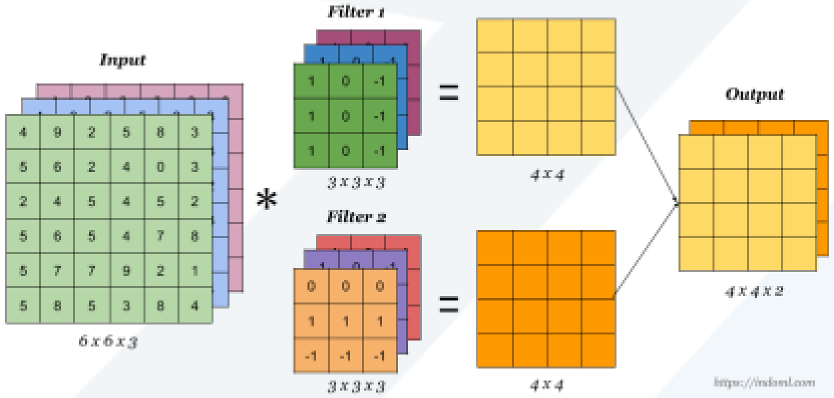
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 4: Euclidean Vector Spaces



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Chapter 3

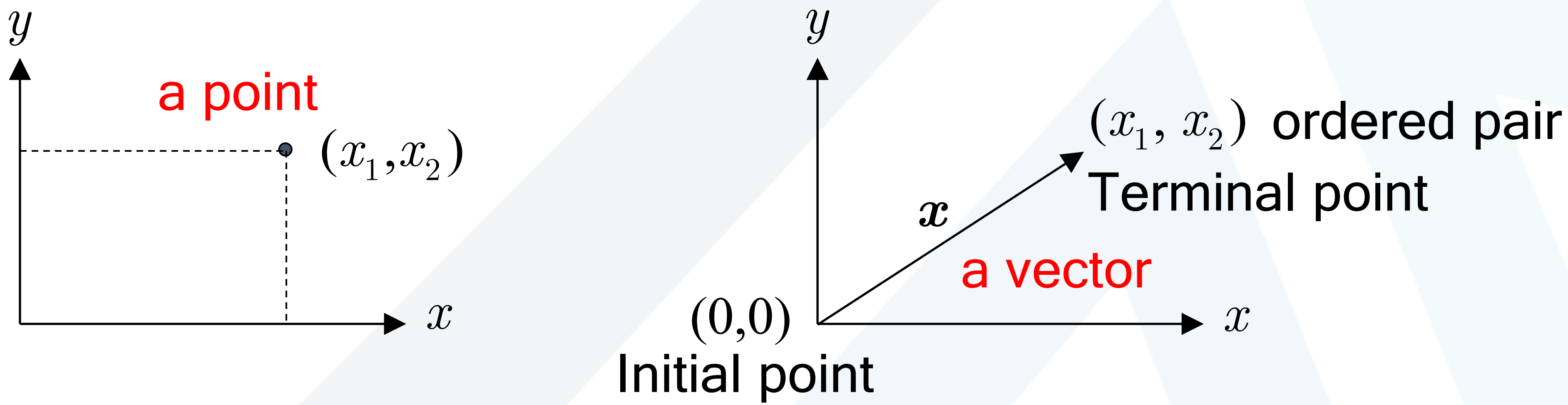
Euclidean Vector Spaces

1. Vectors in 2-Space, 3-Space, and n-Space
2. Norm, Dot Product, and Distance in $\boldsymbol{R}^n$
3. Basis, Spanning Sets and Linear Independence

1. Vectors in 2-Space, 3-Space, and n-Space

Vectors in the plane

- a vector $\boldsymbol{x}$ in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2).

