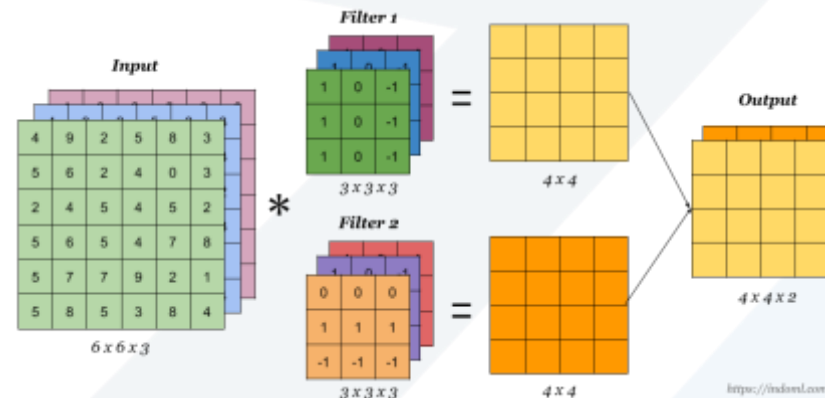


# CECC122: Linear Algebra and Matrix Theory

## Lecture Notes 8: Eigenvalues and Eigenvectors



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## Chapter 7

# Eigenvalues and Eigenvectors

- 1 Eigenvalues and Eigenvectors
- 2 Diagonalization
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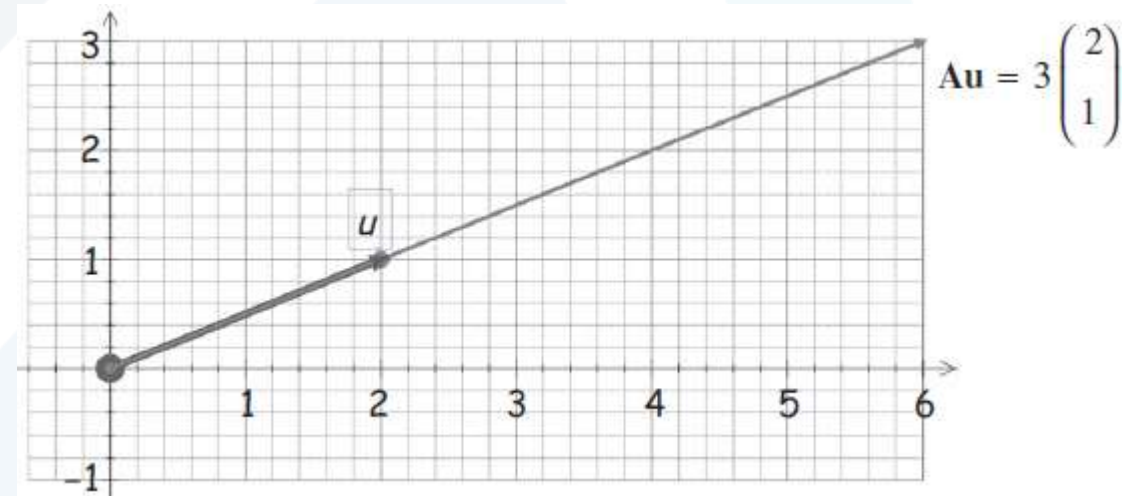
## 1. Eigenvalues and Eigenvectors

- By knowing the **eigenvalues** and **eigenvectors** of a matrix we can easily find its **determinant**, decide whether the matrix has an **inverse** and determine the **powers** of the matrix.
- For an example of linear algebra at work: **Google's search engine**, which relies upon eigenvalues and eigenvectors to **rank** pages with respect to **relevance**.

- Example 1:**

Let  $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$  and  $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Au = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3u$$



- Matrix  $A$  **transforms** the vector  $u$  by **scalar multiplying** it  $\Rightarrow$  the transformation only **changes** the **length** of the vector  $u$  unless  $\lambda = \pm 1$  (length **unchanged**).
- Note:** the relation  $Au = \lambda u$  says that the matrix  $A$  applied to  $u$  gives a vector in the **same** or **opposite** (negative  $\lambda$ ) **direction** of  $u$ .
- Definition:** For a **non-zero** vector  $u$  in  $R^n$  the scalar  $\lambda$  is called an **eigenvalue** of the  $n \times n$  matrix  $A$  and the vector  $u$  is called an **eigenvector** corresponding to  $\lambda$ , which satisfies  $Au = \lambda u$ .

