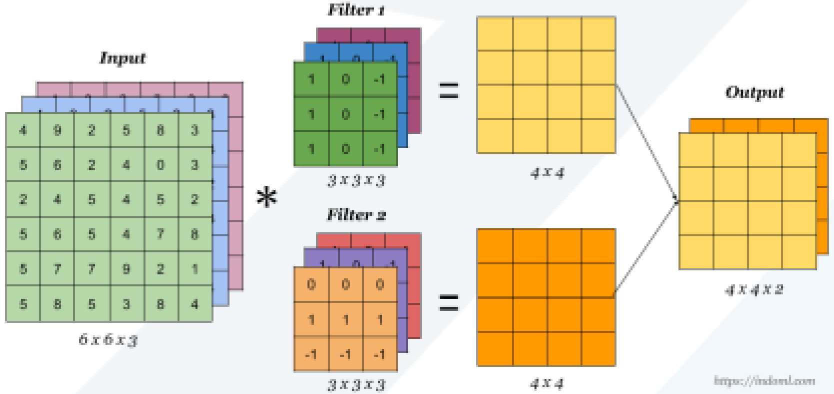
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 7: Linear Transformations



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Chapter 6

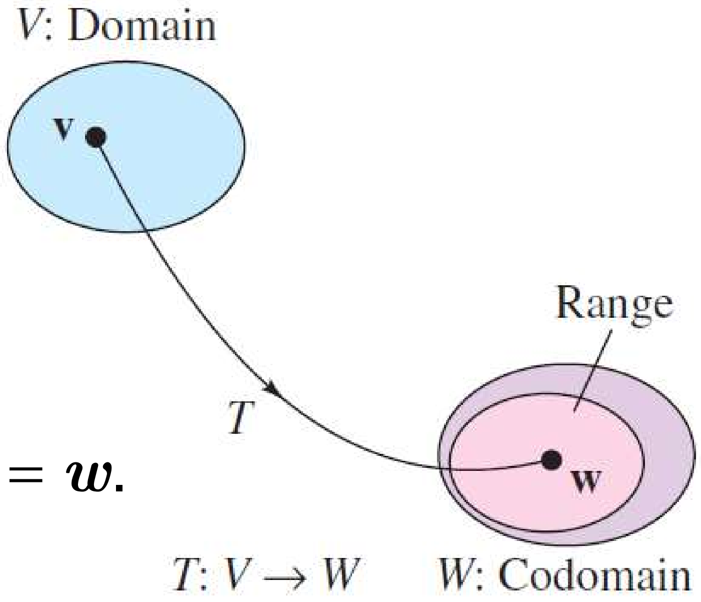
Linear Transformations

1. Introduction to Linear Transformations

Images And Preimages of Functions:

- Function T that maps a vector space V into a vector space W.

$$T: V \xrightarrow{\text{Mapping}} W, \qquad V,\ W: \text{vector spaces}$$



- If v is in V and w is in W such that: $T(v) = w$,
  Then w is called the image of v under T.
- The range of T: The set of all images of vectors in V.
- The preimage of w: The set of all v in V such that $T(v) = w$.
- Example 1: (A function from R^2 into R^2)

$$T: R^2 \rightarrow R^2,\ v = (v_1, v_2) \in R^2 \qquad T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\boldsymbol{v} = (-1, 2)$ (b) Find the preimage of $\boldsymbol{w} = (-1, 11)$

$$(a)\ \boldsymbol{v} = (-1, 2) \Rightarrow T(\boldsymbol{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3,\ 3)$$

$$(b)\ T(\boldsymbol{v}) = \boldsymbol{w} = (-1, 11) \Rightarrow T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1,\ 11)$$

$$\Rightarrow \begin{cases} v_1 - v_2 = -1 \\ v_1 + 2v_2 = 11 \end{cases} \Rightarrow v_1 = 3,\ v_2 = 4$$

Thus {(3, 4)} is the preimage of $\boldsymbol{w} = (-1, 11)$

- Definition: If $T: V \rightarrow W$ is a mapping from a vector space V to a vector space W, then T is called a linear transformation (LT) from V to W if the following two properties hold for all vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V and for all scalars c:

(1) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ [Additivity property]

(2) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$ [Homogeneity property]

When $V = W$, the linear transformation T is called a linear operator on V.

- Example 2: (Verifying a linear transformation T from R^2 into R^2)

$T: R^2 \to R^2, \; \boldsymbol{v} = (v_1, v_2) \in R^2 \qquad T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

$\boldsymbol{u} = (u_1, u_2), \; \boldsymbol{v} = (v_1, v_2)$ vectors in R^2, c: any real

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(u_1 + v_1, u_2 + v_2)$$
$$= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2))$$
$$= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2))$$
$$= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) = T(\boldsymbol{u}) + T(\boldsymbol{v})$$

$$T(c\boldsymbol{u}) = T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) = c(u_1 - u_2, u_1 + 2u_2) = cT(\boldsymbol{u})$$

Therefore, T is a linear transformation.

- Example 3: (A Linear Transformation from P_n to P_{n-1}, $n \geq 1$)

$T: P_n \to P_{n-1}: \; T(\boldsymbol{p}) = T(p(x)) = p'(x)$ derivative

- Example 4: (A Linear Transformation from P_n to P_{n+1})

$$\boldsymbol{p} = p(x) = c_0 + c_1x + \ldots + c_nx^n \in P_n$$

$$T: P_n \to P_{n+1}:\ T(\boldsymbol{p}) = T(p(x)) = xp(x) = c_0x + c_1x^2 + \ldots + c_nx^{n+1}$$

- Example 5: (Functions that are not linear transformations)

$$(a)\ f(x) = \sin x \qquad \sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$(b)\ f(x) = x^2 \qquad (x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(c)\ f(x) = x + 1$$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

$$(d)\ T(\boldsymbol{v}) = \|\boldsymbol{v}\| \qquad \|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\| \Rightarrow T(\boldsymbol{u} + \boldsymbol{v}) \neq T(\boldsymbol{u}) + T(\boldsymbol{v})$$

- Zero transformation: $T: V \to W \quad T(\boldsymbol{v}) = \boldsymbol{0},\ \forall \boldsymbol{v} \in V$
- Identity transformation: $T: V \to V \quad T(\boldsymbol{v}) = \boldsymbol{v},\ \forall \boldsymbol{v} \in V$
- Theorem 1: (Properties of linear transformations)

$T: V \to W, \quad \boldsymbol{u}, \boldsymbol{v} \in V$

(1) $T(\boldsymbol{0}) = \boldsymbol{0}$ (2) $T(-\boldsymbol{v}) = -T(\boldsymbol{v})$ (3) $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$

(4) If $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ then

$$T(\boldsymbol{v}) = T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_nT(\boldsymbol{v}_n)$$

- Example 6: (Functions that are not linear transformations)

Let $T: R^3 \to R^3$ be a linear transformation such that

$$T(1, 0, 0) = (2, -1, 4), \quad T(0, 1, 0) = (1, 5, -2), \quad T(0, 0, 1) = (0, 3, 1)$$

Find $T(2, 3, -2)$

$$(2, 3, -2) = 2(1, 0, 0) + 3(0, 1, 0) - 2(0, 0, 1)$$

$$T(2, 3, -2) = 2T(1, 0, 0) + 3T(0, 1, 0) - 2T(0, 0, 1)$$

$$= 2(2, -1, 4) + 3(1, 5, -2) - 2T(0, 3, 1) = (7, 7, 0)$$

- Example 7: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation from R^2 into R^3

R^2 vector R^3 vector

(a) $\mathbf{v} = (2, -1)$ $\quad T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \Rightarrow T(2,-1) = (6, 3, 0)$

(b) $T(\boldsymbol{u}+\boldsymbol{v}) = A(\boldsymbol{u}+\boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v} = T(\boldsymbol{u}) + T(\boldsymbol{v})$ (vector addition)

$T(c\boldsymbol{u}) = A(c\boldsymbol{u}) = c(A\boldsymbol{u}) = cT(\boldsymbol{u})$ (scalar multiplication)

- Theorem 2: (The linear transformation given by a matrix)
  Let A be an $m\text{x}n$ matrix. The function T defined by $T(\boldsymbol{v}) = A\boldsymbol{v}$ is a linear transformation from R^n into R^m.

