Exercises 8: Linear Transformations

CECC122: Linear Algebra and Matrix Theory

Manara University

2023-2024

Find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$

① $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $\mathbf{v} = (3, -4)$, $\mathbf{w} = (3, 19)$

(a) The image of $\mathbf{v}$ is $T(3, -4) = (3 + (-4), 3 - (-4)) = (-1, 7)$

(b) If $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2) = (3, 19)$, then

$$v_1 + v_2 = 3$$
$$v_1 - v_2 = 19$$

Which implies that $v_1 = 11$ and $v_2 = -8$. So, the preimage of $\mathbf{w}$ is $(11, -8)$

② $T(v_1, v_2) = (2v_2 - v_1, v_1, v_2)$, $\mathbf{v} = (0, 6)$, $\mathbf{w} = (3, 1, 2)$

(a) The image of $\mathbf{v}$ is $T(0, 6) = (2(6) - 0, 0, 6) = (12, 0, 6)$

(b) If $T(v_1, v_2) = (2v_2 - v_1, v_1, v_2) = (3, 1, 2)$, then

$$2v_2 - v_1 = 3$$
$$v_1 = 1$$
$$v_2 = 2$$

Which implies that the preimage of $\boldsymbol{w}$ is (1, 2)

③ $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$, $\boldsymbol{v} = (-4, 5, 1)$, $\boldsymbol{w} = (4, 1, -1)$

(a) The image of $\boldsymbol{v}$ is

$$T(-4, 5, 1) = (2(-4) + 5, 2(5) - 3(-4), -4 - 1) = (-3, 22, -5)$$

(b) If $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3) = (4, 1, -1)$, then

$$\begin{aligned} 2v_1 + v_2 \quad &= 4 \\ -3v_1 + 2v_2 \quad &= 1 \\ v_1 \quad - v_3 &= -1 \end{aligned}$$

Which implies that $v_1 = 1$, $v_2 = 2$ and $v_3 = 2$. So, the preimage of $\boldsymbol{w}$ is (1, 2, 2)

④ $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2)$, $\boldsymbol{v} = (2, -3, -1)$, $\boldsymbol{w} = (3, 9)$

(a) The image of $\boldsymbol{v}$ is

$$T(2, -3, -1) = (4(-3) - 2, 4(2) + 5(-3)) = (-14, -7)$$

(b) If $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2) = (3, 9)$, then

$$-v_1 + 4v_2 = 3$$
$$4v_1 + 5v_2 = 9$$

Which implies that $v_1 = 1$, $v_2 = 1$ and $v_3 = t$, where t is any real number. So, the preimage of $\boldsymbol{w}$ is $\{(1, 1, t)\colon t \text{ is any real number}\}$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 4, -1)$, $T(0, 1, 0) = (1, 3, -2)$, and $T(0, 0, 1) = (0, -2, 2)$. Find $T(2, -4, 1)$

$$(2, -4,1) = 2(1, 0, 0) - 4(0,1, 0) + (0, 0,1)$$

$$T(2, -4,1) = 2\,T(1, 0, 0) - 4\,T(0,1, 0) + T(0, 0,1)$$

$$= 2(2, 4, -1) - 4(1, 3, -2) + (0, -2, 2) = (0, -6, 8)$$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(0, -1, 2) = (-3, 2, -1)$, and $T(1, 0, 1) = (1, 1, 0)$. Find $T(2, 1, 0)$

$$(2, 1, 0) = 0(1, 1, 1) - (0, -1, 2) + 2(0, 0,1)$$

$$(2, 1, 0) = 0.\,T(1, 1, 1) - T(0, -1, 2) + 2\,T(0, 0,1)$$

$$= (0, 0, 0) - (-3, 2, -1)) + 2(1, 1, 0) = (5, 0, 1)$$

Let $T: R^2 \rightarrow R^2$ such that $T(1, 0) = (1, 0)$ and $T(0, 1) = (0, 0)$. (a) Determine $T(x, y)$ for (x, y) in R^2. (b) Give a geometric description of T

