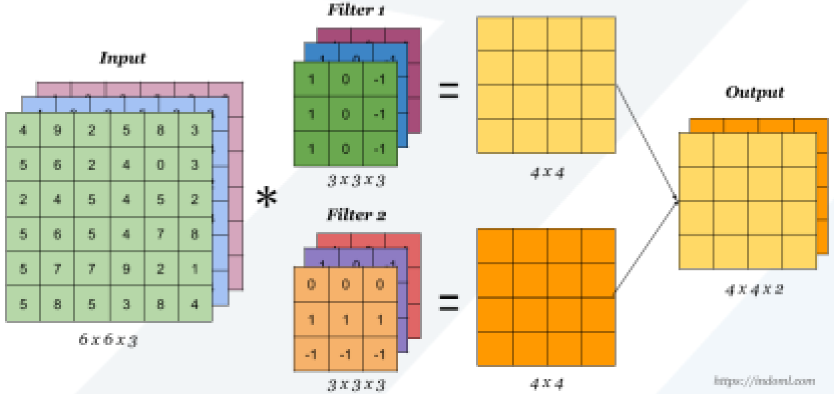
# CECC122: Linear Algebra and Matrix Theory

## Lecture Notes 6: Inner Product Spaces





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# Chapter 5

# Inner Product Spaces

1. General Inner Product
2. Orthonormal Bases: Gram-Schmidt Process
3. Mathematical Models and Least Square Analysis

## 1. General Inner Product

- **Definition:** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a real vector space $V$, and let $c$ be any scalar. An inner product on $V$ is a function that associates a real number $<\mathbf{u}, \mathbf{v}>$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms:

  (1) $<\mathbf{u}, \mathbf{v}> = <\mathbf{v}, \mathbf{u}>$

  (2) $<\mathbf{u}, \mathbf{v} + \mathbf{w}> = <\mathbf{u}, \mathbf{v}> + <\mathbf{u}, \mathbf{w}>$

  (3) $c <\mathbf{u}, \mathbf{v}> = <c\mathbf{u}, \mathbf{v}>$

  (4) $<\mathbf{v}, \mathbf{v}> \geq 0$ and $<\mathbf{v}, \mathbf{v}> = 0$ if and only if $\mathbf{v} = \mathbf{0}$

- **Notes:**

  (1) $\mathbf{u}.\mathbf{v}$ = dot product (Euclidean inner product for $R^n$)

  (2) $<\mathbf{u}, \mathbf{v}>$ = general inner product for vector space $V$

- Note: A vector space $V$ with an inner product is called an inner product space.
  Vector space: $(V, +, .)$
  Inner product space: $(V, +, ., <, >)$

- Example 1: (Euclidean inner product for $R^n$)
  The dot product in $R^n$ satisfies the four axioms of an inner product.
  $$\boldsymbol{u} = (u_1, u_2, \ldots, u_n),\ \boldsymbol{v} = (v_1, v_2, \ldots, v_n)$$
  $$< \boldsymbol{u}, \boldsymbol{v} > = \boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Example 2: (A different inner product for $R^n$)
  Show that the function defines an inner product on $R^2$, where $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$: $<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2$.
  (1) $<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2 = v_1 u_1 + 2 v_2 u_2 = <\boldsymbol{v}, \boldsymbol{u}>$

(2) $\boldsymbol{w} = (w_1, w_2)$

$$< \boldsymbol{u},\ \boldsymbol{v} + \boldsymbol{w} > = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$$

$$= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2$$

$$= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)$$

$$= < \boldsymbol{u},\ \boldsymbol{v} > + < \boldsymbol{u},\ \boldsymbol{w} >$$

(3) $c <\boldsymbol{u},\ \boldsymbol{v}> = c\ (u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = <c\boldsymbol{u},\ \boldsymbol{v}>$

(4) $< \boldsymbol{v},\ \boldsymbol{v} > = {v_1}^2 + 2{v_2}^2 \geq 0$

$$< \boldsymbol{v},\ \boldsymbol{v} > = 0 \Rightarrow {v_1}^2 + 2{v_2}^2 = 0 \Rightarrow v_1 = v_2 = 0 \quad (\boldsymbol{v} = \boldsymbol{0})$$

- Note: (An inner product on $R^n$)

  $< \boldsymbol{u},\ \boldsymbol{v} > = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n, \quad c_i > 0$ (weights)

- Example 3: (A function that is not an inner product)

  Show that the following function is not an inner product on $R^3$

$$< \boldsymbol{u}, \boldsymbol{v} > = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

  Let $\boldsymbol{v} = (1, 2, 1)$, then $<\boldsymbol{v}, \boldsymbol{v}> = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

  Axiom 4 is not satisfied. Thus this function is not an inner product on $R^3$

- Theorem 1: (Properties of inner products)

  Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in an inner product space $V$, and let $c$ be any real number.

