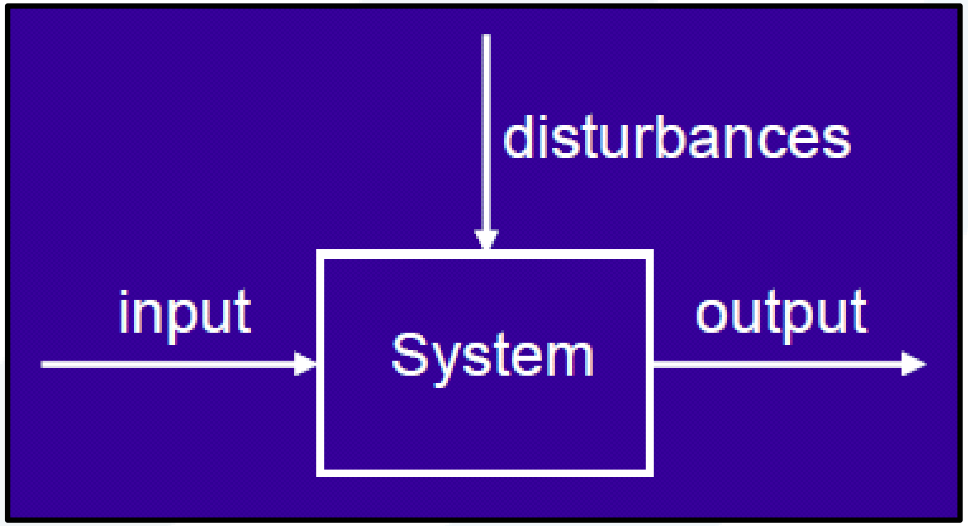
جامعة المنارة
كلية الهندسة
قسم الروبوتيك و الأنظمة الذكية
مقرر النمذجة و المطابقة

Basic Mathematical Techniques
to
Develop Models using Least Squares Method
and
Estimate Parameters of the First Order Model



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د. محمد خير عبدالله محمد

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CONSTRAINING MODELS TO PASS THROUGH A GIVEN POINT

Many applications require a model whose form is dictated by physical principles. For example, the force-extension model of a spring must pass through the origin (0, 0) because the spring exerts no force when it is not stretched. Thus a linear model $y = mx + b$ sometimes must have a zero value for b. However, in general the least-squares method will give a nonzero value for b because of the scatter or measurement error that is usually present in the data.

To obtain a zero-intercept model of the form $y = mx$, we must derive the equation for m from basic principles. The sum of the squared residuals in this case is

$$J = \sum_{i=1}^{n} (mx_i - y_i)^2$$

Computing the derivative $\partial J/\partial m$ and setting it equal to zero gives the result

$$m \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

which can be easily solved for m.

If the model is required to pass through a point not at the origin, say the point (x_0, y_0), subtract x_0 from all the x values, subtract y_0 from all the y values, and then find the coefficient m. The resulting equation will be of the form

$$y = m(x - x_0) + y_0$$

Problem

Consider the data given

x	0	5	10
y	2	6	11

We found that the best-fit line is $y = (9/10)x + 11/6$. Find the best-fit line that passes through the point $x = 10$, $y = 11$.

Solution

Subtracting 10 from all the x values and 11 from all the y values, we obtain a new set of data in terms of the new variables $X = x - 10$ and $Y = y - 11$.

X	-10	-5	0
Y	-9	-5	0

$$m\sum_{i=1}^{3} X_i^2 = \sum_{i=1}^{3} X_i Y_i$$

$$\sum_{i=1}^{3} X_i^2 = (-10)^2 + 5^2 + 0 = 125$$

$$\sum_{i=1}^{3} X_i Y_i = (-10)(-9) + (-5)(-5) + 0 = 115$$

Thus, $m = 115/125 = 23/25$ and the best-fit line is $Y = (23/25)X$. In terms of the original variables, this line is expressed as $y - 11 = (23/25)(x - 10)$ or $y = (23/25)x + 9/5$.