- Example 8: (Rotation in the plane)
  Show that the LT $T: R^2 \to R^2$ given by the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

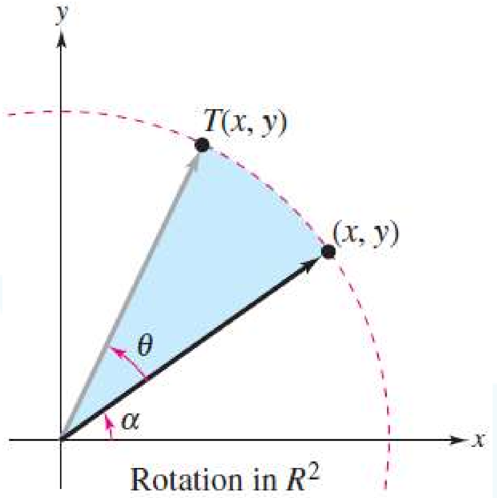
  has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

  $\boldsymbol{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

$$T(\boldsymbol{v}) = A\boldsymbol{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

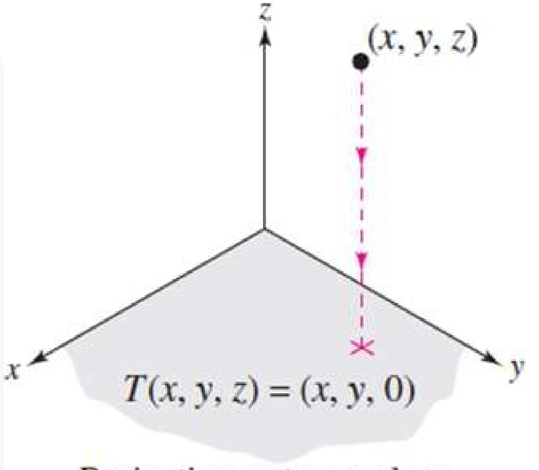


Rotation in R^2

Thus, $T(\boldsymbol{v})$ is the vector that results from rotating the vector $\boldsymbol{v}$ counterclockwise through the angle θ.

- Example 9: (A projection in R^3)

  The LT $T: R^3 \to R^3$ is given by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is called a projection in R^3.



Projection onto xy-plane

2. The Kernel and Range of a Linear Transformation

- **Definition:** Let $T: V \rightarrow W$ be a Linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the kernel of T and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0},\ \forall \mathbf{v} \in V\}$$

- **Example 10: (The kernel of the zero and identity transformations)**

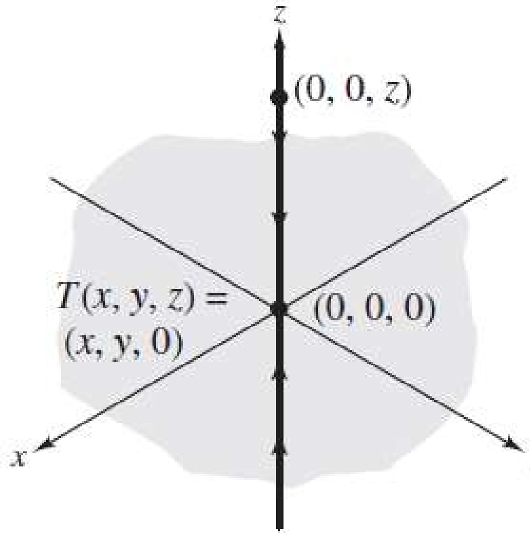
(a) $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation) $\quad \ker(T) = V$

(b) $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation) $\quad \ker(T) = \{\mathbf{0}\}$

- **Example 11: (Finding the kernel of a LT)**

$T(\mathbf{v}) = (x, y, 0) \qquad T: R^3 \rightarrow R^3$

$\ker(T) = \{(0, 0, z) \mid z$ is a real number$\}$



- Example 12: (Finding the kernel of a linear transformation)

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \to R^2)$$

$$\ker(T) = \{(x_1,x_2,x_3) \mid T(x_1,x_2,x_3) = (0,0),\ \boldsymbol{x} = (x_1,x_2,x_3) \in R^3\}$$

$$T(x_1,x_2,x_3) = (0,0) \Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-J. Elimination}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

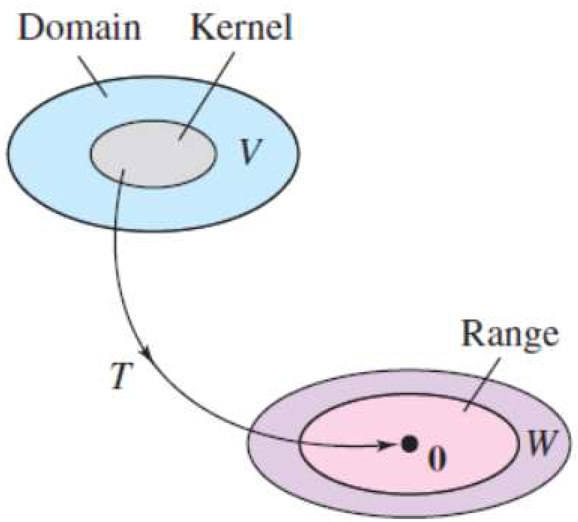
$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\} = \text{span}\{(1,-1,1)\}$$

- **Theorem 3: (The kernel is a subspace of V)**

  The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

- **Definition:** Let $T: V \to W$ be a Linear transformation. Then the set of all vectors $\boldsymbol{w}$ in W that are images of vectors in V is called the **range** of T and is denoted by $\text{range}(T)$ or $R(T)$.

$$\text{range}(T) = \{T(\boldsymbol{v}) \mid \forall \boldsymbol{v} \in V\}$$



- **Theorem 4: (The range is a subspace of W)**

  The range of a LT $T: V \to W$ is a subspace of the W.

