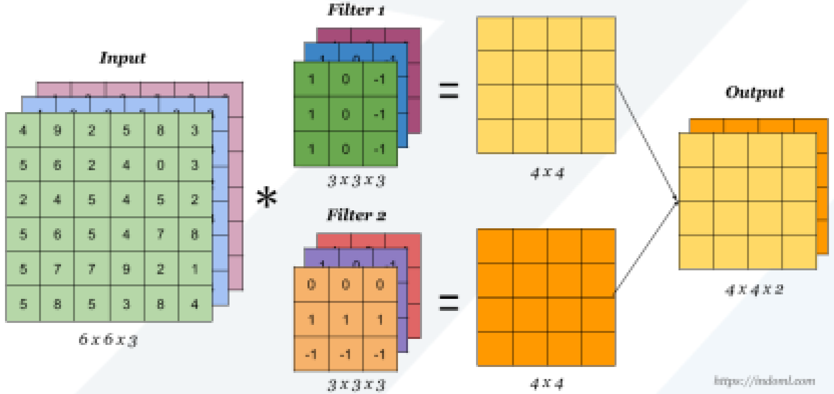
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 3: Determinants



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Chapter 2
Determinants

1. Determinants by Cofactor Expansion
2. Evaluating Determinants by Row Reduction
3. Properties of Determinants
4. Applications of Determinants

- Every square matrix has a unique value associated with it, called the determinant. We can use this value to establish whether the matrix has an inverse or not, also finding whether the linear system has a unique solution.

1. Determinants by Cofactor Expansion

The determinant of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \det(A) = |A| = ad - bc$$

- Example 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7 \qquad \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative.

Inverse of a 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $\det(A) \neq 0$.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

This means that the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $\det(A) \neq 0$

Minors and cofactors

- The determinant M_{ij} of a square matrix determined by deleting the i-th row and j-th column of A is called minor of the entry a_{ij}.

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- The cofactor C_{ij} of the entry a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$
- Example 2: Find all the minors and cofactors of A

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \qquad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5, \qquad M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \qquad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \qquad M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \qquad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3, \qquad M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

$$C_{11} = (-1)^{1+1} M_{11} = -1, \qquad C_{12} = (-1)^{1+2} M_{12} = 5, \qquad C_{13} = (-1)^{1+3} M_{13} = 4$$

$$C_{21} = (-1)^{2+1} M_{21} = -2, \qquad C_{22} = (-1)^{2+2} M_{22} = -4, \qquad C_{23} = (-1)^{2+3} M_{23} = 8$$

$$C_{31} = (-1)^{3+1} M_{31} = 5, \qquad C_{32} = (-1)^{3+2} M_{32} = 3, \qquad C_{33} = (-1)^{3+3} M_{33} = -6$$

- **Theorem 1: (Expansion by cofactors)**

Let A is a square matrix of order n. Then the determinant of A is given by

$$(a)\ \det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(Cofactor expansion along the i-th row, $i = 1, 2, \ldots, n$)

or

$$(b)\ \det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j-th column, $j = 1, 2, \ldots, n$)

- **Example 3: The determinant of a matrix of order 3**

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

From Example 2:

$$\begin{aligned} &C_{11} = -1, \quad C_{12} = 5, \quad C_{13} = 4 \\ &C_{21} = -2, \quad C_{22} = -4, \quad C_{23} = 8 \\ &C_{31} = 5, \quad C_{32} = 3, \quad C_{33} = -6 \end{aligned}$$

$$\begin{aligned}
\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\
&= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\
&= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\
&= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\
&= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14
\end{aligned}$$

- Example 4: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\Rightarrow \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(7) + (2)(5) + (1)(-8) = 2$$

- Note: The row (or column) containing the most zeros is the best choice for expansion by cofactors.