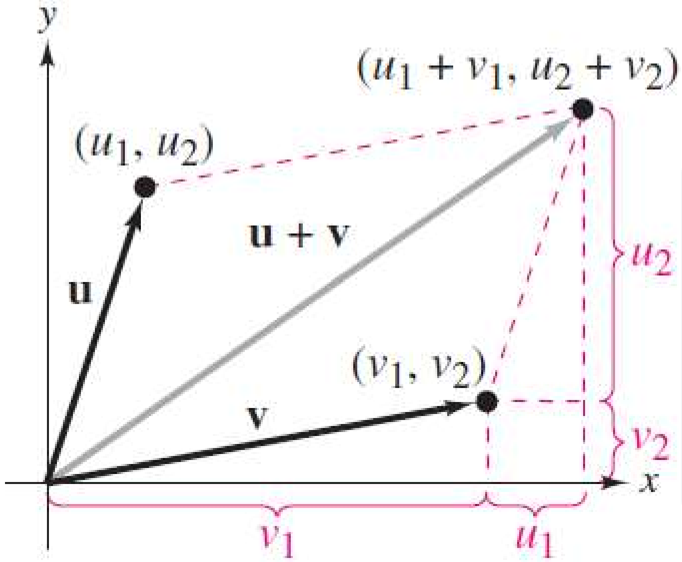


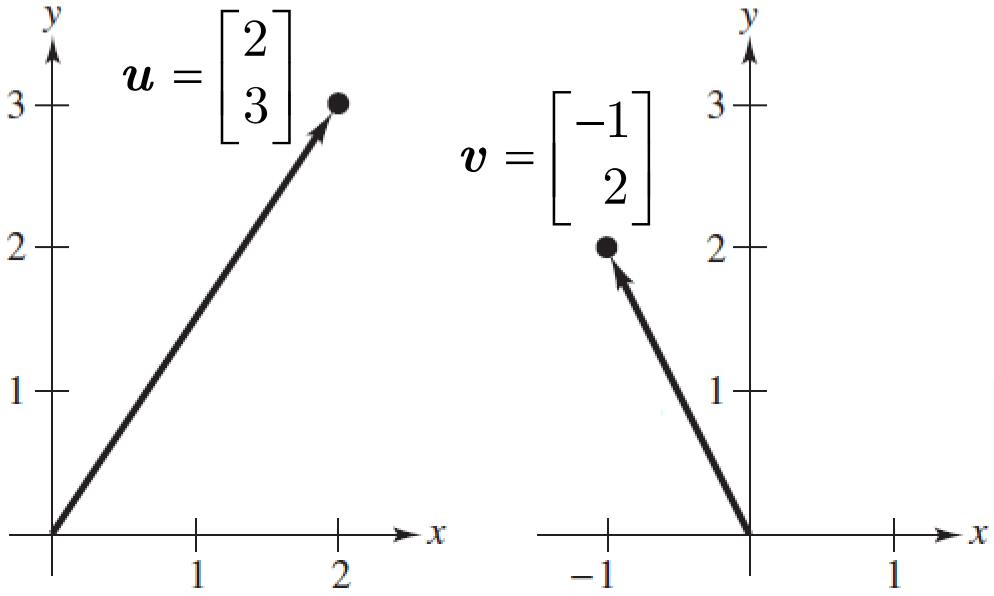
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 =$ first component of $\boldsymbol{x}$

$x_2 =$ second component of $\boldsymbol{x}$

- Vector Addition



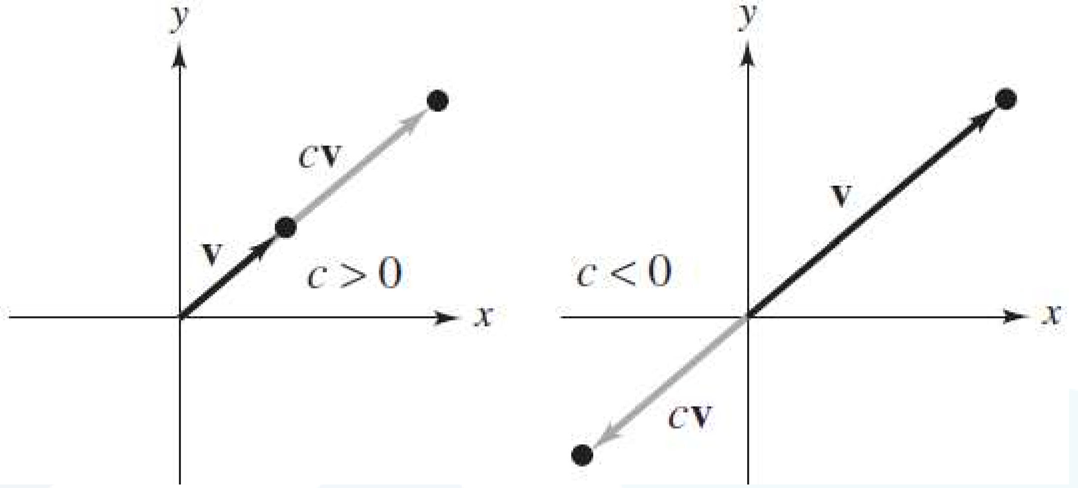


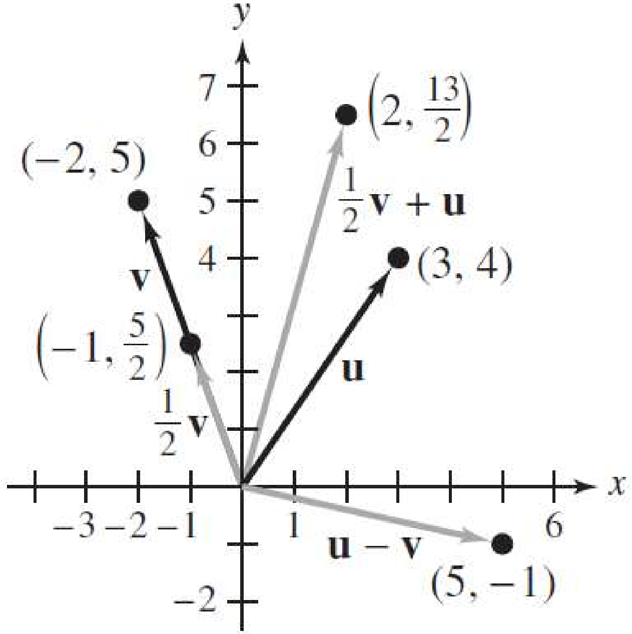
$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