(a) $T(x, y) = T[x(1, 0) + y(0,1)] = xT(1, 0) + yT(0,1) = x(1, 0) + y(0, 0) = (x, 0)$

(b) T is the projection onto the x-axis

Find the kernel of the linear transformation

1. $T: R^2 \rightarrow R^2,\ T(x, y) = (x + 2y, y - x)$

$T(x, y) = (x + 2y, y - x) = (0, 0)$ yields the trivial solution $x = y = 0$. So, $\ker(T) = \{(0, 0)\}$

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② $T: R^2 \rightarrow R^2,\ T(x, y) = (x + y,\ y - x)$

$T(x, y) = (x + y,\ y - x) = (0, 0)$ yields solutions of the form $x = y$. So, $\ker(T) = \{(x, x): x \in R\}$

③ $T: R^3 \rightarrow R^3,\ T(x, y, z) = (x, 0, z)$

$T(x, y, z) = (x, 0, z)$. The kernel consists of all vectors lying on the y-axis. That is, $\ker(T) = \{(0, y, 0) : y$ is a real number$\}$

The linear transformation T is defined by $T(\mathbf{x}) = A\mathbf{x}$. Find the (a) kernel of T, (b) nullity(T), and (c) rank(T)

① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has only the trivial solution $v_1 = v_2 = 0$, the kernel is $\{(0, 0)\}$

(b) nullity(T) = dim(ker(T)) = 0

(c) rank(T) = dim(domain of T) – dim(ker(T)) = 2 – 0 = 2

$$T(\mathbf{v}) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has solutions of the form $(-4t, -2t, t)$, where t is any real number

(b) nullity(T) = dim(ker(T)) = 1

(c) rank(T) = dim(domain of T) – dim(ker(T)) = 3 – 1 = 2

$$T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has only the trivial solution $v_1 = v_2 = 0$, the kernel is $\{(0, 0)\}$

(b) nullity(T) = dim(ker(T)) = 0

(c) rank(T) = dim(domain of T) – dim(ker(T)) = 2 – 0 = 2

Let T: $R^n \to R^m$ be a L.T. given by $T(x) = Ax$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

$$(a)\ A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (b)\ B = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (c)\ C = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 4 & 1 & 8 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

	dim(domain of *T*)	rank(*T*)	nullity(*T*)	one-to-one	onto
(*a*)	2	1	1	No	No
(*b*)	2	2	0	Yes	No
(*c*)	3	2	1	No	Yes

Let $T: R^2 \to R^2$ be a reflection in the line $y = -x$. Find the image of each vector (a) (−1, 2), (b) (2, 3), (c) (a, 0)

The standard matrix for T is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$(a) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \Rightarrow T(-1, 2) = (-2, 1)$$

$$(b) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \quad \Rightarrow T(2, 3) = (-3, -2)$$

$$(c) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \end{bmatrix} \quad \Rightarrow T(a, 0) = (0, -a)$$

(a) find the standard matrix A for the linear transformation T and (b) use A to find the image of the vector v

1. $T(x, y, z) = (2x + 3y - z, 3x - 2z, 2x - y + z)$, $v = (1, 2, -1)$

$T(1, 0, 0) = (2, 3, 2)$, $T(0, 1, 0) = (3, 0, -1)$, $T(0, 0, 1) = (-1, -2, 1)$

$$(a)\ A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & -2 \\ 2 & -1 & 1 \end{bmatrix}, \qquad (b) \begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & -2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -1 \end{bmatrix}$$

So, $T(1, 2, -1) = (9, 5, -1)$

Find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$

1. $T_1: R^2 \rightarrow R^2: T_1(x, y) = (x - 2y, 2x + 3y)$

$T_2: R^2 \rightarrow R^2: T_2(x, y) = (2x, x - y)$

The standard matrices for T_1 and T_2 are $A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

The standard matrix for $T = T_2 \circ T_1$ is $A = A_2 A_1 = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & -5 \end{bmatrix}$

The standard matrix for $T' = T_1 \circ T_2$ is $A = A_1 A_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 7 & -3 \end{bmatrix}$