- Example 5: The determinant of a matrix of order 4

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = 3\left[(2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3[(2)(1)(-4) + (3)(-1)(-7)] = (3)(13) = 39$$

- The determinant of a matrix of order 3 (Sarrus Rule)

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Subtract these three products

Add these three products

$$\det(A)=|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-(a_{31}a_{22}a_{13}+a_{32}a_{23}a_{11}+a_{33}a_{21}a_{12})$$

- Example 6: (Using Sarrus Rule)

$$\begin{matrix} & & & -4 & 0 & 6 \end{matrix}$$

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}\begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & -4 \end{matrix}$$

$$\begin{matrix} & & & 0 & 16 & -12 \end{matrix}$$

$$\Rightarrow \det(A)=|A|=0+16-12-(-4+0+6)=2$$

- Upper triangular matrix: All the entries below the main diagonal are zeros.
- Lower triangular matrix: All the entries above the main diagonal are zeros.
- Diagonal matrix: All the entries above and below the main diagonal are zeros.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular lower triangular diagonal

- Theorem 2: (Determinant of a Triangular Matrix)

  If A is an $n\times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$|A| = (2)(-2)(1)(3) = -12$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$|B| = (-1)(3)(2)(4)(-2) = 48$$

2. Evaluating Determinants by Row Reduction

- **Theorem 3: (Elementary row operations and determinants)**

  Let A and B be square matrices

$$(a)\ B = r_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b)\ B = r_i^{(k)}(A) \Rightarrow \det(B) = k\det(A) \quad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c)\ B = r_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \quad \Rightarrow \quad \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \quad \Rightarrow \quad \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \quad \Rightarrow \quad \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- **Notes:** $\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$

  $\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$

  $\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$

- Note: A row-echelon form of a square matrix is always upper triangular.
- Example 7: (Determinant using elementary row operations)

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \det(A) = ?$$

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{12}^{(-2)}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{23}^{(-1)}}{=} (7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} \overset{r_3^{(-1)}}{=} (7)(-1)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (7)(-1)(1)(1)(1) = -7$$

Determinants and elementary column operations

- **Theorem 4: (Elementary columns operations and determinants)**

  Let A and B be square matrices

$$(a)\ B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b)\ B = c_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |c_{i}^{(k)}(A)| = k|A|)$$

$$(c)\ B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \Rightarrow \det(A_1) = \det(c_1^{(\frac{1}{2})}(A)) = \tfrac{1}{2}\det(A) = (\tfrac{1}{2})(-8) = -4$$

$$A_2 = c_{12}(A) \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- **Theorem 5: (Conditions that yield a zero determinant)**

  If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$

  (a) An entire row (or an entire column) consists of zeros.

  (b) One row (or column) is a multiple of another row (or column).

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Example 8: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{r_{25}^{(-1)}}{\overset{r_{24}^{(1)}}{=}} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{c_{41}^{(-3)}}{=}\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4}\begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \overset{r_{21}^{(1)}}{=} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3}\begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix} = (5)(-27) = -135$$

3. Properties of Determinants

- **Theorem 6: (Determinant of a matrix product)**

  If A and B are square matrices of order n, then $\det(AB) = \det(A)\det(B)$

- **Note:** $\det(A + B) \neq \det(A) + \det(B)$

- Example 9: (Determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$ Find $|A|$, $|B|$, and $|AB|$

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7,\quad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}=\begin{bmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{bmatrix}\Rightarrow |AB|=\begin{vmatrix}8 & 4 & 1\\ 6 & -1 & -10\\ 5 & 1 & -1\end{vmatrix}=-77$$

Check: $|AB|=|A|\,|B|$ $\qquad -77=-7 \text{ x } 11$

- Theorem 7: (Determinant of a scalar multiple of a matrix)

  If A is an $n\text{x}n$ matrix and c is a scalar, then: $\det(cA) = c^n \det(A)$

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \qquad \text{Find } |A|$$

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Theorem 8: (Determinant of an invertible a matrix)

  A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$

- Example 10: (Classifying square matrices as singular or nonsingular)

$$A=\begin{bmatrix}0 & 2 & -1\\ 3 & -2 & 1\\ 3 & 2 & -1\end{bmatrix},\quad B=\begin{bmatrix}0 & 2 & -1\\ 3 & -2 & 1\\ 3 & 2 & 1\end{bmatrix}$$

$|A|=0 \Rightarrow A$ has no inverse (it is singular).

$|B|=-12\neq 0 \Rightarrow B$ has an inverse (it is nonsingular).

- Theorem 9: (Determinant of an inverse matrix)

If A is invertible, then $\det(A^{-1})=\dfrac{1}{\det(A)}$.