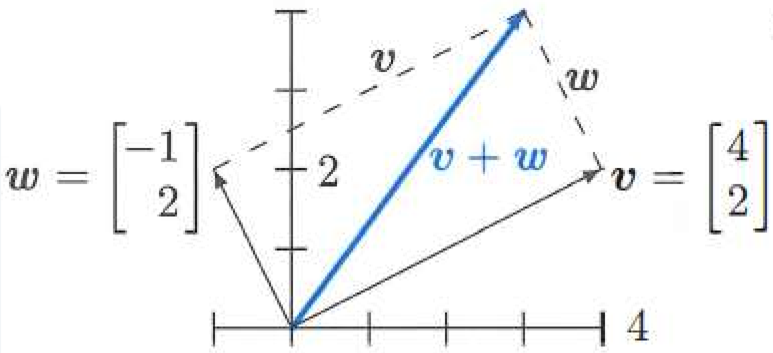
- Scalar Multiplication

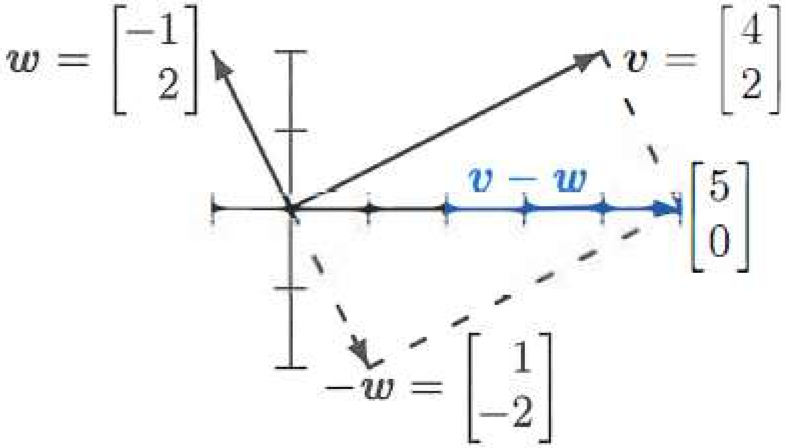
$$c\boldsymbol{v} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

$$-\boldsymbol{v} = (-1)\boldsymbol{v} \Rightarrow \boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$$









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Vectors in the n-space

R^1 = 1-space = set of all real number

R^2 = 2-space = set of all ordered pair of real numbers (x_1, x_2)

R^3 = 3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)

$\vdots$

R^n = n-space = set of all ordered n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$

- Notes: An n-tuple $(x_1, x_2, \ldots, x_n)$ can be viewed as:

(1) a point in R^n with the x_i's as its coordinates.

(2) a vector $\boldsymbol{x}$ in R^n with the x_i's as its components.

(3) a vector $\boldsymbol{x}$ in Rn will be represented also as $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Operations on Vectors in R^n

Let $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ and $\boldsymbol{v} = (v_1, v_2, ..., v_n)$ two vectors in R^n, and if c is any scalar

- Equal: $\boldsymbol{u} = \boldsymbol{v}$ if and only if $u_1 = v_1,\ u_2 = v_2,\ ...,\ u_n = v_n$
- Vector addition (the sum of $\boldsymbol{u}$ and $\boldsymbol{v}$): $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\ u_2 + v_2,\ ...,\ u_n + v_n)$
- Scalar multiplication (the scalar multiple of $\boldsymbol{u}$ by c): $c\boldsymbol{u} = (cu_1,\ cu_2,\ ...,\ cu_n)$
- Note: The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.
- Negative: $-\boldsymbol{u} = (-u_1, -u_2, ..., -u_n)$
- Difference: $\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1,\ u_2 - v_2,\ ...,\ u_n - v_n)$
- Zero vector: $\mathbf{0} = (0, 0, ..., 0)$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the additive identity in R^n.

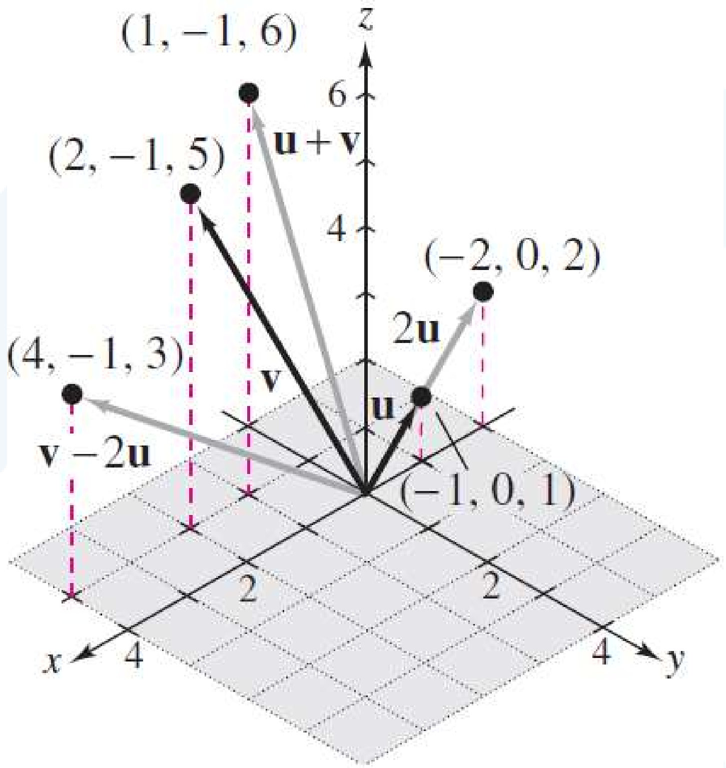
(2) The vector $-v$ is called the additive inverse of v.

- Example 1: Vector operations in R^3

Let $u = (-1, 0, 1)$ and $v = (2, -1, 5)$ in R^3.