CONSTRAINING A COEFFICIENT

Sometimes we know from physical theory that the data can be described by a function with a specified form and specified values of one or more of its coefficients. In such cases, we can modify the least-squares method to find the best-fit function of a specified form.

Problem

Fit the power function $y = bx^m$ to the data y_i. The value of m is known.

Solution

The least-squares criterion is $$J = \sum_{i=1}^{n} (bx_i^m - y_i)^2$$

To obtain the value of b that minimizes J, we must solve $\partial J/\partial b = 0$.

$$\frac{\partial J}{\partial b} = 2\sum_{i=1}^{n} x_i^m \left(bx_i^m - y_i\right) = 0$$

This gives $$b = \frac{\sum_{i=1}^{n} x_i^m y_i}{\sum_{i=1}^{n} x_i^{2m}}$$

Problem

A hole 6 mm in diameter was made in a translucent milk container . A series of marks 1 cm apart was made above the hole. While adjusting the tap flow to keep the water height constant, the time for the outflow to fill a 250 ml cup was measured (1 ml $= 10^{-6}$ m^3). This was repeated for several heights. The data are given in the following table.

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

Obtain a functional description of the volume outflow rate f as a function of water height h above the hole.

Solution

First obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

$$f = \frac{250}{t}$$

We learned that the following power function can describe the data:

$$f = bh^m$$

We can find the values of m and b by using `p=polyfit(log10(x),log10(y),1)`. The first element p_1 of the vector `p` will be m, and the second element p_2 will be $\log b$. We can find b from $b = 10^{p_2}$. The following MATLAB program performs the calculations.

```
h = (1:11);
time = [26, 19, 14, 12, 11, 9.5, 9, 8.5, 8, 7.5, 7];
flow = 250./time;
logflow = log10(flow);logheight = log10(h);
p = polyfit(logheight,logflow, 1);
m = p(1),b = 10^p(2)
F = b*h.^m;
J = sum((F - flow).^2)
S = sum((flow - mean(flow)).^2)
r2 = 1 - J/S
```

```
m =
   0.5499
b =
   9.4956
J =
   2.5011
S =
 698.2203
r2 =
   0.9964
```

The results are $m = 0.5499$ and $b = 9.4956$, and the corresponding function is

$$f = bh^m = 9.4956h^{0.5499}$$

The quality-of-fit values are $J = 2.5011$, $S = 698.2203$, and $r^2 = 0.9964$, which indicates a very good fit.

Sometimes we know from physical theory that the data can be described by a power function with a specified exponent. For example, Torricelli's principle of hydraulic resistance states that the volume flow rate f of a liquid through a restriction is proportional to the square root of the pressure drop p across the restriction; that is, $f = c\sqrt{p} = cp^{1/2}$. In many applications, the pressure drop is due to the weight of liquid in a container.

In such situations, Torricelli's principle states that the flow rate is proportional to the square root of the height h of the liquid above the orifice. Thus,

$$f = b\sqrt{h} = bh^{1/2}$$

where b is a constant that must be determined from data.

Problem

Determine the best-fit value of the coefficient b in the square-root function

$$f = bh^{1/2}$$

Height h (cm)	11	10	9	8	7	6	5	4	3	2	1
Time t (s)	7	7.5	8	8.5	9	9.5	11	12	14	19	26

Solution

First obtain the flow rate data in ml/s by dividing the 250 ml volume by the time to fill:

$$f = \frac{250}{t}$$

Referring to Example, whose model is $y = bx^m$, we see here that $y = f, h = x, m = 0.5$,
From Equation of that example,

$$b = \frac{\sum_{i=1}^{n} h_i^{0.5} y_i}{\sum_{i=1}^{n} h_i}$$

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```
h = (1:11);
time = [26, 19, 14, 12, 11, 9.5, 9, 8.5, 8, 7.5, 7];
flow = 250./time;
b = sum(sqrt(h).*flow)/sum(h)
f = b*sqrt(h);
J = sum((f - flow).^2)
S = sum((flow - mean(flow)).^2)
r2 = 1 - J/S
```