  (1) $<\mathbf{0}, \boldsymbol{v}> = <\boldsymbol{v}, \mathbf{0}> = 0$

  (2) $<\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}> = <\boldsymbol{u}, \boldsymbol{w}> + <\boldsymbol{v}, \boldsymbol{w}>$

  (3) $<\boldsymbol{u}, c\boldsymbol{v}> = c<\boldsymbol{u}, \boldsymbol{v}>$

- Example 4: (The Standard Inner Product on $M_n(R)$)

$$A, B \in M_n(R), \quad \langle A, B\rangle = \operatorname{tr}(AB^T)$$

for the 2×2 matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$\langle A, B\rangle = \operatorname{tr}(AB^T) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\langle A, B\rangle = \operatorname{tr}(AB^T) = 1(-1) + 2(0) + 3(3) + 4(2) = 16$$

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}, \quad \|B\| = \sqrt{\langle B, B\rangle} = \sqrt{14}$$

- Example 5: (The Standard Inner Product on $P_n$)

$p, q \in P_n \qquad p = a_0 + a_1 x + \cdots + a_n x^n \qquad q = b_0 + b_1 x + \cdots + b_n x^n$

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n \qquad \|p\| = \sqrt{\langle p, p \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$

- Norm (length) of $\boldsymbol{u}$: $\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}$
- Note: $\|\boldsymbol{u}\|^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle$
- Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$: $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle}$
- Angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$: $\cos\theta = \dfrac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}, \ 0 \le \theta \le \pi$
- Orthogonal: $(\boldsymbol{u} \perp \boldsymbol{v})$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $<\boldsymbol{u}, \boldsymbol{v}> = 0$

- Notes:

(1) If $\|\boldsymbol{v}\| = 1$, then $\boldsymbol{v}$ is called a unit vector

(2) $\|\boldsymbol{v}\| \neq 1$, $\boldsymbol{v} \neq \mathbf{0}$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ (the unit vector in the direction of $\boldsymbol{v}$)

- Properties of norm:

(1) $\|\boldsymbol{u}\| \geq 0$ (2) $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$ (3) $\|c\boldsymbol{u}\| = |c|\,\|\boldsymbol{u}\|$

- (Properties of distance)

(1) $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$ (2) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{u} = \boldsymbol{v}$ (3) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$

- Note: Norm, Distance and Orthogonality depend on the inner product being used.

- Example 6: $\boldsymbol{u}=(1, 0)$ and $\boldsymbol{v}=(0, 1)$ in $R^2$

  Euclidean inner product:

$$\|\boldsymbol{u}\|=\sqrt{1^2+0^2}=1, \quad \|\boldsymbol{v}\|=\sqrt{0^2+1^2}=1, \quad d(\boldsymbol{u},\boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{1^2+(-1)^2}=\sqrt{2}$$

  Weighted Euclidean inner product: $\langle \boldsymbol{u},\boldsymbol{v}\rangle = 3u_1v_1+2u_2v_2$

$$\|\boldsymbol{u}\|=\sqrt{3(1)^2+2(0)^2}=\sqrt{3}, \quad \|\boldsymbol{v}\|=\sqrt{3(0)^2+2(1)^2}=\sqrt{2}$$

$$d(\boldsymbol{u},\boldsymbol{v})=\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{3(1)^2+2(-1)^2}=\sqrt{5}$$

- Example 7: $\boldsymbol{u}=(1, 1)$ and $\boldsymbol{v}=(1, -1)$ in $R^2$

  Euclidean inner product: $\boldsymbol{u}.\boldsymbol{v}=1(1)+(-1)(1)=0 \quad \Rightarrow \quad \boldsymbol{u}\perp\boldsymbol{v}$

  Weighted Euclidean inner product: $\langle \boldsymbol{u},\boldsymbol{v}\rangle = 3u_1v_1+2u_2v_2$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle = 3u_1v_1+2u_2v_2 = 3(1)(1)+2(-1)(1)=1\neq 0$$

- Theorem 2: Let $u$ and $v$ be vectors in an inner product space $V$.

  (1) Cauchy-Schwarz inequality: $|\langle u, v\rangle| \le \|u\|\|v\|$

  (2) Triangle inequality: $\|u+v\| \le \|u\| + \|v\|$

  (3) Pythagorean theorem: $u$ and $v$ are orthogonal iff $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

## Orthogonal Complements

- Definition: If $W$ is a subspace of a real inner product $V$, then the set of all vectors in $V$ that are orthogonal to every vector in $W$ is called the orthogonal complement of $W$ and is denoted by the symbol $W^{\perp}$.