Perform each vector operation:

(a) $u + v$ (b) $2u$ (c) $v - 2u$



(a) $u + v = (-1, 0, 1) + (2, -1, 5) = (1, -1, 6)$

(b) $2u = 2\,(-1, 0, 1) = (-2, 0, 2)$

(c) $v - 2u = (2, -1, 5) - (-2, 0, 2) = (4, -1, 3)$

- **Theorem 1: (Properties of vector addition and scalar multiplication)**

  Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in R^n, and let c and d be scalars

(1)	$\boldsymbol{u}+\boldsymbol{v}$ is a vector in R^n	Closure under addition
(2)	$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$	Commutative property of addition
(3)	$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$	Associative property of addition
(4)	$\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$	Additive identity property
(5)	$\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$	Additive inverse property
(6)	$c\boldsymbol{u}$ is a vector in R^n	Closure under scalar multiplication
(7)	$c(\boldsymbol{u}+\boldsymbol{v})=c\boldsymbol{u}+c\boldsymbol{v}$	Distributive property
(8)	$(c+d)\boldsymbol{u}=c\boldsymbol{u}+d\boldsymbol{u}$	Distributive property
(9)	$c(d\boldsymbol{u})=(cd)\boldsymbol{u}$	Associative property of multiplication
(10)	$1(\boldsymbol{u})=\boldsymbol{u}$	Multiplicative identity property

- Example 2: Vector operations in R^4

Let $u = (2, -1, 5, 0)$, $v = (4, 3, 1, -1)$ and $w = (-6, 2, 0, 3)$ be vectors in R^4. Solve x for each of the following: (a) $x = 2u - (v + 3w)$, (b) $3(x + w) = 2u - v + x$

(a) $x = 2u - (v + 3w) = 2u - v - 3w$

$$= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9)$$

$$= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9)$$

$$= (18, -11, 9, -8)$$

(b) $3(x + w) = 2u - v + x \Leftrightarrow 3x + 3w = 2u - v + x \Leftrightarrow 3x - x = 2u - v - 3w$

$$\Leftrightarrow 2x = 2u - v - 3w \Leftrightarrow x = u - \tfrac{1}{2}v - \tfrac{3}{2}w$$

$$x = (2, 1, 5, 0) + (-2, -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}) + (9, -3, 0, -\tfrac{9}{2})$$

$$= (9, -\tfrac{11}{2}, \tfrac{9}{2}, -4)$$

- Theorem 2: (Properties of additive identity and additive inverse)

  Let v be a vector in R^n, and c be a scalar. Then the properties below are true:

  (1) The additive identity is unique. That is, if $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v}$, then $\boldsymbol{u} = \mathbf{0}$

  (2) The additive inverse of v is unique. That is, if $\boldsymbol{v} + \boldsymbol{u} = \mathbf{0}$, then $\boldsymbol{u} = -\boldsymbol{v}$

  (3) $0\boldsymbol{v} = \mathbf{0}$ (4) $c\mathbf{0} = \mathbf{0}$

  (5) If $c\boldsymbol{v} = \mathbf{0}$, then $c = 0$ or $\boldsymbol{v} = \mathbf{0}$

  (6) $-(-\boldsymbol{v}) = \boldsymbol{v}$

Linear combination

- The vector $\boldsymbol{x}$ is called a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ if it can be expressed in the form $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_k\boldsymbol{v}_k$ where $c_1, c_2, \ldots, c_k$ are scalars.

- Example 3: linear combination

Given $\boldsymbol{x} = (-1, -2, -2)$, $\boldsymbol{u} = (0, 1, 4)$, $\boldsymbol{v} = (-1, 1, 2)$, and $\boldsymbol{w} = (3, 1, 2)$ in R^3. Find a, b, and c such that $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$.

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned} \quad \Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\boldsymbol{x} = \boldsymbol{u} - 2\boldsymbol{v} - \boldsymbol{w}$

- Example 4: not a linear combination

Given $\boldsymbol{x} = (1, -2, 2)$, $\boldsymbol{u} = (1, 2, 3)$, $\boldsymbol{v} = (0, 1, 2)$, and $\boldsymbol{w} = (-1, 0, 1)$ in R^3. Prove that $\boldsymbol{x}$ is not a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Gauss-J. Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right] \Rightarrow \boldsymbol{x} \neq a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$$

2. Norm, Dot Product, and Distance in R^n

- **Norm (Length) of a Vector:** The norm of a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in R^n is given by: $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

- **Example 5: Norm of a vector**

  (a) In R^5, the length of $\boldsymbol{v} = (0, -2, 1, 4, -2)$ is given by:

$$\|\boldsymbol{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

  (b) In R^3 the length of $\boldsymbol{v} = \left(\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ is given by:

$$\|\boldsymbol{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(-\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

  ($\boldsymbol{v}$ is a unit vector)

- Notes: Properties of length

$(1)\ \|\boldsymbol{v}\| \geq 0$

$(2)\ \|\boldsymbol{v}\| = 1 \Rightarrow \boldsymbol{v}$ is called a unit vector

$(3)\ \|\boldsymbol{v}\| = 0$ iff $\boldsymbol{v} = 0$

- Notes:

(1) the standard unit vector in R^2: $\{\boldsymbol{i}, \boldsymbol{j}\} = \{(1, 0), (0, 1)\}$

(2) the standard unit vector in R^3: $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

- Notes: Two nonzero vectors are parallel $\boldsymbol{u} = c\boldsymbol{v}$

(1) $c > 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the same direction.

(2) $c < 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the opposite direction.

- Theorem 3: (Length of a scalar multiple)

  Let v be a vector in R^n and c be a scalar, then $\|cv\| = |c|\,\|v\|$

- Theorem 4: (Unit vector in the direction of v)

  If v is a nonzero vector in R^n, then the vector $u = \dfrac{v}{\|v\|}$ has length 1 and has the same direction as v.

  This vector u is called the unit vector in the direction of v.

- Note: The process of finding the unit vector in the direction of v is called normalizing the vector v.