$$\begin{array}{ccc} \text{Eigenvalue} & \downarrow & \\ & & Au = \lambda u \\ \text{Eigenvector} & \uparrow & \uparrow \end{array}$$

- Example 2: (Verifying eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A\mathbf{u} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\mathbf{u}$$

Eigenvalue  
↓  
↑  
Eigenvector

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\mathbf{v}$$

Eigenvalue  
↓  
↑  
Eigenvector

■ Example 3: (Verifying eigenvalues and eigenvectors)

$$\text{Let } A = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A\mathbf{u} = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 2\mathbf{u}$$

Eigenvalue  
↓  
↑  
Eigenvector

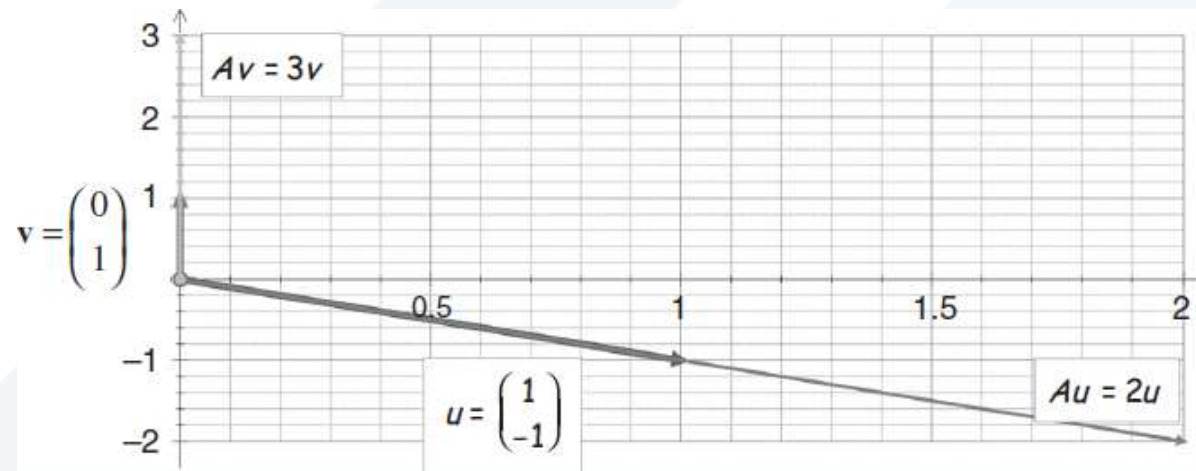
- **Theorem 1: (Finding eigenvalues and eigenvectors of a matrix  $A \in M_{n \times n}$ )**
  - (1) An eigenvalue of  $A$  is a scalar  $\lambda$  such that  $\det(\lambda I - A) = 0$ .
  - (2) The eigenvectors of  $A$  corresponding to  $\lambda$  are the **nonzero solutions** of the **homogeneous system**  $(\lambda I - A)u = \mathbf{0}$ .
- **Note:**  $Au = \lambda u \Rightarrow (\lambda I - A)u = \mathbf{0}$  has nonzero solutions iff  $\det(\lambda I - A) = 0$ .
- The equation  $\det(\lambda I - A) = 0$  is called the **characteristic equation** of  $A$ .
- The polynomial  $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$  is called the **characteristic polynomial** of  $A$ .
- **Example 4: (Finding eigenvalues and eigenvectors)**

Let  $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

Characteristic equation:  $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3) = 0$   
**Eigenvalues:  $\lambda_1 = 2, \lambda_2 = 3$**

(1)  $\lambda_1 = 2$ :  $(\lambda_1 I - A)\mathbf{u} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, t \neq 0$

(2)  $\lambda_1 = 3$ :  $(\lambda_1 I - A)\mathbf{u} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}, t \neq 0$



- Example 5: (Finding eigenvalues and eigenvectors)

Let  $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 & -3 \\ -1 & \lambda - 5 & -1 \\ -3 & -1 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 6)(\lambda + 2) = 0$$

Eigenvalues:  $\lambda_1 = 3$ ,  $\lambda_2 = 6$  and  $\lambda_3 = -2$

$$\lambda_1 = 3: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$





$$\lambda_2 = 6: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = -2: (\lambda I - A)\mathbf{u} = \begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \neq 0$$