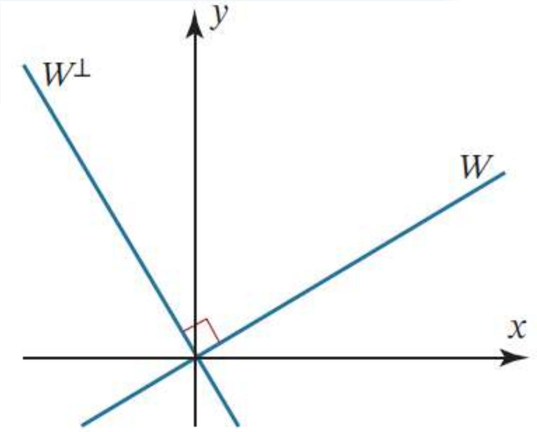
- Theorem 3: (Properties of Orthogonal Complements)

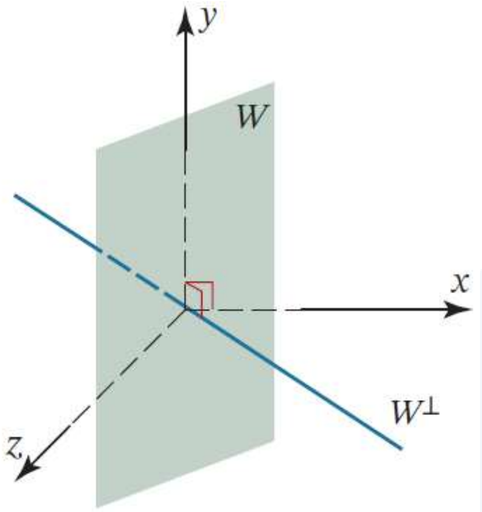
  If $W$ is a subspace of a real inner product $V$, then:

  (a) $W^{\perp}$ is a subspace of $V$ (b) $W^{\perp} \cap W = \{\mathbf{0}\}$

- Example 8: (Orthogonal Complements)

in $R^2$




in $R^3$

## 2. Orthonormal Bases: Gram-Schmidt Process

Definition: A set $S$ of vectors in an inner product space $V$ is called an orthogonal set if every pair of vectors in the set is orthogonal.

$$S = \{v_1, v_2, \cdots, v_n\} \subseteq V \Rightarrow \langle v_i, v_j \rangle = 0, \quad i \neq j$$

An orthogonal set in which each vector is a unit vector is called orthonormal.

$$S=\{v_1, v_2, \cdots, v_n\} \subseteq V \Rightarrow \langle v_i, v_j\rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

- Note: If $S$ is a basis, then it is called an orthogonal/orthonormal basis.
- Example 9: (A nonstandard orthonormal basis for $R^3$)

  Show that the following set is an orthonormal basis.

$$S=\{(v_1=\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (v_2=-\tfrac{\sqrt{2}}{6}, \tfrac{\sqrt{2}}{6}, \tfrac{2\sqrt{2}}{3}), (v_3=\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3})\}$$

$$v_1 . v_2 = -\tfrac{1}{6}+\tfrac{1}{6}+0=0 \qquad \|v_1\| = \sqrt{v_1 . v_1} = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}+0}=1$$

$$v_1 . v_3 = \tfrac{2}{3\sqrt{2}}-\tfrac{2}{3\sqrt{2}}+0=0 \qquad \|v_2\| = \sqrt{v_2 . v_2} = \sqrt{\tfrac{2}{36}+\tfrac{2}{36}+\tfrac{8}{9}}=1$$

$$v_2 . v_3 = -\tfrac{\sqrt{2}}{9}-\tfrac{\sqrt{2}}{9}+\tfrac{2\sqrt{2}}{9}=0 \qquad \|v_3\| = \sqrt{v_3 . v_3} = \sqrt{\tfrac{4}{9}+\tfrac{4}{9}+\tfrac{1}{9}}=1$$

Thus $S$ is an orthonormal set

- Example 10: (An orthonormal basis for $P_3$)
  with the inner product $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$
  the standard basis $B = \{1, x, x^2, x^3\}$ is orthonormal
- Theorem 4: (Orthogonal sets are linearly independent)
  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space $V$, then $S$ is linearly independent.
- Theorem 5: (Coordinates relative to an orthonormal basis)
  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthogonal/orthonormal basis for an inner product space $V$, and if $\boldsymbol{u}$ is any vector in $V$, then

$$\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 + \cdots + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_n\rangle}{\|\boldsymbol{v}_n\|^2}\boldsymbol{v}_n$$

$S$ orthogonal

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \cdots + \langle u, v_n\rangle v_n$$ $S$ orthonormal

- Note: If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal/orthonormal basis for an inner product space $V$ and $w \in V$, then the corresponding coordinate matrix of $w$ relative to $B$ is

$$[u]_S = \left(\frac{\langle u, v_1\rangle}{\|v_1\|^2}, \frac{\langle u, v_2\rangle}{\|v_2\|^2}, \cdots, \frac{\langle u, v_n\rangle}{\|v_n\|^2}\right)^T$$