- Example 6: Finding a unit vector

  Find the unit vector in the direction of $v = (3, -1, 2)$.

$$\|v\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

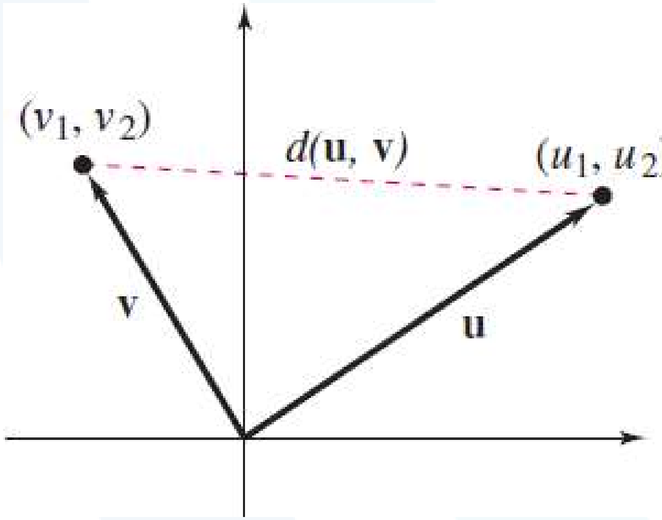
$$\Rightarrow \frac{v}{\|v\|} = \frac{(3\,,-1\,,2)}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{1}{\sqrt{14}}(3\,,-1\,,2) = \left(\frac{3}{\sqrt{14}}\,,\frac{-1}{\sqrt{14}}\,,\frac{2}{\sqrt{14}}\right)$$

- **Distance between two vectors:** The distance between two vectors u and v in R^n is: $d(u, v) = \|u - v\|$
- **Notes:** (Properties of distance)

(1) $d(u, v) \geq 0$

(2) $d(u, v) = 0$ if and only if $u = v$

(3) $d(u, v) = d(v, u)$



- ## Example 7: Distance between 2 vectors

  The distance between $\boldsymbol{u} = (0, 2, 2)$ and $\boldsymbol{v} = (2, 0, 1)$ is

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \|(0-2), 2-0, 2-1)\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- **Dot product in R^n**: The dot product of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is the scalar quantity: $\boldsymbol{u}.\boldsymbol{v} = u_1\boldsymbol{v}_1 + u_2\boldsymbol{v}_2 + \ldots + u_n\boldsymbol{v}_n$

- ## Theorem 5: (Properties of the dot product)

  If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in R^n and c is a scalar, then:

  (1) $\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{v}.\boldsymbol{u}$

  (2) $\boldsymbol{u}.(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u}.\boldsymbol{v} + \boldsymbol{u}.\boldsymbol{w}$

  (3) $c(\boldsymbol{u}.\boldsymbol{v}) = (c\boldsymbol{u}).\boldsymbol{v} = \boldsymbol{u}.(c\boldsymbol{v})$

  (4) $\boldsymbol{v}.\boldsymbol{v} = \|\boldsymbol{v}\|^2$

  (5) $\boldsymbol{v}.\boldsymbol{v} \geq 0$, and $\boldsymbol{v}.\boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$

- Example 8: Finding the dot product of two vectors

  The dot product of $\boldsymbol{u} = (1, 2, 0, -3)$ and $\boldsymbol{v} = (3, -2, 4, 2)$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Euclidean n-space: R^n was defined to be the set of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called Euclidean n-space.

- Example 9: Finding dot product

  $\boldsymbol{u} = (2, -2)$, $\boldsymbol{v} = (5, 8)$, $\boldsymbol{w} = (-4, 3)$

  (a) $\boldsymbol{u}.\boldsymbol{v}$ (b) $(\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w}$ (c) $\boldsymbol{u}.(2\boldsymbol{v})$ (d) $\|\boldsymbol{w}\|^2$ (e) $\boldsymbol{u}.(\boldsymbol{v} - 2\boldsymbol{w})$

  (a) $\boldsymbol{u}.\boldsymbol{v} = (2)(5) + (-2)(8) = -6$ (b) $(\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w} = -\boldsymbol{w} = -6(-4, 3) = (24, -18)$

(c) $u.(2v) = 2(u.v) = 2(-6) = -12$ (d) $\|w\|^2 = w.w = (-4)(-4) + (3)(3) = 25$

(e) $(v - 2w) = (5 - (-8), 8 - 6) = (13, 2)$

$u.(v - 2w) = (2)(13) + (-2)(2) = 22$

- Example 10: Using the properties of the dot product

Given $u.u = 39$, $u.v = -3$, $v.v = 79$. Find $(u + 2v).(3u + v)$

$$
\begin{aligned}
(u + 2v).(3u + v) &= u.(3u + v) + 2v.(3u + v) \\
&= u.(3u) + u.v + (2v).(3u) + (2v).v \\
&= 3(u.u) + u.v + 6(v.u) + 2(v.v) \\
&= 3(u.u) + 7(u.v) + 2(v.v) \\
&= 3(39) + 7(-3) + 2(79) = 254
\end{aligned}
$$

- **Theorem 6: (The Cauchy-Schwarz inequality)**

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, then $|\boldsymbol{u}.\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$