- Definition:** Let  $A$  be an  $n \times n$  matrix and have the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . If  $\lambda$  occurs only once then we say  $\lambda$  is a **simple eigenvalue**. If  $\lambda$  occurs  $m$  times where  $m > 1$  then we say  $\lambda$  is an eigenvalue with **multiplicity** of  $m$ .
- In the equation  $(\lambda - 1)^3(\lambda - 2) = 0$  we have  $\lambda_{1,2,3} = 1$  is an eigenvalue of multiplicity 3 and  $\lambda_4 = 2$  is a simple eigenvalue.
- Example 6: (Finding eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -3 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0 \quad \text{Eigenvalue: } \lambda_{1,2,3} = 2$$

$$\lambda_{1,2,3} = 2: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

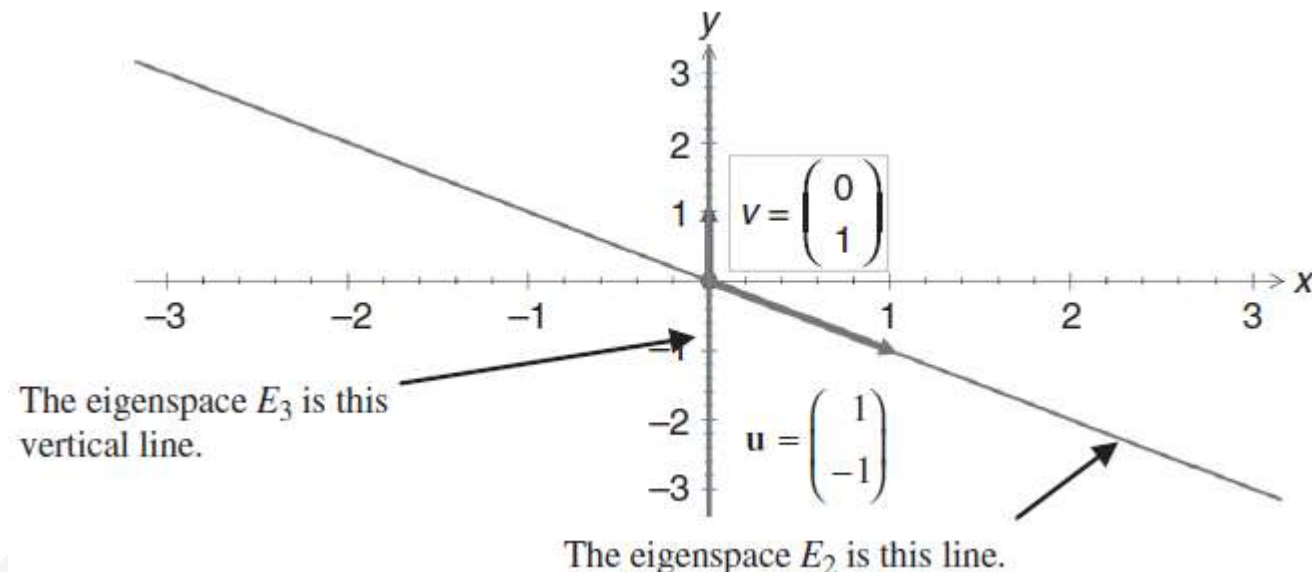
$$\begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -3t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

## Eigenspace

- If  $\lambda$  is an **eigenvalue** of a square matrix  $A$  with an **eigenvector**  $u$  then every non-zero **scalar multiplication** of  $u$ , is also an eigenvector corresponding to  $\lambda$ .
- If  $A$  is an  $n \times n$  matrix with an eigenvalue of  $\lambda$ , then the set  $S$  of all eigenvectors of  $A$  corresponding to  $\lambda$  **together** with the zero vector,  $0$ , is a subspace of  $R^n$ :

$$S = \{0\} \cup \{u \mid u \text{ is an eigenvector corresponding to } \lambda\}$$

- This subspace  $S$  is called an **eigenspace** of  $\lambda$  and is denoted by  $E_\lambda$ , that is  $E_\lambda = S$ .
- For example, the **eigenspace** associated with **Example 4** for the eigenvalue  $\lambda_1 = 2$  is the eigenvector  $u = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and for  $\lambda_2 = 3$  the eigenvector  $v = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