$S$ orthogonal

$$[w]_S = (\langle u, v_1\rangle, \langle u, v_2\rangle, \cdots, \langle u, v_n\rangle)^T$$

$S$ orthonormal

- Example 11: (Representing vectors relative to an orthonormal basis)

Find the coordinates of vector $w = (5, -5, 2)$ relative to the following orthonormal basis for $R^3$ $S = \{(\frac{3}{5}, \frac{4}{5}, 0), (-\frac{4}{5}, \frac{3}{5}, 0), (0, 0, 1)\}$

$$< w, v_1 > = w.v_1 = (5, -5, 2) \,.\, (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$

$$< w, v_2 > = w.v_2 = (5, -5, 2) \,.\, (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7 \Rightarrow [w]_S = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

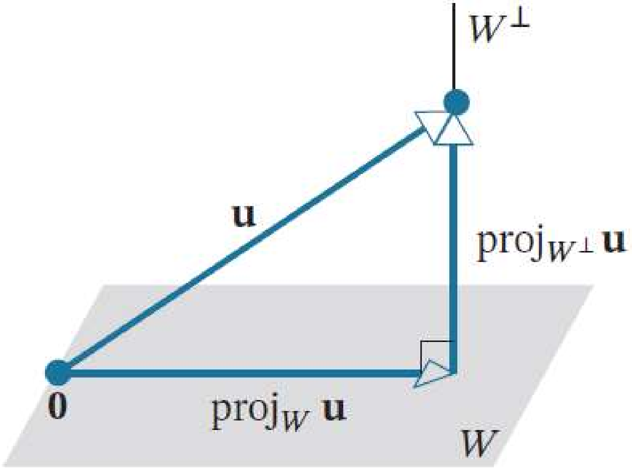
$$< w, v_3 > = w.v_3 = (5, -5, 2) \,.\, (0, 0, 1) = 2$$

## Orthogonal Projections

- **Theorem 6: (Projection Theorem)**

  If $W$ is a finite-dimensional subspace of an inner product space $V$, then every vector $u$ in $V$ can be expressed in exactly one way as $u = w_1 + w_2$, where $w_1$ is in $W$ and $w_2$ is in $W^{\perp}$.

$$u = \text{proj}_W u + \text{proj}_{W^{\perp}} u = \text{proj}_W u + (u - \text{proj}_W u)$$



- **Theorem 7: (formulas for calculating orthogonal projection)**

  Let $W$ be a finite-dimensional subspace of an inner product space $V$. If $S = \{v_1, v_2, \ldots, v_r\}$ is an orthogonal/orthonormal basis for $W$, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \qquad S \text{ orthogonal}$$

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r \qquad S \text{ orthonormal}$$

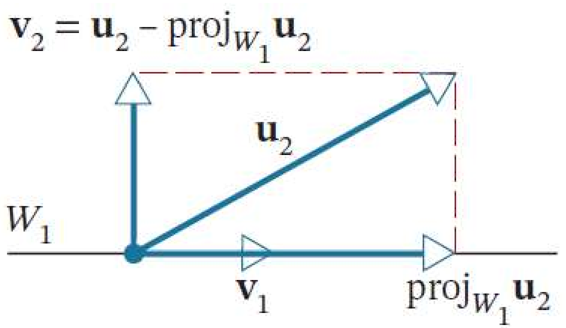
## The Gram-Schmidt Process

- **Theorem 8: (Projection Theorem)**

  Every nonzero finite-dimensional inner product space has an orthonormal basis.

**Proof** (Gram-Schmidt orthonormalization construction)

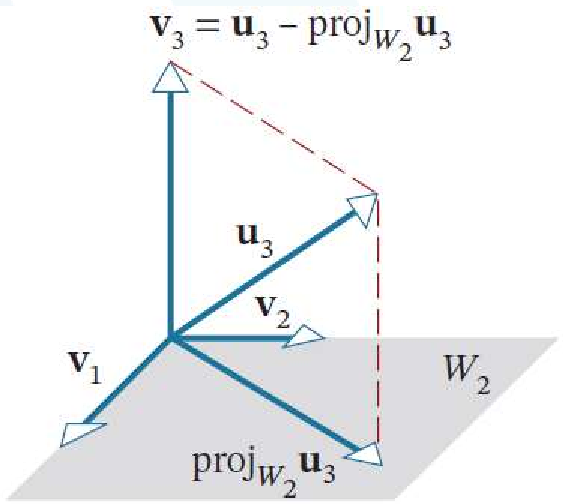
Let $W$ be any nonzero finite-dimensional subspace of an inner product space, and suppose that $\{u_1, u_2, \ldots, u_r\}$ is any basis for $W$.