- **Example 11: (An example of the Cauchy-Schwarz inequality)**

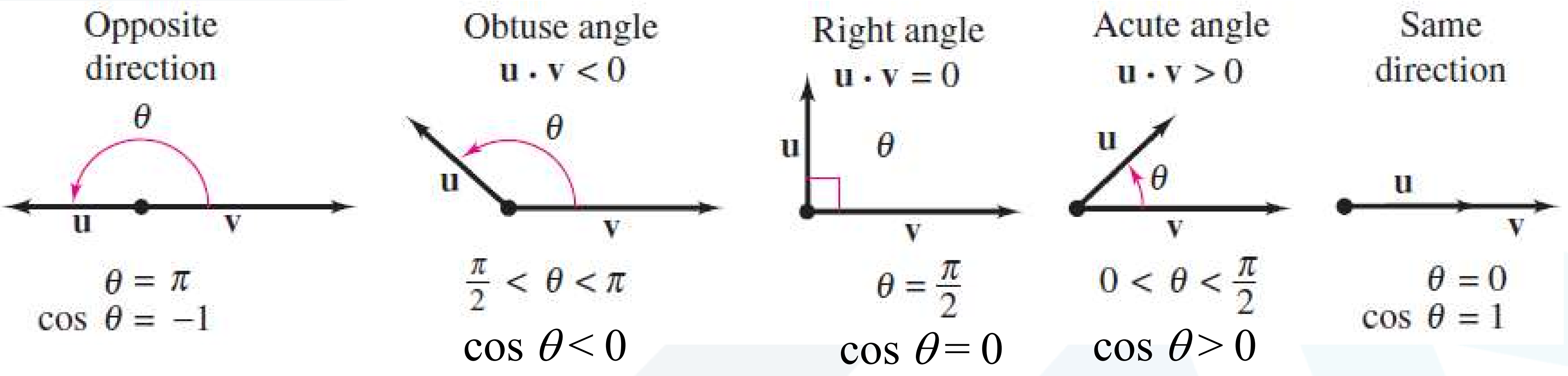
  Verify the Cauchy-Schwarz inequality for $\boldsymbol{u} = (1, -1, 3)$ and $\boldsymbol{v} = (2, 0, -1)$

  $$\boldsymbol{u}.\boldsymbol{u} = 11,\ \boldsymbol{u}.\boldsymbol{v} = -1,\ \boldsymbol{v}.\boldsymbol{v} = 5$$

  $$|\boldsymbol{u}.\boldsymbol{v}| = |-1| = 1, \quad \|\boldsymbol{u}\|\|\boldsymbol{v}\| = \sqrt{\boldsymbol{u}.\boldsymbol{u}}\sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{11}\sqrt{5} = \sqrt{55}, \quad |\boldsymbol{u}.\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- **The angle between two vectors in R^n:**

  $$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|},\ 0 \leq \theta \leq \pi$$



- Note: The angle between the zero vector and another vector is not defined.
- Example 12: Finding the angle between $\boldsymbol{u} = (-4, 0, 2, -2)$, $\boldsymbol{v} = (2, 0, -1, 1)$

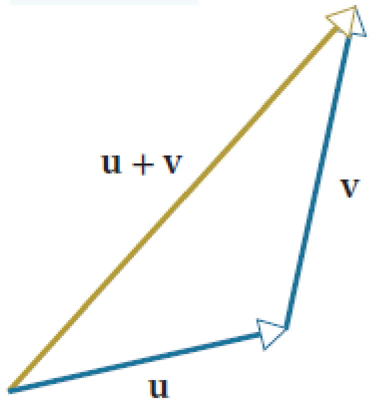
$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}.\boldsymbol{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{(2)^2 + 0^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$u.v = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{144}} = -1 \Rightarrow \theta = \pi$$

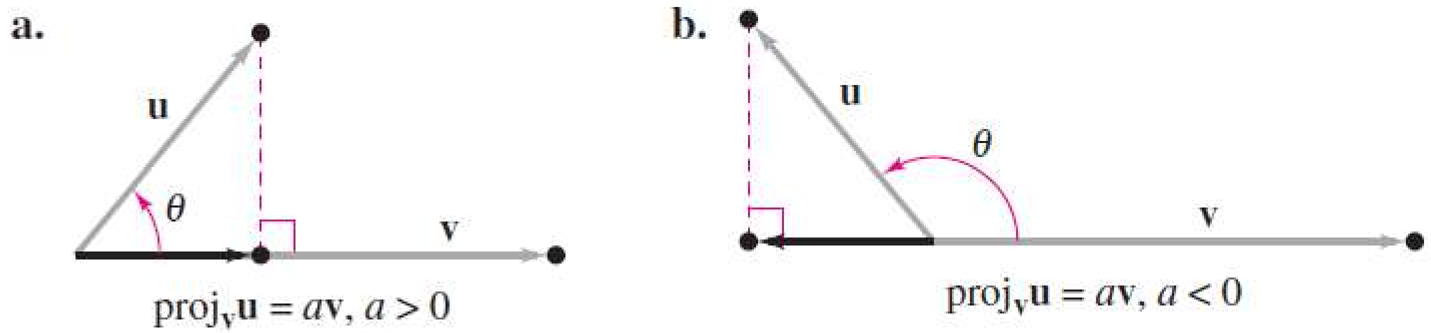
- Note: u and v have opposite directions ($u = -2v$).
- Orthogonal vectors: Tow vectors u and v in R^n are orthogonal if $u.v = \mathbf{0}$.
- Note: The vector **0** is said to be orthogonal to every vector.
- Theorem 7: (The Triangle inequality)
  If u and v are vectors in R^n, then $\|u + v\| \leq \|u\| + \|v\|$



- Note: Equality occurs in the triangle inequality if and only if the vectors u and v have the same direction.

