collapse.

2. Draw a dimensionless diagram showing the variation of W with θ.

Ex.6. Consider a rigid horizontal beam suspended from the roof by three bars of the same material and sectional properties (A, $\pm\sigma_y$) but of different length as shown in the figure. The beam translates and rotates by small quantities, under the action of a force W, acting as shown.



1. If the force W increases in magnitude, from zero, determine the evolution of W to the total collapse.

2. Draw a dimensionless diagram showing the variation of W with v, the displacement of its point of application