**Step 1:** Let $v_1 = u_1$

**Step 2:** $v_2 = u_2 - \text{proj}_{W_1} u_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$W_1 = \text{span}(v_1)$ and $v_2 \perp v_1,\ v_2 \neq \mathbf{0}$




**Step 3:** $v_3 = u_3 - \text{proj}_{W_2} u_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \dfrac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

$W_2 = \text{span}(v_1, v_2)$ and $v_3 \perp W_2,\ v_3 \neq \mathbf{0}$

Continuing in this way we will produce after $r$ steps an orthogonal set of nonzero vectors $\{v_1, v_2, \ldots, v_r\}$.

By normalizing these basis vectors we can obtain an orthonormal basis.

- Theorem 9: (Gram-Schmidt orthonormalization process)

(1) Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for an inner product space $V$

(2) Let $B' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, where

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_n = \mathbf{u}_n - \sum_{i=1}^{n-1} \frac{\langle \mathbf{u}_n, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}\mathbf{v}_i$$

Then $B'$ is an orthogonal basis for $V$

(3) Let $\boldsymbol{w}_i = \dfrac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}$

Then $B'' = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ is an orthonormal basis for $V$

Also, span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\} =$ span$\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ for $k = 1, 2, ..., n$

- Example 12: (Applying the Gram-Schmidt orthonormalization process)

  Apply the Gram-Schmidt orthonormalization process to the basis $B$ for $R^2$

$$B = \{\boldsymbol{u}_1 = (1, 1), \boldsymbol{u}_2 = (0, 1)\}$$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1, 1)$$

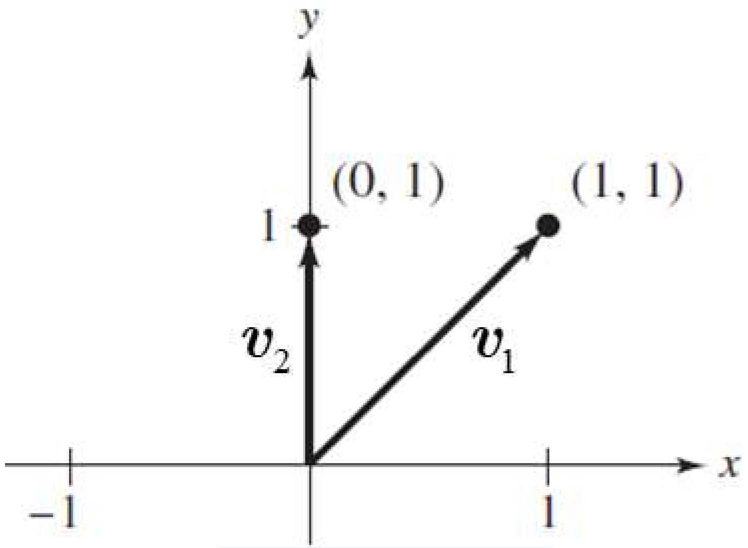
$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}\boldsymbol{v}_1 = (0, 1) - \frac{1}{2}(1, 1) = (-\frac{1}{2}, \frac{1}{2})$$

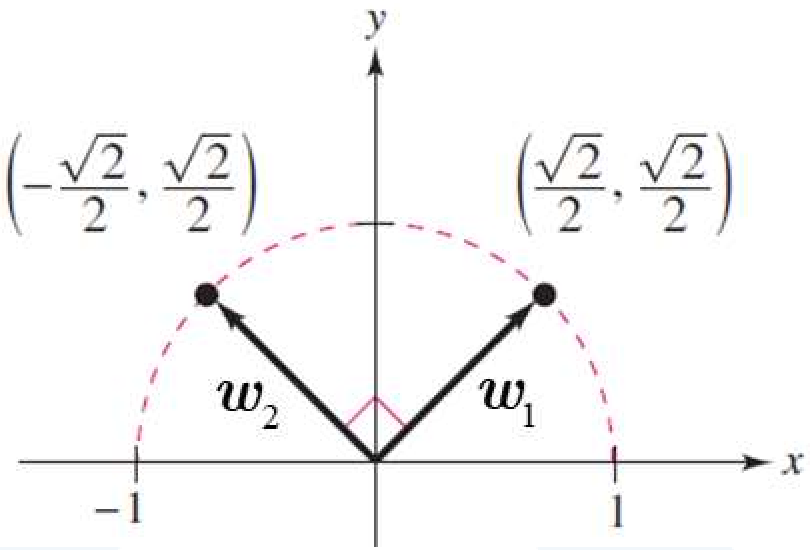
The set $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is an orthogonal basis for $R^2$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{1/\sqrt{2}}(-\frac{1}{2}, \frac{1}{2}) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

The set $B'' = \{w_1, w_2\}$ is an orthonormal basis for $R^2$



- Example 13: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis $B$ for $R^3$

$$B = \{v_1 = (1, 1, 0),\ v_2 = (1, 2, 0),\ v_3 = (0, 1, 2)\}$$

$$v_1 = u_1 = (1, 1, 0)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = (1, 2, 0) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0, 0, 2)$$

The set $B' = \{v_1, v_2, v_3\}$ is an orthogonal basis for $R^3$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1)$$