- **Rank of a linear transformation $T: V \to W$:**

$$\text{rank}(T) = \text{the dimension of the range of } T$$

- **Nullity of a linear transformation $T: V \to W$:**

$$\text{nullity}(T) = \text{the dimension of the kernel of } T$$

- Note: Let $T: R^n \to R^m$ be the LT given by $T(\mathbf{x}) = A\mathbf{x}$. Then

  $\text{rank}(T) = \text{rank}(A)$, $\qquad \text{nullity}(T) = \text{nullity}(A)$

- Theorem 5: (Sum of rank and nullity)

  Let $T: V \to W$ be a LT from an n-dimensional vector space V into a vector space W. Then:

  $$\text{rank}(T) + \text{nullity}(T) = n$$

  $$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

- Example 13: (Finding rank and nullity of a linear transformation)

  Find the rank and nullity of the LT $T: R^3 \to R^3$ defined by

  $$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  $$\text{rank}(T) = \text{rank}(A) = 2$$

  $$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Example 14: (Finding rank and nullity of a linear transformation)

  Let $T: R^5 \to R^7$ be a linear transformation

  (a) Find the dimension of the kernel of T if the dimension of the range is 2

  (b) Find the rank of T if the nullity of T is 4

  (c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

  (a) $\dim(\text{domain of } T) = 5 \Rightarrow \dim(\ker \text{ of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

  (b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

  (c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

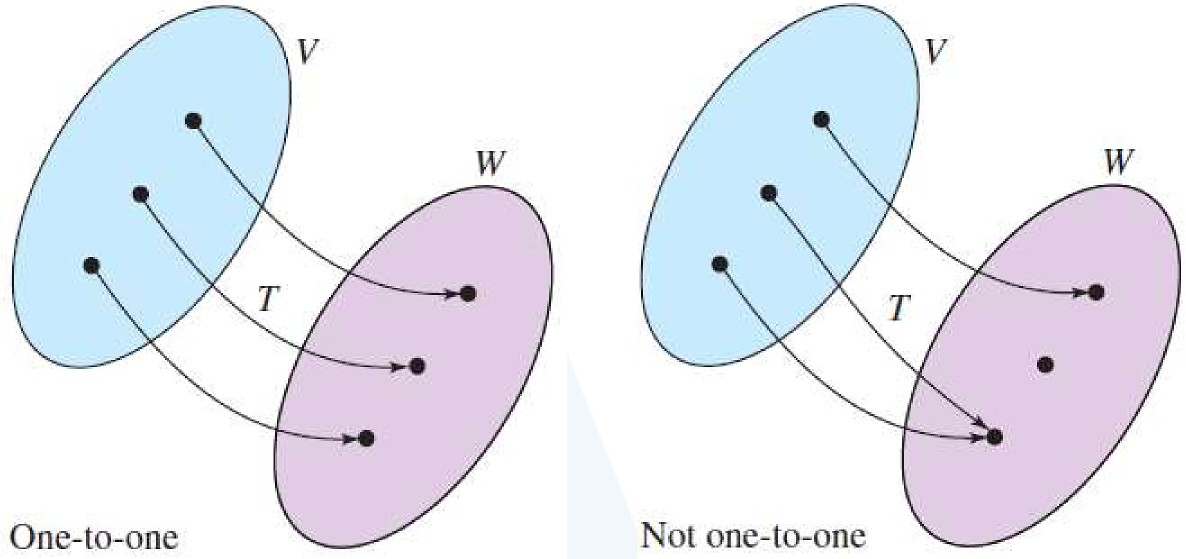
3. Compositions and Inverse Transformations

- Definition: A function $T: V \to W$ is one-to-one when the preimage of every $\boldsymbol{w}$ in the range consists of a single vector.

  T is one-to-one if and only if, for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in V, $T(\boldsymbol{u}) = T(\boldsymbol{v})$ implies $\boldsymbol{u} = \boldsymbol{v}$.

- **Definition:** A function $T: V \rightarrow W$ is onto when every element in W has a preimage in V. (T is onto W when W is equal to the range of T).

- **Theorem 6: (One-to-one LT)**
  Let $T: V \rightarrow W$ be a LT. Then T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$.



- **Example 15: (One-to-one and not one-to-one linear transformation)**
  (a) The linear transformation $T: M_{3x2} \rightarrow M_{2x3}$ given by $T(A) = A^T$ is one-to-one because its kernel consists of only the $m x n$ zero matrix.
  (b) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one because its kernel is all of R^3.

- Example 16: (One-to-one and onto linear transformation)

  The LT $T: P_3 \to R^4$ given by $T(a + bx + cx^2 + dx^3) = (a,\ b,\ c,\ d)$.

- Example 17: (One-to-one and not onto linear transformation)

  $T: P_n \to P_{n+1}:\ T(\boldsymbol{p}) = T(p(x)) = xp(x)$

- Theorem 7: (Onto linear transformation)

  Let $T: V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto iff the rank of T is equal to the dimension of W.

- Theorem 8: (One-to-one and onto linear transformation)

  Let $T: V \to W$ be a linear transformation, with vector space V and W both of dimension n. Then T is one-to-one iff it is onto.

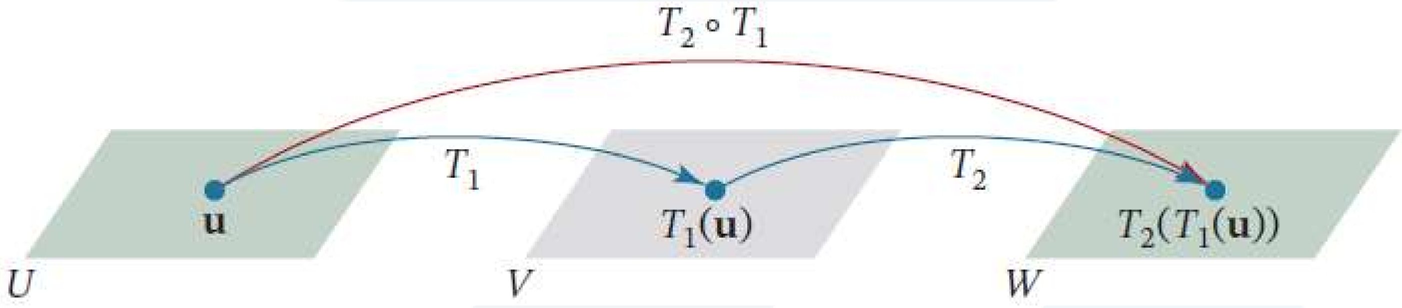
- **Example 18: (One-to-one and not one-to-one linear transformation)**

Let $T: R^n \to R^m$ be a LT given by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither.

$$(a)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ (c)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ (d)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$T: R^n \to R^m$	dim(domain of T)	rank(T)	nullity(T)	one-to-one	onto
(a) $T: R^3 \to R^3$	3	3	0	Yes	Yes
(b) $T: R^2 \to R^3$	2	2	0	Yes	No
(c) $T: R^3 \to R^2$	3	2	1	No	Yes
(d) $T: R^3 \to R^3$	3	2	1	No	No

Definition: If $T_1 \colon U \to V$ and $T_2 \colon V \to W$ are linear transformations, then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$ is the function defined by the formula $(T_2 \circ T_1)(u) = T_2(T_1(u))$, where u is a vector in U.



- **Theorem 9: (Composition of linear transformations)**
  If $T_1 \colon U \to V$ and $T_2 \colon V \to W$ are linear transformations, then $(T_2 \circ T_1) \colon U \to W$ is also a linear transformations.