- Theorem 10: (Determinant of a transpose)

If A is a square matrix, then $\det(A^{T})=\det(A)$.

$$A=\begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 2\\ 2 & 1 & 0\end{bmatrix} \Rightarrow |A| = \begin{vmatrix}1 & 0 & 3\\ 0 & -1 & 2\\ 2 & 1 & 0\end{vmatrix} = 4, \quad |A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, \quad |A^T| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an nxn matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\boldsymbol{x} = b$ has a unique solution for every $n \times 1$ matrix $\boldsymbol{b}$.

(3) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Example 11: Which of the following system has a unique solution?

$$(a)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array} \qquad (b)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & -4 \end{array}$$

$(a)\ A\boldsymbol{x}=\boldsymbol{b} \Rightarrow |A|=0$. This system does not have a unique solution.

$(b)\ B\boldsymbol{x}=\boldsymbol{b} \Rightarrow |B|=-12 \neq 0$. This system has a unique solution.

4. Applications of Determinants

- The Adjoint of a Matrix $\operatorname{adj}(A)=\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$

- The Inverse of a Matrix Using its Adjoint

If A is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det A}\operatorname{adj}(A)$

- Example 12: Find the inverse of a matrix using its adjoint

Use the adjoint of A to find A^{-1}, where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{vmatrix} = 3 \Rightarrow A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$$

Cramer's Rule

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$$A\boldsymbol{x} = \boldsymbol{b}, \quad A = \left[a_{ij}\right]_{n\times n} = \left[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \cdots, \boldsymbol{A}^{(n)}\right], \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \cdots, \boldsymbol{A}^{(j-1)}, \boldsymbol{b}, \boldsymbol{A}^{(j+1)}, \cdots, \boldsymbol{A}^{(n)} \right] = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e. $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, \cdots, n$$

- Example 13: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{rcrcrcl} -x & + & 2y & - & 3z & = & \boxed{1} \\ 2x & & & + & z & = & 0 \\ 3x & - & 4y & + & 4z & = & 2 \end{array}$$

$$\det(A)=\begin{vmatrix}-1 & 2 & -3\\ 2 & 0 & 1\\ 3 & -4 & 4\end{vmatrix}=10 \qquad \det(A_1)=\begin{vmatrix}1 & 2 & -3\\ 0 & 0 & 1\\ 2 & -4 & 4\end{vmatrix}=8$$

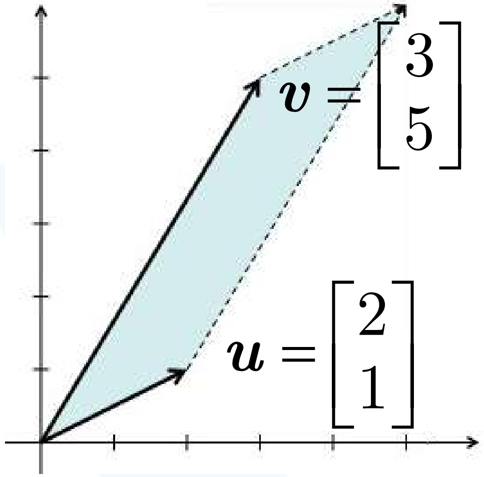
$$\det(A_2)=\begin{vmatrix}-1 & 1 & -3\\ 2 & 0 & 1\\ 3 & 2 & 4\end{vmatrix}=-15, \qquad \det(A_3)=\begin{vmatrix}-1 & 2 & 1\\ 2 & 0 & 0\\ 3 & -4 & 2\end{vmatrix}=-16$$

$$x=\frac{\det(A_1)}{\det(A)}=\frac{4}{5}, \quad y=\frac{\det(A_2)}{\det(A)}=\frac{-3}{2}, \quad z=\frac{\det(A_3)}{\det(A)}=\frac{-8}{5}$$

- **Determinants as Area or Volume**

  If A is a 2x2 matrix, the area of the parallelogram determined by the columns of A is: $\text{Area}=\left|\det(A)\right|$

If A is a 3x3 matrix, the volume of the parallelepiped determined by the columns of A is: $\text{Volume} = |\det(A)|$



- The area of parallelogram formed by $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is

$$\text{Area} = |\det(A)| = \left|\det \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\right| = |10 - 3| = 7$$

- The volume of parallelepiped formed by $u = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is

$$\text{Volume} = |\det(A)| = \left|\det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}\right| = |2 + 3 + 4 - (1 + 1 + 24)| = 17$$