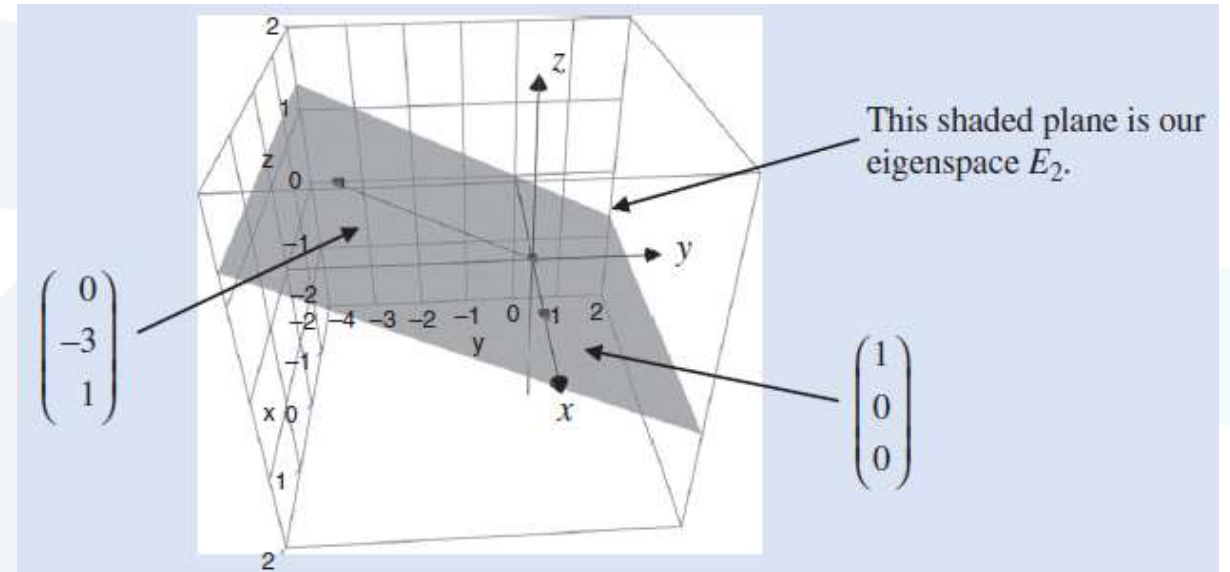


- The **eigenspace** associated with **Example 6** for the eigenvalue  $\lambda_{1,2,3} = 2$  is:

$$E_2 = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

- Note:** A set of basis vectors  $B$  of the eigenspace  $E_2$  are given by:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$



- Theorem 2: (Eigenvalues of triangular matrices)**

If  $A$  is an  $n \times n$  diagonal or triangular matrix, then its eigenvalues are the entries on its main diagonal.

- **Example 7: (Finding eigenvalues)**

$$\text{Let } A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}, \quad |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3) = 0$$

$\lambda_1 = 2, \lambda_2 = 1 \text{ and } \lambda_3 = -3$

- **Theorem 3: (Eigenvalues and Eigenvectors of Matrix Powers)**

If  $A$  is an  $n \times n$  matrix with eigenvector  $u$  corresponding to eigenvalue  $\lambda$ .

(1) If  $m$  is a positive integer, then  $\lambda^m$  is an eigenvalue of the matrix  $A^m$  with the same eigenvector  $u$ .

(2) If the matrix  $A$  is invertible, then the eigenvalue of the inverse matrix  $A^{-1}$  is  $1/\lambda$  with the same eigenvector  $u$ .

- **Example 8: (Finding eigenvalues)**

Find the eigenvalues of  $A^3$  and  $A^{-1}$  of **Example 5**

The eigenvalues of  $A$  are:  $\lambda_1 = 3$ ,  $\lambda_2 = 6$  and  $\lambda_3 = -2$

The eigenvalues of  $A^3$  are:  $\lambda_1 = 3^3 = 27$ ,  $\lambda_2 = 6^3 = 216$  and  $\lambda_3 = (-2)^3 = -8$

The eigenvalues of  $A^{-1}$  are:  $\lambda_1 = 1/3$ ,  $\lambda_2 = 1/6$  and  $\lambda_3 = -1/2$

- **Theorem 4: (Properties of the eigenvalues)**

If  $A$  is an  $n \times n$  matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

(1) The determinant of the matrix  $A$  is given by  $\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$ .