The set $B'' = \{w_1, w_2, w_3\}$ is an orthonormal basis for $R^3$

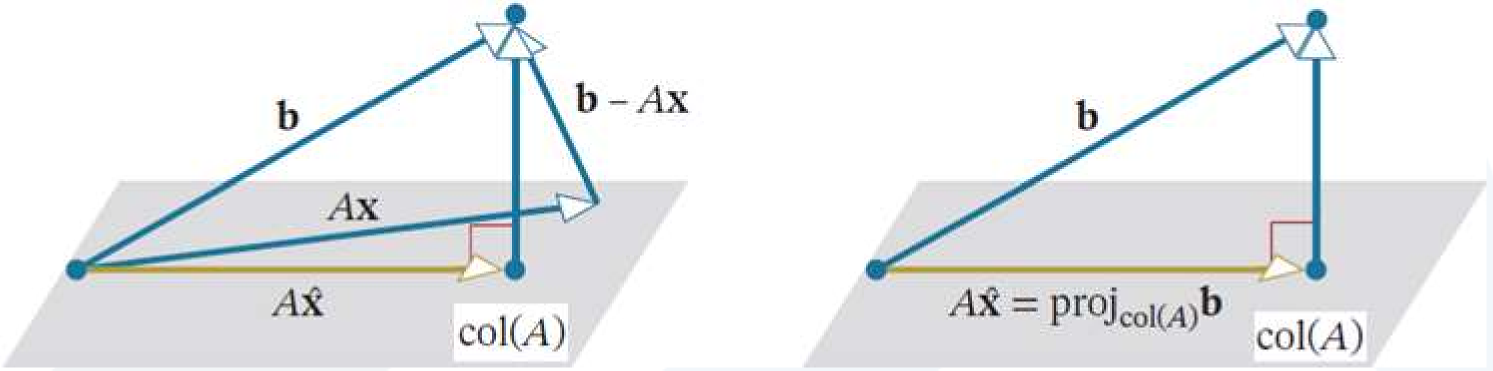
## 3. Mathematical Models and Least Square Analysis

### Best Approximation; Least Squares

- **Least Squares Problem:** Given $Ax = b$ of $m$ equations in $n$ unknowns, find $x$ in $R^n$ that minimizes $\|b - Ax\|$ with respect to the Euclidean inner product on $R^m$. We call $x$, if it exists, a least squares solution of $Ax = b$, $b - Ax$ the least squares error vector, and $\|b - Ax\|$ the least squares error.

$$\boldsymbol{b} - A\boldsymbol{x} = \begin{bmatrix} e_1 \\ e_1 \\ \vdots \\ e_m \end{bmatrix} \Rightarrow \|\boldsymbol{b} - A\boldsymbol{x}\|^2 = e_1^2 + e_2^2 + \cdots + e_m^2$$

- Note: For every vector $\boldsymbol{x}$ in $R^n$, the product $A\boldsymbol{x}$ is in the column space of $A$ because it is a linear combination of the column vectors of $A$. Find a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is equivalent to find a vector $A\widehat{\boldsymbol{x}}$ in the col($A$) that is closest to $\boldsymbol{b}$ (it minimizes the length of the vector $\boldsymbol{b} - A\boldsymbol{x}$) $\Rightarrow A\widehat{\boldsymbol{x}} = \text{proj}_{\text{col}(A)}\boldsymbol{b}$.



- Theorem 10: (Best Approximation Theorem)

  If $W$ is a finite-dimensional subspace of an inner product space $V$, and if $\boldsymbol{b}$ is a vector in $V$, then $\text{proj}_W \boldsymbol{b}$ is the best approximation to $\boldsymbol{b}$ from $W$ in the sense that $\|\boldsymbol{b} - \text{proj}_W \boldsymbol{b}\| < \|\boldsymbol{b} - \boldsymbol{w}\|$ for every vector $\boldsymbol{w}$ in $W$ that is different from $\text{proj}_W \boldsymbol{b}$.

- If $V = R^n$ and $W = \text{col}(A)$, then the best approximation to $\boldsymbol{b}$ from $\text{col}(A)$ is $\text{proj}_{\text{col}(A)} \boldsymbol{b}$.

- Finding Least Squares Solutions: $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$

  This is called the normal equations associated with $A\boldsymbol{x} = \boldsymbol{b}$.