- **Example 19: (Composition of linear transformations)**
  Let T_1 and T_2 be linear transformations from R^3 into R^3 such that:

$$T_1(x,y,z) = (2x + y,\ 0,\ x + z), \quad T_2(x,y,z) = (x - y,\ z,\ y)$$

Find the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$

$$(T_2 \circ T_1)(x,\ y,\ z) = T_2(T_1(x,\ y,\ z)) = T_2(2x + y,\ 0,\ x + z) = (2x + y,\ x + z,\ 0)$$

$$(T_1 \circ T_2)(x,\ y,\ z) = T_1(T_2(x,\ y,\ z)) = T_1(x - y,\ z,\ y) = (2x - 2y + z,\ 0,\ x)$$

- Note: $T_2 \circ T_1 \neq T_1 \circ T_2$
- Composition with the Identity Operator

If $T: V \to V$ is any linear operator, and if $I: V \to V$ is the identity, then for all vectors $\boldsymbol{v}$ in V, we have

$$(T \circ I)(\boldsymbol{v}) = T(I(\boldsymbol{v})) = T(\boldsymbol{v})$$

$$(I \circ T)(\boldsymbol{v}) = I(T(\boldsymbol{v})) = T(\boldsymbol{v})$$

$T \circ I = T$ and $I \circ T = T$

- Note: Let $T_1: R^n \to R^m$ and $T_2: R^m \to R^p$ be LT where $T_1(\boldsymbol{u}) = A_1\boldsymbol{u}$ and $T_2(\boldsymbol{v}) = A_2\boldsymbol{v}$, then

  (1) The composition $T: R^n \to R^p$, defined by $T(\boldsymbol{v}) = T_2(T_1(\boldsymbol{v}))$, is a LT.

  (2) The matrix A for T is given the matrix product $A = A_2A_1$, where $T(\boldsymbol{u}) = A\boldsymbol{u}$

- Example 20: (Composition of linear transformations)

  Let T_1 and T_2 be linear transformations from R^3 into R^3 such that:

$$T_1(x,y,z) = (2x + y,\ 0,\ x + z), \qquad T_2(x,y,z) = (x - y,\ z,\ y)$$

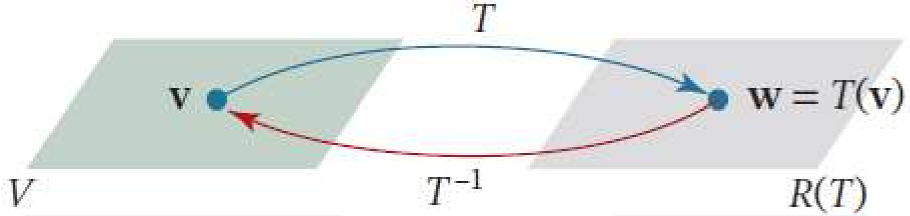
  Find the composition $T = T_2 \circ T_1$

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow A = A_2A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T(x, y, z) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (2x + y, x + z, 0)$$

Definition: If $T: V \to W$ is a linear transformations, then T is invertible if there is a transformation T^{-1} such that: $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$. We call T^{-1} the inverse of T.

$$T^{-1}(T(\boldsymbol{u})) = \boldsymbol{u}, \ \forall \boldsymbol{u} \in U \qquad T(T^{-1}(\boldsymbol{w})) = \boldsymbol{w}, \ \forall \boldsymbol{w} \in R(T)$$



- Notes:

  (1) The inverse transformation $T^{-1}: R(T) \to V$ exists iff T is one-to-one.

  (2) If $T: V \to W$ is a linear transformations, then $T^{-1}: R(T) \to V$ is also a LT.

- Example 21: (An Inverse Transformation)

  $T: P_n \to P_{n+1}: T(\boldsymbol{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}$

  is a one-to-one LT $\Rightarrow T^{-1}(c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}) = c_0 + c_1 x + \ldots + c_n x^n$

- Note: Consider $T: R^n \to R^n$ where $T(\boldsymbol{u}) = A\boldsymbol{u}$

  (1) T is one-to-one if and only if A is invertible.

  (2) T^{-1} exists if and only if A is invertible.

  The inverse transformation is the matrix transformation given by A^{-1}.

- Example 22: (Finding the inverse of a linear transformation)

  The linear transformations $T: R^3 \to R^3$ defined by:

  $$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$$

  Show that T is invertible, and find its inverse.

4. Geometry of Matrix Operators

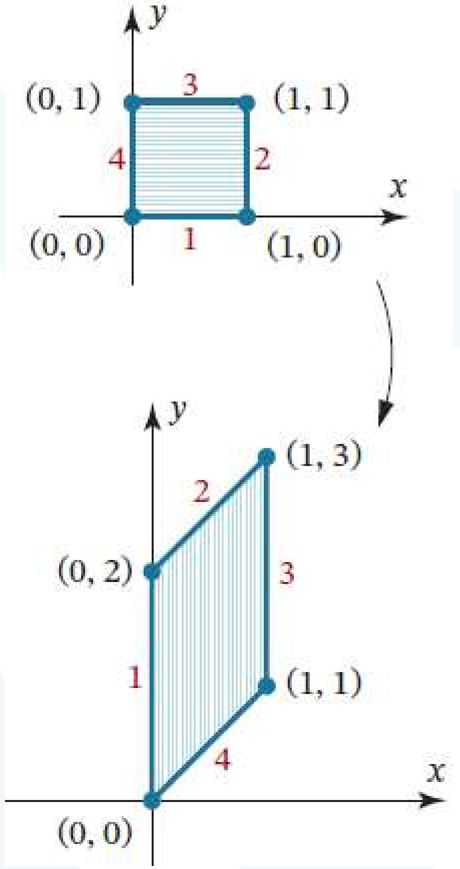
- Example 23: (Transformation of the Unit Square)

Sketch the image of the unit square under multiplication by the invertible matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



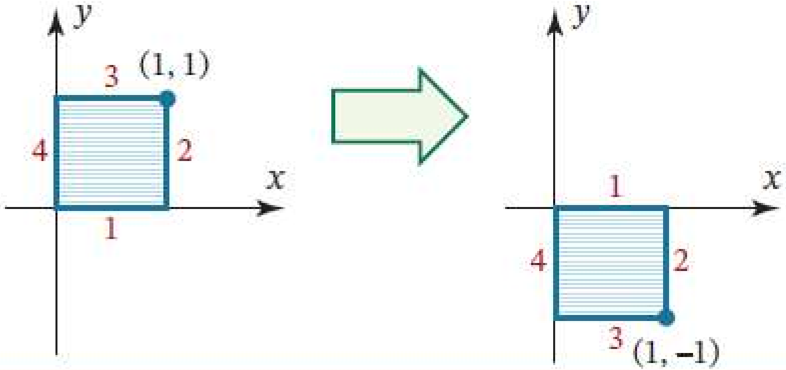
The image of the unit square is a parallelogram with vertices (0, 0), (0, 2), (1, 1), and (1, 3).