(2) The trace of the matrix  $A$  is given by  $\text{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$ .

- **Example 9:**

Find the determinant and trace of  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$  given that the eigenvalues of  $A$  are 1, 4 and  $-1$ .

$$\det(A) = (1)(4)(-1) = -4,$$

$$\text{tr}(A) = 1 + 4 - 1 = 4$$

- **Theorem 5: (Eigenvalues and Invertibility)**

A square matrix  $A$  is invertible iff  $\lambda = 0$  is not an eigenvalue of  $A$

- **Theorem 6: (linearly independent eigenvectors)**

Let  $A$  be an  $n \times n$  matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ . and corresponding eigenvectors  $p_1, p_2, \dots, p_m$  where  $1 \leq m \leq n$ . Then these eigenvectors  $u_1, u_2, \dots$  and  $u_m$  are linearly independent.

- **Theorem 7: (The Cayley Hamilton Theorem)**

Let  $A$  be a square matrix of size  $n \times n$ , and let  $p(\lambda)$  be its characteristic polynomial, i.e.  $p(\lambda) = \det(\lambda I - A)$ , then:

$$p(A) = A^n + c_{n-1}A^{n-1} + \dots + c_1A + c_0I = O$$



- **Note:** A consequence of the Cayley-Hamilton theorem is a new method for finding the inverse of a nonsingular matrix.

$$A^n + c_{n-1}A^{n-1} + \dots + c_1A + c_0I = O \Rightarrow A \left[ \frac{-1}{c_0} (A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1) \right] = I$$

$$\Rightarrow A^{-1} = \frac{-1}{c_0} (A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1)$$

- **Example 10: (Finding the inverse of a matrix using Cayley-Hamilton theorem)**

Find the inverse of  $A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$

$$p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$$

$$\Rightarrow A^{-1} = \frac{1}{4} (A^2 - 5A + 8I)$$

$$A^{-1} = \frac{1}{4} \left( \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}^2 - 5 \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} 18 & 6 & 2 \\ 6 & 20 & 2 \\ 6 & -6 & 4 \end{bmatrix}$$

## 2. Diagonalization

- **Definition:** If  $A$  and  $B$  are square matrices, then we say that  $B$  is **similar** to  $A$  if there is an invertible matrix  $P$  such that  $B = P^{-1}AP$ .
- **Definition:** A square matrix  $A$  is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix  $P$  such that  $P^{-1}AP$  is diagonal. In this case the matrix  $P$  is said to diagonalize  $A$ .
- **Theorem 8: (Similar matrices have the same eigenvalues)**  
An  $n \times n$  matrix  $A$  is **diagonalizable** if and only if it has  $n$  **linearly independent** eigenvectors.
- **Steps for diagonalizing an  $n \times n$  square matrix:**  
Step 1: Find  $n$  linearly independent eigenvectors  $p_1, p_2, \dots, p_n$  for  $A$  with corresponding eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$

Step 2: Let  $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

Step 3: Let  $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ , where  $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ ,  $i = 1, 2, \dots, n$

▪ **Example 11: (A diagonalizable matrix)**

Show that the matrix  $A$  is diagonalizable  $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ . Then find a matrix  $P$  such that  $P^{-1}AP$  is diagonal.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

**Eigenvalues:  $\lambda_1 = 4, \lambda_{2,3} = -2$**

$$\lambda_1 = 4, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_{2,3} = -2, \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = [\mathbf{p}_2 \ \mathbf{p}_1 \ \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = [\mathbf{p}_2 \ \mathbf{p}_3 \ \mathbf{p}_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- **Note:** The order of the eigenvalues used to form  $P$  will determine the order in which the eigenvalues appear on the main diagonal of  $D$ .
- **Example 12: (A matrix that is not diagonalizable)**

Show that the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  is not diagonalizable

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0 \quad \text{Eigenvalue: } \lambda_{1,2} = 1$$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$A$  does not have two ( $n = 2$ ) linearly independent eigenvectors, so  $A$  is not diagonalizable.

- **Theorem 9: (Sufficient conditions for diagonalization)**

If an  $n \times n$  matrix  $A$  has  $n$  distinct eigenvalues, then the corresponding eigenvectors are linearly independent and  $A$  is diagonalizable.