- Example 14: (Finding Least Squares Solutions)

  Find the Least Squares Solution, the least squares error vector, and the least squares error of the linear system:

$$\begin{array}{rcrcr} x & - & y & = & 4 \\ 3x & + & 2y & = & 1 \\ -2x & + & 4y & = & 3 \end{array}$$

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \boldsymbol{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$A^T A\boldsymbol{x} = A^T \boldsymbol{b} \Rightarrow \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17/95 \\ 143/285 \end{bmatrix}$$

$$\boldsymbol{b} - A\boldsymbol{x} = \begin{bmatrix} 1232/285 \\ -154/285 \\ 77/57 \end{bmatrix}, \quad \text{and} \quad \left\| \boldsymbol{b} - A\boldsymbol{x} \right\| \approx 4.556$$

- Theorem 11: (a unique least squares solution)

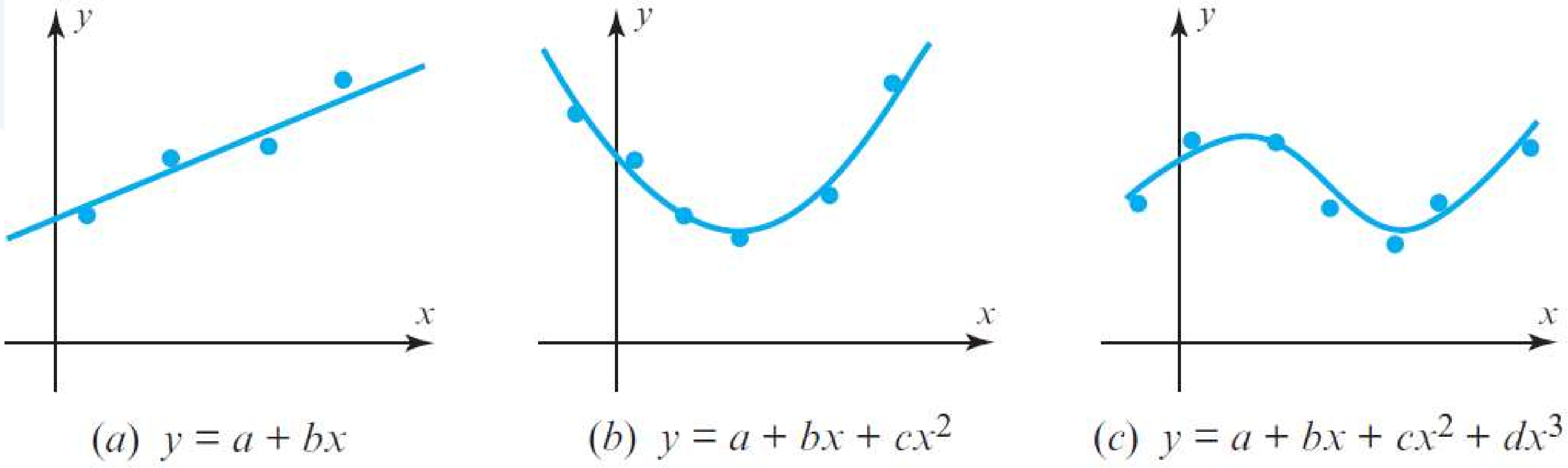
  If $A$ is an $m\text{x}n$ matrix with linearly independent column vectors, then for every $m\text{x}1$ matrix $\boldsymbol{b}$, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique least squares solution. This solution is given by: $\boldsymbol{x} = (A^TA)^{-1}A^T\boldsymbol{b}$.

  Moreover, if $W$ is the column space of $A$, then the orthogonal projection of $\boldsymbol{b}$ on $W$ is: $\text{proj}_W\,\boldsymbol{b} = A\boldsymbol{x} = A(A^TA)^{-1}A^T\boldsymbol{b}$.

## Mathematical Modeling Using Least Squares

- Fitting a Curve to Data

  A common problem in experimental work is to find a mathematical relationship $y = f(x)$ between two variables $x$ and $y$ by "fitting" a curve to points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.




mathematical model

## Least Squares Fit of a Straight Line $y = a + bx$

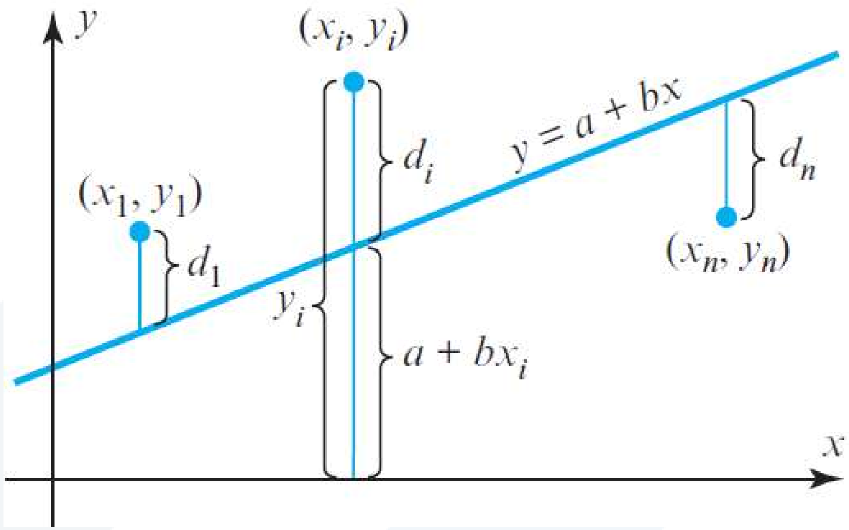
$$\begin{aligned} y_1 &= a + bx_1 \\ y_2 &= a + bx_2 \\ &\vdots \\ y_n &= a + bx_n \end{aligned} \quad \Rightarrow \quad M\boldsymbol{v} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

$$M\boldsymbol{v} = \boldsymbol{y} \quad \Rightarrow \quad M^T M\boldsymbol{v} = M^T \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{v}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (M^T M)^{-1} M^T \boldsymbol{y}$$