Determine whether the linear transformation is invertible. If it is, find its inverse

1. $T: R^2 \rightarrow R^2: T(x, y) = (x + 2y, x - y)$
2. $T: R^2 \rightarrow R^2: T(x, y) = (2x, 0)$

1. The standard matrices for T is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Because $\det(A) = -2 \neq 0$, A is invertible $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

So, $T^{-1} = (\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y)$

2. The standard matrices for T is $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ Because $\det(A) = 0$, A is not invertible

Find (a) the image of $\boldsymbol{v}$ and (b) the preimage of $\boldsymbol{w}$

1. $T(v_1, v_2) = (2v_1 + v_2, v_1 - 2v_2)$, $\boldsymbol{v} = (1, -1)$, $\boldsymbol{w} = (3, -1)$
2. $T(v_1, v_2) = (v_1, 2v_2 - v_1, v_2)$, $\boldsymbol{v} = (0, 4)$, $\boldsymbol{w} = (2, 4, 3)$
3. $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$, $\boldsymbol{v} = (2, 3, 0)$, $\boldsymbol{w} = (-11, -1, 10)$
4. $T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2)$, $\boldsymbol{v} = (2, 1, 4)$, $\boldsymbol{w} = (-1, 2)$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 0, -1)$, $T(0, 1, 0) = (-1, 3, 0)$, and $T(0, 0, 1) = (0, -2, 0)$. Find $T(2, -4, 1)$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(1, 1, 0) = (-3, 2, -1)$, and $T(1, 0, 0) = (1, 1, 0)$. Find $T(3, 2, 1)$

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Let $T: R^2 \rightarrow R^2$ such that $T(1, 0) = (0, 0)$ and $T(0, 1) = (0, 1)$. (a) Determine $T(x, y)$ for (x, y) in R^2. (b) Give a geometric description of T

Find the kernel of the linear transformation

1. $T: R^2 \rightarrow R^2,\ T(x, y) = (x + y,\ x - y)$
2. $T: R^2 \rightarrow R^2,\ T(x, y) = (x + y,\ x - 2y)$
3. $T: R^3 \rightarrow R^3,\ T(x, y, z) = (x,\ y,\ x + y)$

The linear transformation T is defined by $T(x) = Ax$. Find the (a) kernel of T, (b) nullity(T), and (c) rank(T)

① $A = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$ ② $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ ③ $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$

Let $T: R^n \to R^m$ be a L.T. given by $T(x) = Ax$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

① $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

Let $T: R^2 \to R^2$ be a reflection in the line $y = x$. Find the image of each vector (a) (–1, 3), (b) (1, 3), (c) (a, b)

Let $T(1, 0) = (0, 2)$ and $T(0, 1) = (1, 0)$. (a) Determine $T(x, y)$ for any (x, y), (b) Give a geometric description of T

Let $T(1, 0) = (1, 1)$ and $T(0, 1) = (1, 0)$. (a) Determine $T(x, y)$ for any (x, y), (b) Give a geometric description of T

(a) find the standard matrix A for the linear transformation T and (b) use A to find the image of the vector v

1. $T(x, y, z) = (x + 2y - 3z, 3x - 5y, y - 3z),$ $\quad v = (3, 13, 4)$
2. $T(x, y, z) = (2x + y, 3y - z),$ $\quad v = (0, 1, -1)$

Find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$

1. $T_1: R^2 \to R^2: T_1(x, y) = (x - 2y, 2x + 3y)$
   $T_2: R^2 \to R^2: T_2(x, y) = (y, 0)$

2. $T_1: R^2 \to R^3: T_1(x, y) = (x, y, y)$
   $T_2: R^3 \to R^2: T_2(x, y, z) = (y, z)$

Determine whether the linear transformation is invertible. If it is, find its inverse

1. $T: R^2 \to R^2: T(x, y) = (x + y, x - y)$
2. $T: R^2 \to R^2: T(x, y) = (x - y, y - x)$
3. $T: R^3 \to R^3: T(x, y, z) = (x, x + y, x + y + z)$