- **Example 13: (Determining whether a matrix is diagonalizable)**

Determine whether the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$  is diagonalizable

$A$  is a triangular matrix, its eigenvalues are the main diagonal entries  $\lambda_1 = 1$ ,  $\lambda_2 = 0$  and  $\lambda_3 = -3$ . These three values are distinct, so  $A$  is diagonalizable.

- **Theorem 10: (Powers of matrices)**

If an  $n \times n$  matrix  $A$  is diagonalizable with  $P^{-1}AP = D$  where  $D$  is a diagonal matrix then  $A^k = PD^kP^{-1}$  where  $k$  is a positive integer, and

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Example 14: (Powers of matrices)

Let  $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ . Find  $A^6$  given that  $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$  and  $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$$A^6 = PD^6P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 2^6 & 0 \\ 0 & 0 & 1^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$

### 3. Symmetric Matrices and Orthogonal Diagonalization

- A square matrix  $A$  is **symmetric** if it is **equal** to its **transpose**:  $A = A^T$ .
- **Example 15: (Symmetric matrices and nonsymmetric matrices)**

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

(symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$

(symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

(nonsymmetric)

- **Theorem 11: (Eigenvalues of symmetric matrices)**

If  $A$  is an  $n \times n$  symmetric matrix, then the following properties are true:

- (1)  $A$  is **diagonalizable**.
- (2) All **eigenvalues** of  $A$  are **real**.



(3) If  $\lambda$  is an **eigenvalue** of  $A$  with **multiplicity**  $k$ , then  $\lambda$  has  $k$  **linearly independent eigenvectors**.

- A square matrix  $P$  is called **orthogonal** if it is **invertible** and  $P^{-1} = P^T$ .
- **Example 16: (Orthogonal matrices)**

(a)  $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  is orthogonal because  $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b)  $P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$  is orthogonal because  $P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$

- **Theorem 12: (Properties of orthogonal matrices)**

An  $n \times n$  matrix  $P$  is orthogonal iff its column vectors form an orthonormal set.

- Example 17: (Orthogonal matrices)

Show that the matrix  $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$  is orthogonal

If  $P$  is a orthogonal matrix, then  $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\{p_1, p_2, p_3\}$

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0, \quad \|p_1\| = \|p_2\| = \|p_3\| = 1$$

- **Theorem 13: (Properties of symmetric matrices)**

Let  $A$  be an  $n \times n$  symmetric matrix. If  $\lambda_1$  and  $\lambda_2$  are distinct eigenvalues of  $A$ , then their corresponding eigenvectors  $u$  and  $v$  are orthogonal.

- **Example 18: (Eigenvectors of a symmetric matrix)**

Show that any two eigenvectors of matrix  $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$  corresponding to distinct eigenvalues are orthogonal.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

$\Rightarrow$  Eigenvalues:  $\lambda_1 = 2, \lambda_2 = 4$

$$(1) \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad s \neq 0$$

$$(2) \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal}$$

## Orthogonal Diagonalization

- matrix  $A$  is **orthogonally diagonalizable** when there exists an **orthogonal** matrix  $P$  such that  $P^{-1}AP = D$  is **diagonal**.

- Theorem 14: (Fundamental theorem of symmetric matrices)**

Let  $A$  be an  $n \times n$  matrix. Then  $A$  is orthogonally diagonalizable and has real eigenvalues if and only if  $A$  is symmetric.

- **Orthogonal diagonalization of a symmetric matrix:**

Let  $A$  be an  $n \times n$  symmetric matrix.

- (1) Find all **eigenvalues** of  $A$  and determine the **multiplicity** of each.
- (2) For each eigenvalue of multiplicity 1, choose a **unit eigenvector**. (Find any eigenvector and then normalize it).
- (3) For each eigenvalue of multiplicity  $k \geq 2$ , find a set of  $k$  **linearly independent** eigenvectors. If this set is not orthonormal, apply **Gram-Schmidt** orthonormalization process.
- (4) Steps 2 and 3 produces an **orthonormal** set of  $n$  eigenvectors. Use these eigenvectors to form the columns of  $P$ . The matrix  $P^{-1}AP = P^TAP = D$  will be **diagonal**.