Reflections, Rotations, and Projections

Reflection about the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

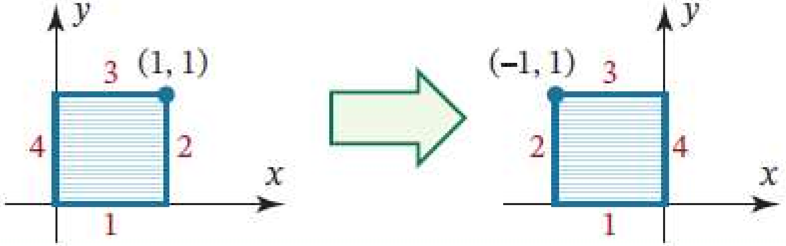
$$T(x, y) = (x, -y)$$



Reflection about the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

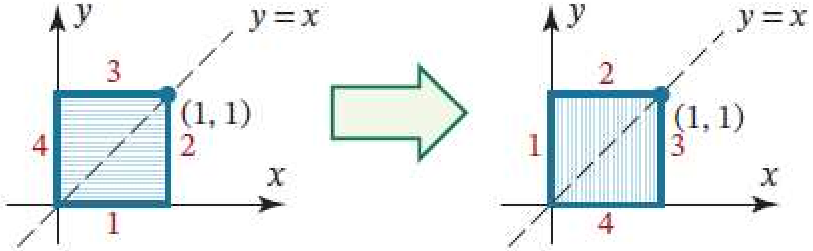
$$T(x, y) = (-x, y)$$



Reflection about the line $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

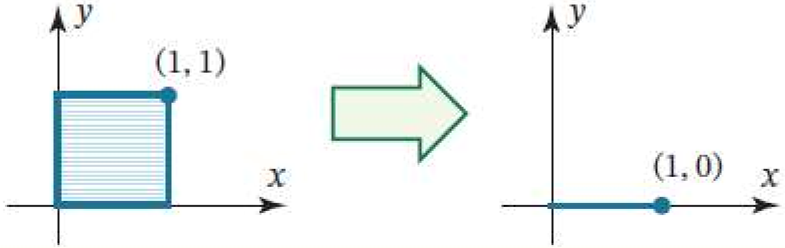
$$T(x, y) = (y, x)$$



Orthogonal projection onto the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

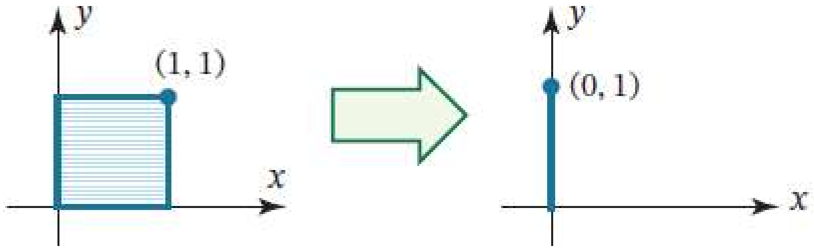
$$T(x, y) = (x, 0)$$



Orthogonal projection onto the y-axis

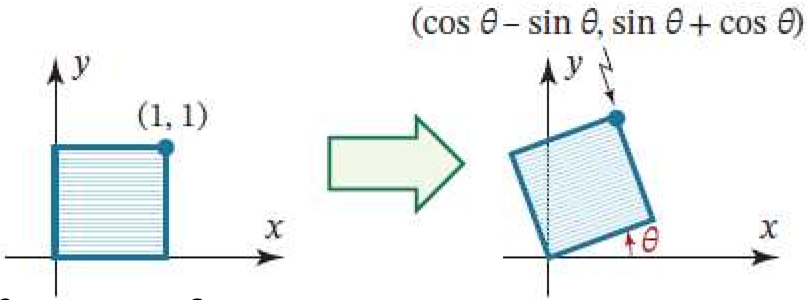
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x, y) = (0, y)$$



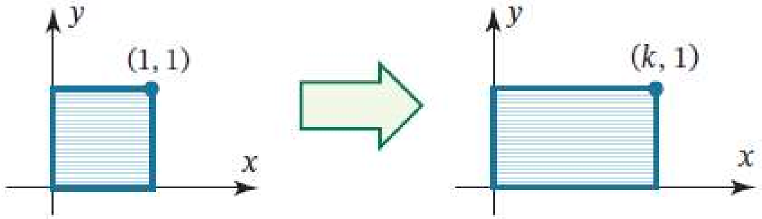
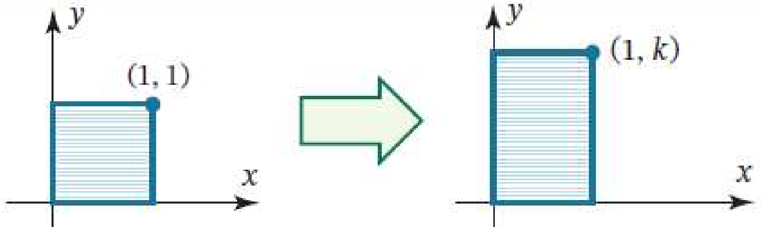
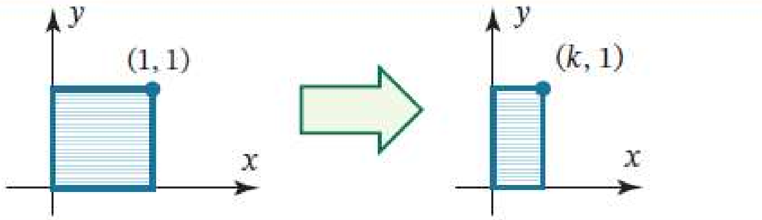
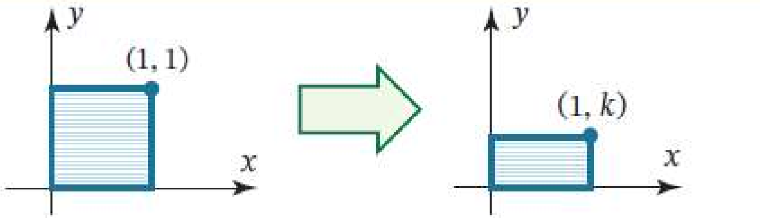
Rotation about the origin through a positive angle θ

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$T(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

Expansions and Compressions

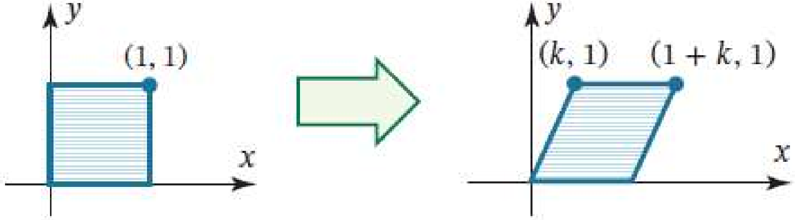
Expansion in the x-direction with factor k $(k > 1)$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $T(x, y) = (kx, y)$		
Expansion in the y-direction with factor k $(k > 1)$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ $T(x, y) = (x, ky)$		
Compression in the x-direction with factor k $(0 < k < 1)$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$		
Compression in the y-direction with factor k $(0 < k < 1)$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$		