$y = a^* + b^* x$ Least squares line of best fit or the regression line

It minimizes $\|\boldsymbol{y} - M\boldsymbol{v}\|^2 = [y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + \cdots + [y_n - (a + bx_n)]^2$

$d_1 = |y_1 - (a + bx_1)|, d_2 = |y_2 - (a + bx_2)|, \cdots, d_n = |y_n - (a + bx_n)|$ residuals

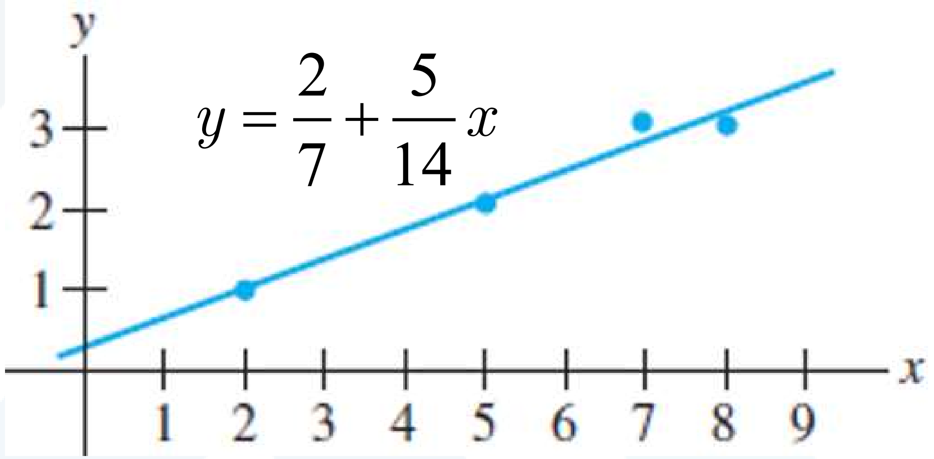


- Example 15: (Least Squares Straight Line Fit)

Find the least squares straight line fit to the points (2, 1), (5, 2), (7, 3), and (8, 3)

$$M^T M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

$$M^T \boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$




$$\boldsymbol{v}^* = (M^T M)^{-1} M^T \boldsymbol{y} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{5}{14} \end{bmatrix}$$

# Least Squares Fit of a Polynomial $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$\begin{aligned} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_m x_1^m &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_m x_2^m &= y_2 \\ &\vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_m x_n^m &= y_n \end{aligned}$$

$$M\boldsymbol{v} = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^m \\ 1 & x_2 & x_2^2 & \ldots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

$$M\boldsymbol{v} = \boldsymbol{y} \quad \Rightarrow \quad M^T M\boldsymbol{v} = M^T \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{v}^* = (M^T M)^{-1} M^T \boldsymbol{y}$$

## ▪ Example 16: (Fitting a Quadratic Curve to Data)

Newton's second law of motion $s = s_0 + v_0 t + \frac{1}{2} g t^2$

Laboratory experiment

| Time $t$ (sec) | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|
| Displacement $s$ (ft) | $-0.18$ | 0.31 | 1.03 | 2.48 | 3.73 |

Approximate $g$

Let $s = a_0 + a_1 t + a_2 t^2$

(0.1,–0.18), (0.2, 0.31), (0.3, 1.03), (0.4, 2.48), (0.5, 3.73)

$$M = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & 0.01 \\ 1 & 0.2 & 0.04 \\ 1 & 0.3 & 0.09 \\ 1 & 0.4 & 0.16 \\ 1 & 0.5 & 0.25 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} -0.18 \\ 0.31 \\ 1.03 \\ 2.48 \\ 3.73 \end{bmatrix}$$

$$\boldsymbol{v}^* = \begin{pmatrix} a_0^* \\ a_1^* \\ a_2^* \end{pmatrix} = (M^T M)^{-1} M^T \boldsymbol{y} = \begin{pmatrix} -0.4 \\ 0.35 \\ 16.1 \end{pmatrix}$$

$$g = 2a_2^* = 2(16.1) = 32.2 \text{ feet}/s^2$$

$$s_0 = a_0^* = -0.4 \text{ feet} \qquad v_0 = a_1^* = 0.35 \text{ feet/s}$$