■ **Example 19: (Orthogonal diagonalization)**

Find a matrix  $P$  that orthogonally diagonalizes  $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

$$(1) |\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$$

Eigenvalues:  $\lambda_1 = -6$ ,  $\lambda_2 = 3$  (has a multiplicity of 2)

$$(2) \lambda_1 = -6, \quad v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$(3) \lambda_2 = 3, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

linearly Independent

## Gram-Schmidt Process:

$$w_2 = v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad u_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$(4) P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^T AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

## 4. Singular Value Decomposition

- The **singular value decomposition** (SVD) is one of the most important factorizations of a matrix. SVD factorization can be applied to **any matrix**; it does not need to be a **square** or **symmetric** matrix.
- **Theorem 15:**  
If  $A$  is an  $m \times n$  matrix, then:
  - (a)  $A^T A$  is orthogonally diagonalizable.
  - (b) The eigenvalues of  $A^T A$  are nonnegative.
- **Definition:** If  $A$  is an  $m \times n$  matrix, and if  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the **eigenvalues** of  $A^T A$ , then the numbers  $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots, \sigma_n = \sqrt{\lambda_n}$  are **called the singular values** of  $A$ .



- **Example 20: (Singular Values)**

Find the singular values of the matrix  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial of  $A^T A$  is  $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

So the eigenvalues of  $A^T A$  are:  $\lambda_1 = 3$ ,  $\lambda_2 = 1$ , and the singular values of  $A$  are:  $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ ,  $\sigma_2 = \sqrt{\lambda_2} = 1$

- Theorem 16: (Singular Value Decomposition )**

If  $A$  is an  $m \times n$  matrix of **rank**  $k$ , then  $A$  can be expressed in the form  $A = U \Sigma V^T$ , where  $\Sigma$  has size  $m \times n$  and can be expressed in partitioned form as:

$$\Sigma = \begin{bmatrix} D & \vdots & 0_{k \times (n-k)} \\ \hline 0_{(m-k) \times k} & \vdots & 0_{(m-k) \times (n-k)} \end{bmatrix}$$

in which  $D$  is a **diagonal**  $k \times k$  matrix whose successive entries are the first  $k$  singular values of  $A$  in **nonincreasing** order ( $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0$ ),  $U$  is an  $m \times m$  **orthogonal** matrix, and  $V$  is an  $n \times n$  **orthogonal** matrix.

$$\mathbf{u}_i = \frac{1}{\sigma_i} A \mathbf{v}_i \quad (i = 1, 2, \dots, k)$$

$V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]$  orthogonally diagonalizes  $A^T A$

$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m\}$  is an extension of  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$

to an orthonormal basis for  $R^m$ .

$$D = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_k \end{bmatrix}$$

- **Example 21: (Singular Value Decomposition if  $A$  Is Not Square)**

Find a singular value decomposition of the matrix of **Example 20**

The eigenvalues of  $A^T A$  are:  $\lambda_1 = 3$ ,  $\lambda_2 = 1$ , and the singular values of  $A$  are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = 1$$

The unit eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$  are

$$\mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad V \text{ orthogonally diagonalizes } A^T A$$

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sigma_2} A \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$\mathbf{u}_1$  and  $\mathbf{u}_2$  are two of the three column vectors of  $U$

To extend the orthonormal set  $\{u_1, u_2\}$  to an orthonormal basis for  $R^3$

The vector  $u_3 = [x \ y \ z]^T$  needs to be orthogonal to both vectors  $u_1$  and  $u_2$

$$\frac{\sqrt{6}}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

In matrix form, 
$$\begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$A = U \Sigma V^T$

## Reduced Singular Value Decomposition

- The **zero rows** and **columns** of the matrix  $\Sigma$  are superfluous and can be eliminated

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \end{bmatrix}$$

which is called a **reduced singular value decomposition RSVD** of  $A$ .

$$A = U_1 \Sigma_1 V_1^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$$

which is called a **reduced singular value expansion** of  $A$ .

- Example 22: (Reduced Singular Value Decomposition)**  
Find a RSVD and a reduced singular value expansion of **Example 21**



Since  $A$  has rank 2, with  $k = 2$  the reduced SVD of  $A$  corresponding to is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T = \sqrt{3} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} + (1) \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \sqrt{3} \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \text{ the reduced singular value expansion}$$