Shears

Shear in the positive x-direction by a factor k

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

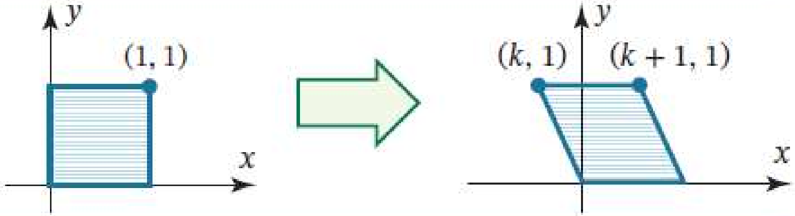
$(k > 0)$ $T(x, y) = (x + ky, y)$



Shear in the negative x-direction by a factor k

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

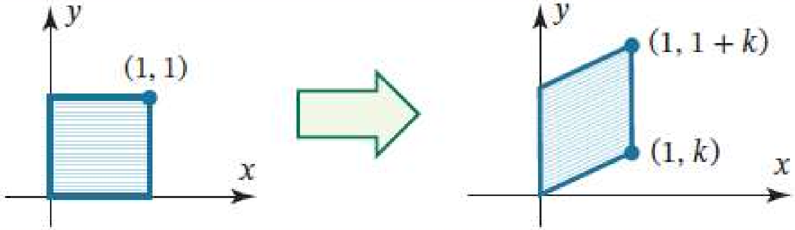
$(k < 0)$



Shear in the positive y-direction by a factor k

$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

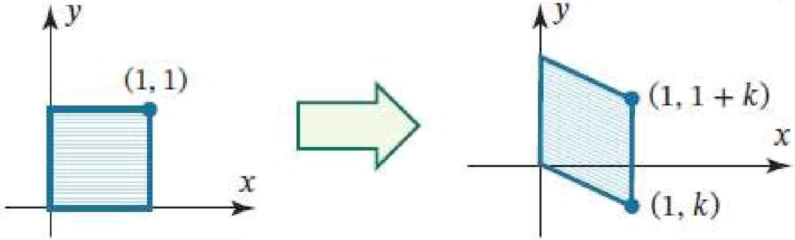
$(k > 0)$ $T(x, y) = (x, kx + y)$



Shear in the negative y-direction by a factor k

$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

$(k < 0)$



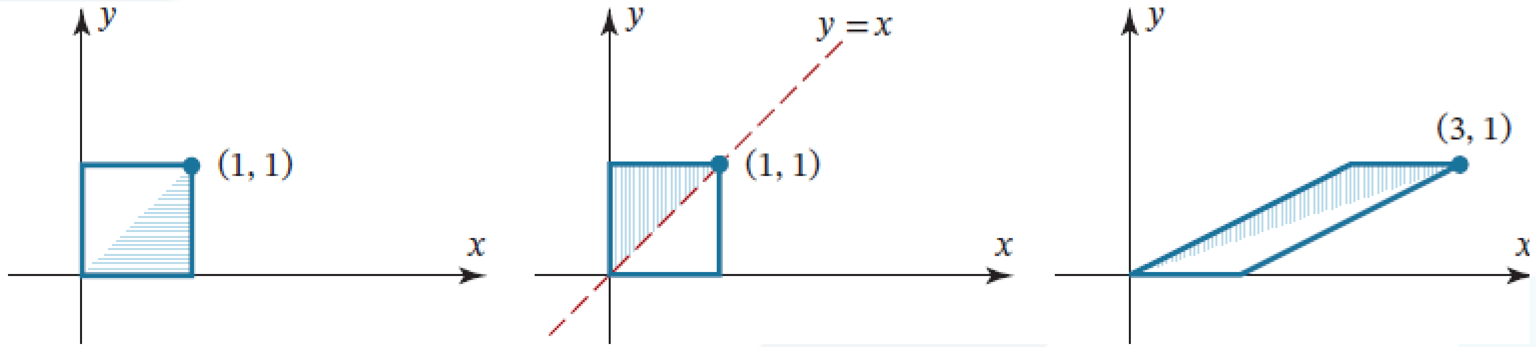
- Example 24: (Transformation of the Unit Square)

(a) Find the standard matrix for the operator on R^2 that first shears by a factor of 2 in the x-direction and then reflects the result about the line $y = x$. Sketch the image of the unit square under this operator.

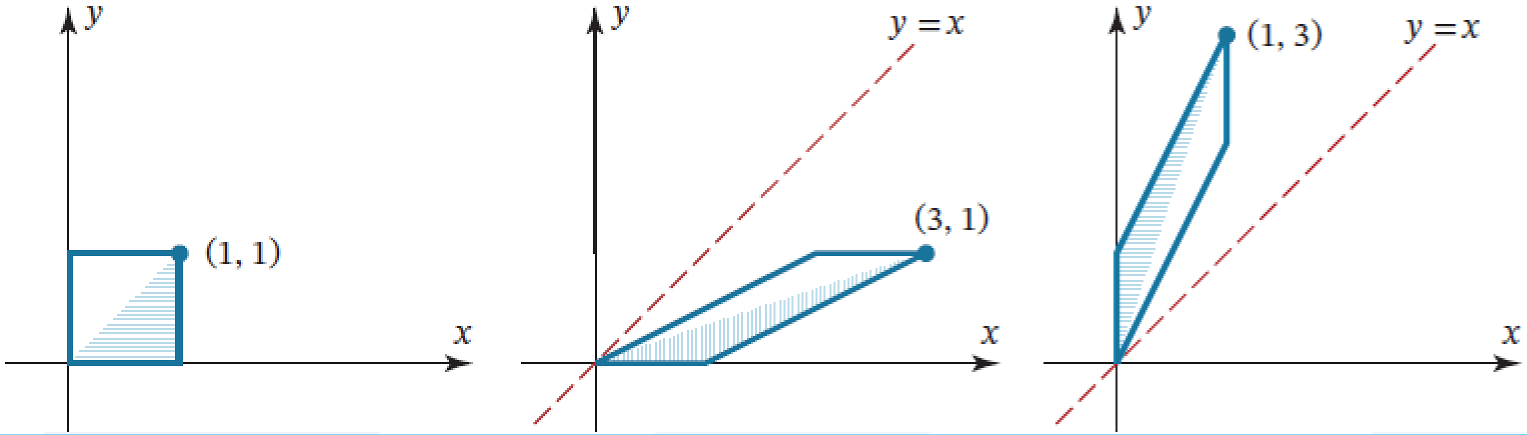
(b) Find the standard matrix for the operator on R^2 that first reflects about $y = x$ and then shears by a factor of 2 in the x-direction. Sketch the image of the unit square under this operator. Conclude.

(a) The matrix for the shear is $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and for the reflection is $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

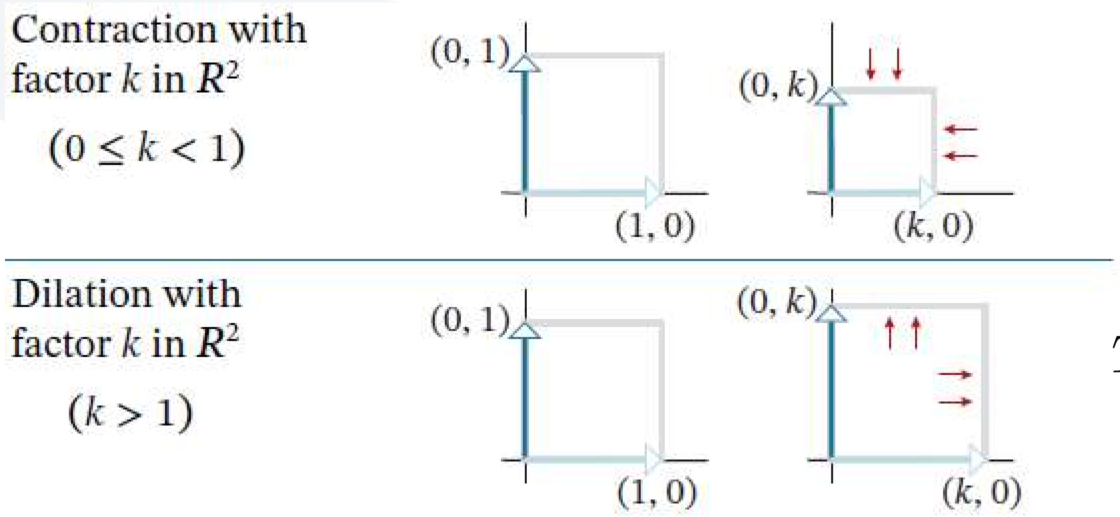


(b) $A_1A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$



$A_1A_2 \neq A_2A_1$

Dilations and Contractions



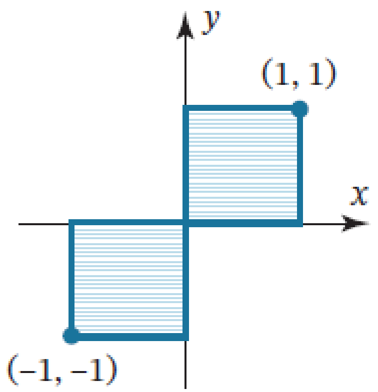
$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$T(x, y) = (kx, ky)$$

$$A = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Note: The multiplication by A causes a compression or expansion of the unit square by a factor of k_1 in the x-direction followed by an expansion or compression of the unit square by a factor of k_2 in the y-direction.