Orthogonal projections

- Let u and v be two vectors in R^n, such that $v \neq \mathbf{0}$. Then the orthogonal projection of u onto v is given by $\text{proj}_v u = \dfrac{u \cdot v}{v \cdot v} v = av$



- Note: If v is a unit vector, then $v \cdot v = \|v\|^2 = 1$. The formula for the orthogonal projection of u onto v takes the following simpler form:

$$\text{proj}_v u = (u \cdot v)v$$

- Example 13: (Finding an orthogonal projection in R^3)

  Find the orthogonal projection of $\boldsymbol{u} = (6, 2, 4)$ onto $\boldsymbol{v} = (1, 2, 0)$.

  $$\boldsymbol{u}.\boldsymbol{v} = (6)(1) + (2)(2) + (4)(0) = 10 \qquad \boldsymbol{v}.\boldsymbol{v} = 1^2 + 2^2 + 0^2 = 5$$

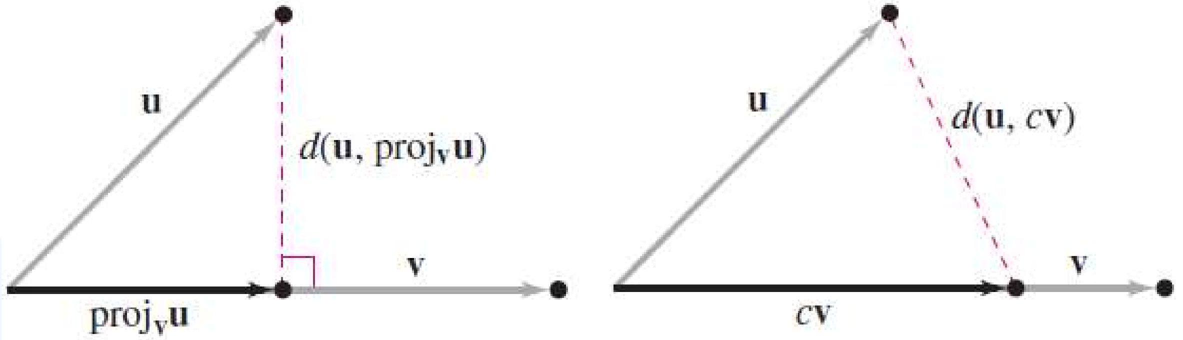
  $$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}\boldsymbol{v} = \frac{10}{5}(1, 2, 0) = (2, 4, 0)$$

- Note: $\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = (6,2,4) - (2,4,0) = (4,-2,4)$ is orthogonal to $\boldsymbol{v} = (1,2,0)$

- Theorem 8: (Orthogonal Projection and Distance)

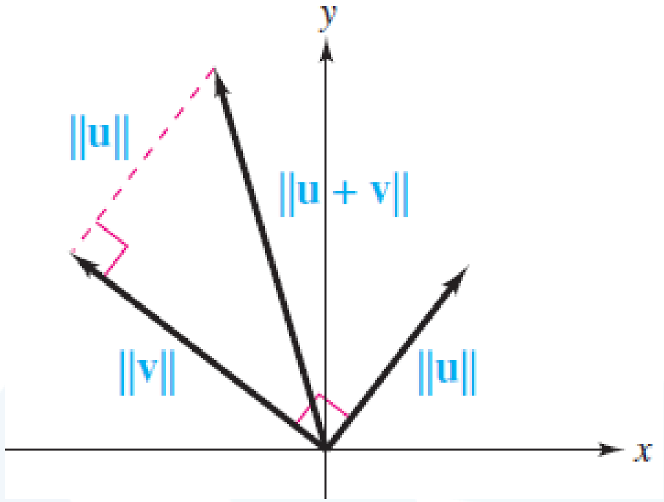
  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, such that $\boldsymbol{v} \neq \mathbf{0}$. Then

  $$d(\boldsymbol{u}, \text{proj}_{\boldsymbol{v}}\boldsymbol{u}) < d(\boldsymbol{u}, c\boldsymbol{v}), \qquad c \neq \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}$$



- Theorem 9: (The Pythagorean theorem)

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if: $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$



- Dot product and matrix multiplication:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ in R^n is represented as an $n \times 1$ column matrix)

$$\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 + u_2v_2 + \cdots + u_nv_n \end{bmatrix}$$

3. Basis, Spanning Sets and Linear Independence

- Definition: Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a non empty set of vectors in R^n and let the vector equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_k\boldsymbol{v}_k = \mathbf{0}$.

  (1) If the equation has only the trivial solution $(c_1 = c_2 = \ldots\ c_k = 0)$, then S is called linearly independent (LI).

  (2) If the equation has a non trivial solution (i.e. not all zeros), then S is called linearly dependent (LD).

- Notes:

  (1) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent. (2) $v \neq \mathbf{0} \Rightarrow \{v\}$ is linearly independent.

  (3) $S_1 \subseteq S_2$ if S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent.

  if S_2 is linearly independent $\Rightarrow S_1$ is linearly independent.

- Example 14: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is LI or LD

$$S=\{v_1=(1,2,3),\ v_2=(0,1,2),\ v_3=(-2,0,1)\}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0} \Rightarrow \begin{array}{rcrcrl} c_1 & & & - & 2c_3 & = 0 \\ 2c_1 & + & c_2 & & & = 0 \\ 3c_1 & + & 2c_2 & + & c_3 & = 0 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-J. Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow c_1 = c_2 = c_3 = 0 \\ \Rightarrow S \text{ is LI} \end{array}$$

- Independence of two vectors: Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in R^n are linearly dependent if and only if one is a scalar multiple of the other.

(1) $S = \{v_1, v_2\} = \{(1, 2, 0), (-2, 2, 1)\}$ is LI because v_1 and v_2 are not scalar multiples of each other.

(2) $S = \{v_1, v_2\} = \{(4, -4, -2), (-2, 2, 1)\}$ is LD because $v_1 = -2v_2$

- Theorem 10: (dependence in R^n)

  Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of different vectors in R^n. If $n < k$, then the set S is linearly dependent.

- Note: Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of k vectors that are LI in R^n, then $k \leq n$.