- **Reflection About the Origin:**



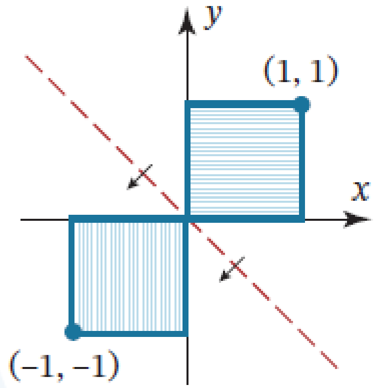
Multiplication by the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has the geometric effect of reflecting the unit square about the origin.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The same result can be obtained by first reflecting the unit square about the x-axis and then reflecting that result about the y-axis.

- **Reflection About the Line $y = -x$**

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



- **Theorem 10: (Elementary matrix transformations)**
  If E is an elementary matrix, then $T_E: R^2 \to R^2$ is one of the following:
  (a) A shear along a coordinate axis.
  (b) A reflection about $y = x$.
  (c) A compression along a coordinate axis.
  (d) An expansion along a coordinate axis.
  (e) A reflection about a coordinate axis.
  (f) A compression or expansion along a coordinate axis followed by a reflection about a coordinate axis.

- **Theorem 11: (Invertible matrix transformations)**
  If $T_A: R^2 \to R^2$ is multiplication by an invertible matrix A, then the geometric effect of T_A is the same as an appropriate succession of shears, compressions, expansions, and reflections.

- Example 25: (Decomposing a Matrix Operator)

In Example 23 we illustrated the effect on the unit square of multiplication by:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

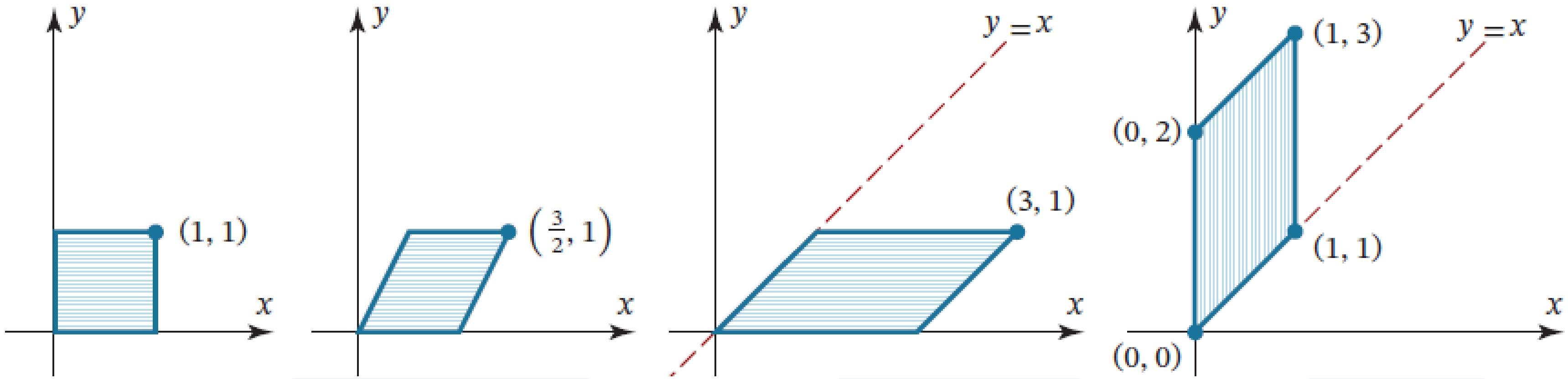
Express this matrix as a product of elementary matrices, and then describe the effect of multiplication by the matrix A in terms of shears, compressions, expansions, and reflections.

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(1/2)}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-1/2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

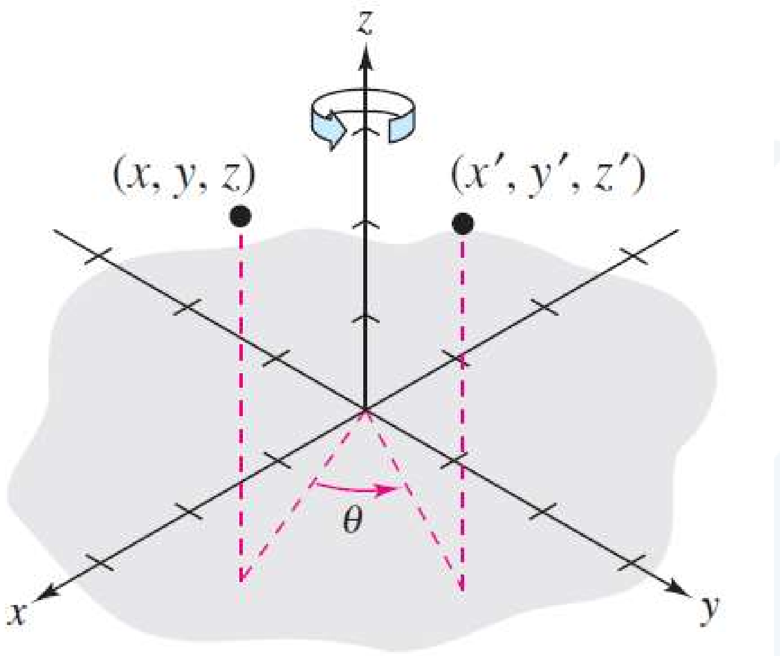
$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Reading from right to left we can now see that the geometric effect of multiplying by A is equivalent to successively:

1. shearing by a factor of ½ in the x-direction;
2. expanding by a factor of 2 in the x-direction;
3. reflecting about the line $y = x$.