- Theorem 11: (Independence in R^n)

  Let $S = \{v_1, v_2, \ldots, v_n\}$ be n vectors in R^n. Let A be the $n\text{x}n$ matrix whose columns are given by $v_1, v_2, \ldots, v_n$. Then vectors $v_1, v_2, \ldots, v_n$ are linearly independent $\Leftrightarrow$ matrix A is invertible.

Spanning sets

- Definition: Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of k vectors in R^n. The set S is a spanning set of R^n if every vector in R^n can be written as a linear combination of vectors in S. In such cases it is said that S spans or generates the n-space R^n.

- Example 15: (A spanning set for R^3)

  The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in R^3 can be written as:

$$\boldsymbol{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3)$$

- Note: Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of k vectors in R^n that spans R^n, then $k \geq n$.

- Example 16: (A spanning set for R^3)

Show that the set $S_1 = \{\boldsymbol{v}_1 = (1, 2, 3),\ \boldsymbol{v}_2 = (0, 1, 2),\ \boldsymbol{v}_3 = (-2, 0, 1)\}$ spans R^3

We must determine whether an arbitrary vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in R^3 can be as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

$$\boldsymbol{u} \in R^3 \Rightarrow \boldsymbol{u} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 \Rightarrow \begin{array}{rcrcrcl} c_1 & & & - & 2c_3 & = & u_1 \\ 2c_1 & + & c_2 & & & = & u_2 \\ 3c_1 & + & 2c_2 & + & c_3 & = & u_3 \end{array}$$

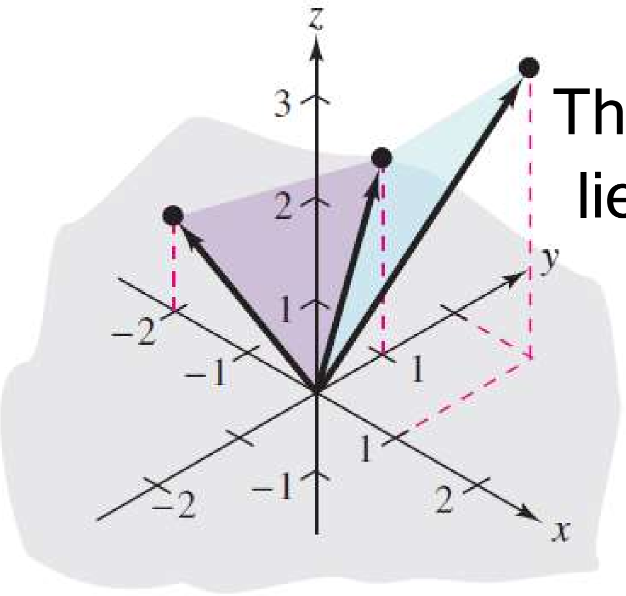
$$|A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{u}$

$\Rightarrow$ spans$(S_1) = R^3$

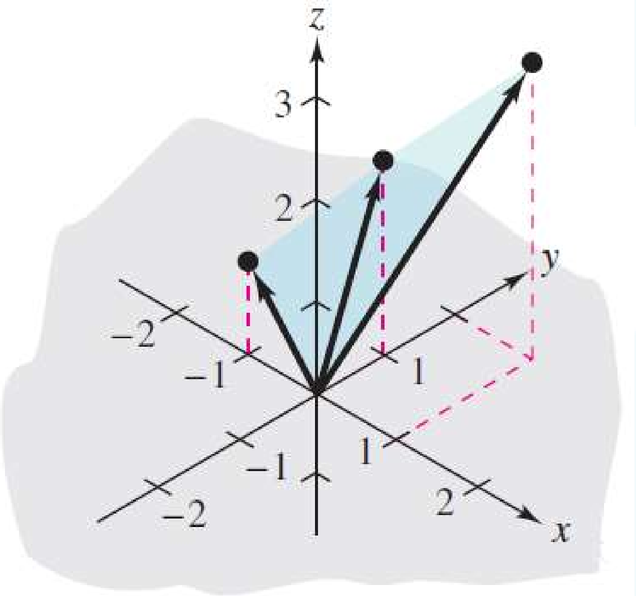
- Example 17: (A Set Does Not Span R^3)

From Example 4: the set $S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$ does not span R^3 because w = (1, –2, 2) is in R^3 and cannot be expressed as a linear combination of the vectors in S_2.



The vectors in S_1 do not lie in a common plane

The vectors in S_2 lie in a common plane



$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ $S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$

Basis

- **Definition:** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a set of n vectors in R^n. The set S form a **basis** for $R^n \Leftrightarrow$

  (i) $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ span R^n and

  (ii) $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent.

- The standard basis for R^3: $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

- A nonstandard Basis for R^3: $S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$.

- **Notes:**

  (1) Any n linearly independent vectors in R^n form a basis for R^n.

  (2) Any n vectors which span R^n form a basis for R^n.

  (3) Every basis of R^n contains exactly n vectors.

- **Theorem 12: (Uniqueness of basis representation)**

  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a basis for R^n, then every vector in R^n can be written in one and only one way as a linear combination of vectors in S.

- **Example 18: (Basis for R^3)**

  Show that the set $S = \{\boldsymbol{v}_1 = (1, 2, 1), \boldsymbol{v}_2 = (2, 9, 0), \boldsymbol{v}_3 = (3, 3, 4)\}$ form a basis for R^3.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$$

$A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{u} \Rightarrow$ spans$(S) = R^3$.

$A\boldsymbol{x} = \boldsymbol{0}$ has exactly one (trivial) solution $\Rightarrow S$ is linearly independent.

$\Rightarrow S$ form a basis for R^3.