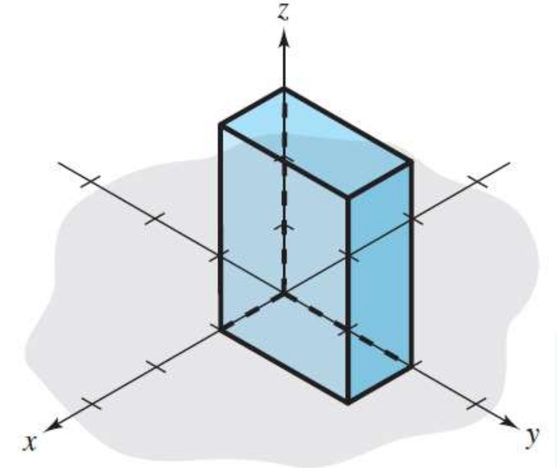
- Rotation In R^3

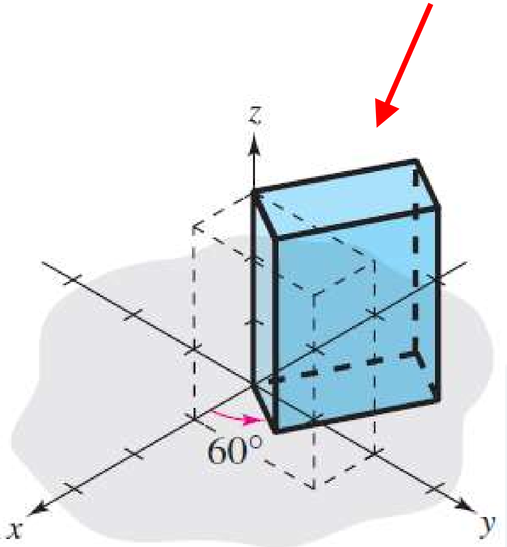


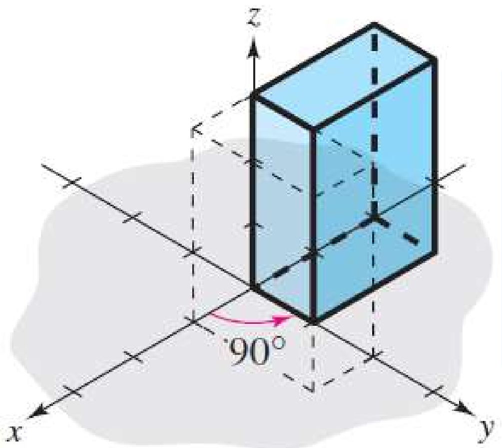
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

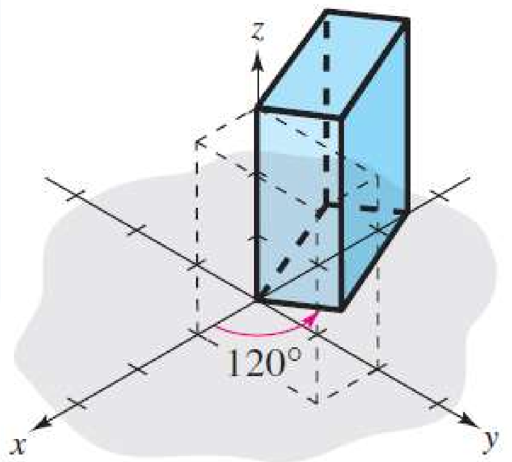
Rotation about the z-axis

$$A = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$









Rotation about the x-axis

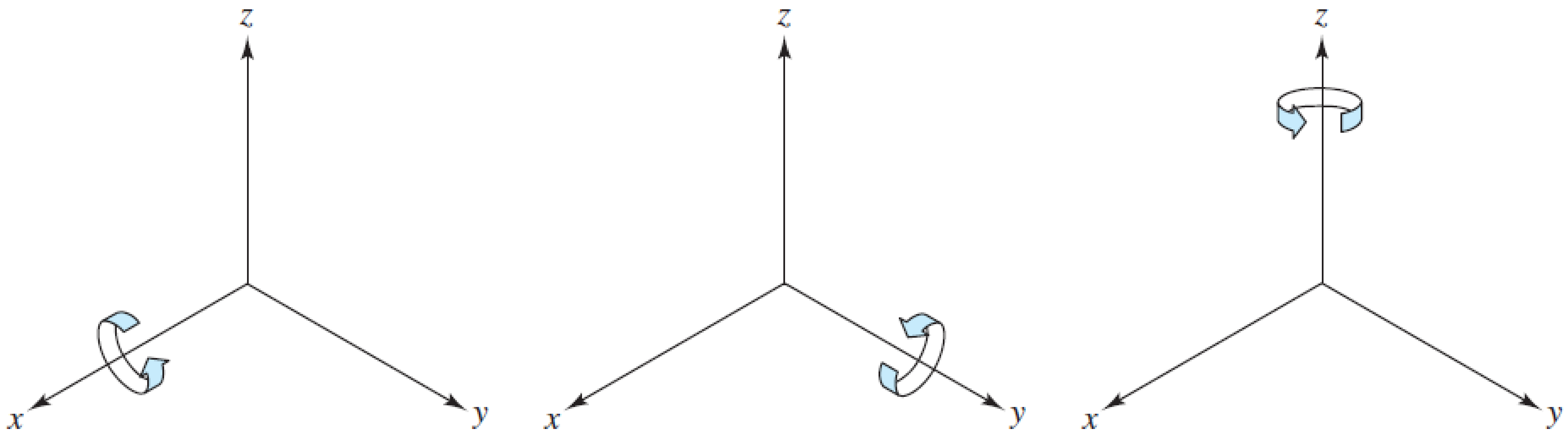
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about the y-axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

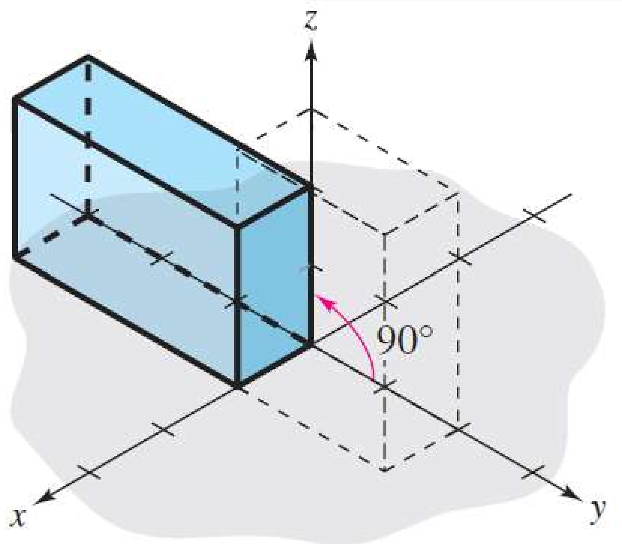
Rotation about the z-axis

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



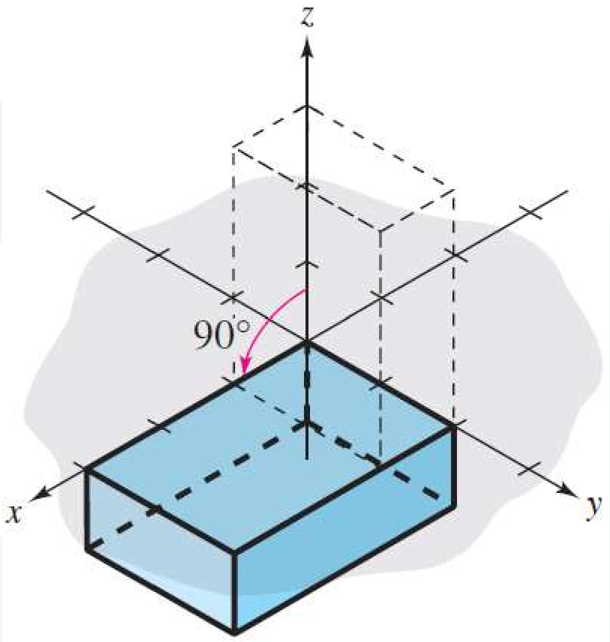
Rotation about x-axis   Rotation about y-axis   Rotation about z-axis

Rotation of 90° about the x-axis



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Rotation of 90° about the y-axis



$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$