```
b =
   10.4605
J =
    5.5495
S =
  698.2203
r2 =
    0.9921
```

The result is $b = 10.4604$ and the flow model is $f = 10.4604\sqrt{h}$. The quality-of-fit values are $J = 5.5495$, $S = 698.2203$, and $r^2 = 0.9921$, which indicates a very good fit.

Parameter Estimation using the Free Response of the First Order Model

The free and the step response can be used to estimate one or more of the parameters of a dynamic model. For example, consider the first-order model

$$m\frac{dv}{dt} + cv = f(t)$$

where $f(t)$ is the input. The time constant is $\tau = m/c$, and the free response is

$$v(t) = v(0)e^{-t/\tau}$$

If we take the natural logarithm of both sides we obtain

$$\ln v(t) = \ln v(0) - \frac{t}{\tau}$$

By defining $V(t) = \ln v(t)$, we can transform this equation into the equation of a straight line, as follows:

$$V(t) = V(0) - \frac{t}{\tau}$$

This describes a straight line in terms of $V(t)$ and t. Its slope is $-1/\tau$ and its intercept is $V(0)$. These quantities may be estimated by drawing a straight line through the tranformed data points $[V(t_i), t_i]$ if the scatter in the data is not too large. Otherwise we can use the least-squares method to estimate the parameters.

If the measurement of $v(0)$ is subject to random measurement error, then $V(0)$ is not known precisely, and we can use the least squares method to compute estimates of the coefficients τ and $V(0)$. Using the least-squares equations for a first-order polynomial, we can derive the following equations:

$$-\frac{1}{\tau}\sum_{i=1}^{n} t_i^2 + V(0)\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} V_i t_i$$

$$-\frac{1}{\tau}\sum_{i=1}^{n} t_i + V(0)n = \sum_{i=1}^{n} V_i$$

These are two linear algebraic equations, which can be solved for τ and $V(0)$.

On the other hand, in many applications the starting value $v(0)$ can be measured without significant error. In this case, we can transform the data by using $z(t) = V(t) - V(0)$ so that the model becomes $z(t) = -t/\tau$, which is a linear equation constrained to pass through the origin. We can then use

$$-\frac{1}{\tau}\sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} t_i z_i$$

to find the time constant τ.

Example

The temperature of liquid cooling in a porcelain mug at room temperature (68°F) was measured at various times. The data are given below.

Time t (sec)	Temperature T (°F)
0	145
620	130
2266	103
3482	90

Develop a model of the liquid temperature as a function of time, and use it to estimate how long it takes the temperature to reach 120°F.

Solution

We will model the liquid as a single lumped thermal mass with a representative temperature T. From conservation of heat energy we have

$$\frac{dE}{dt} = -\frac{T - T_o}{R}$$

where E is the heat energy in the liquid, $T_o = 68°F$ is the temperature of the air surrounding the cup, and R is the total thermal resistance of the cup. We have $E = mc_p(T - T_o) = C(T - T_o)$, where m is the liquid mass, c_p is its specific heat, and $C = mc_p$ is the thermal capacitance. Assuming that m, c_p, and T_o are constant, we obtain

$$C\frac{dT}{dt} = -\frac{T - T_o}{R}$$

If we let $\Delta T = T - T_o$ and note that

$$\frac{d(\Delta T)}{dt} = \frac{dT}{dt}$$

we obtain

$$RC\frac{d(\Delta T)}{dt} + \Delta T = 0 \qquad (1)$$

The time constant is $\tau = RC$, and the solution has the form

$$\Delta T(t) = \Delta T(0)e^{-t/\tau}$$

Thus,

$$\ln \Delta T(t) = \ln \Delta T(0) - \frac{t}{\tau} \qquad (2)$$

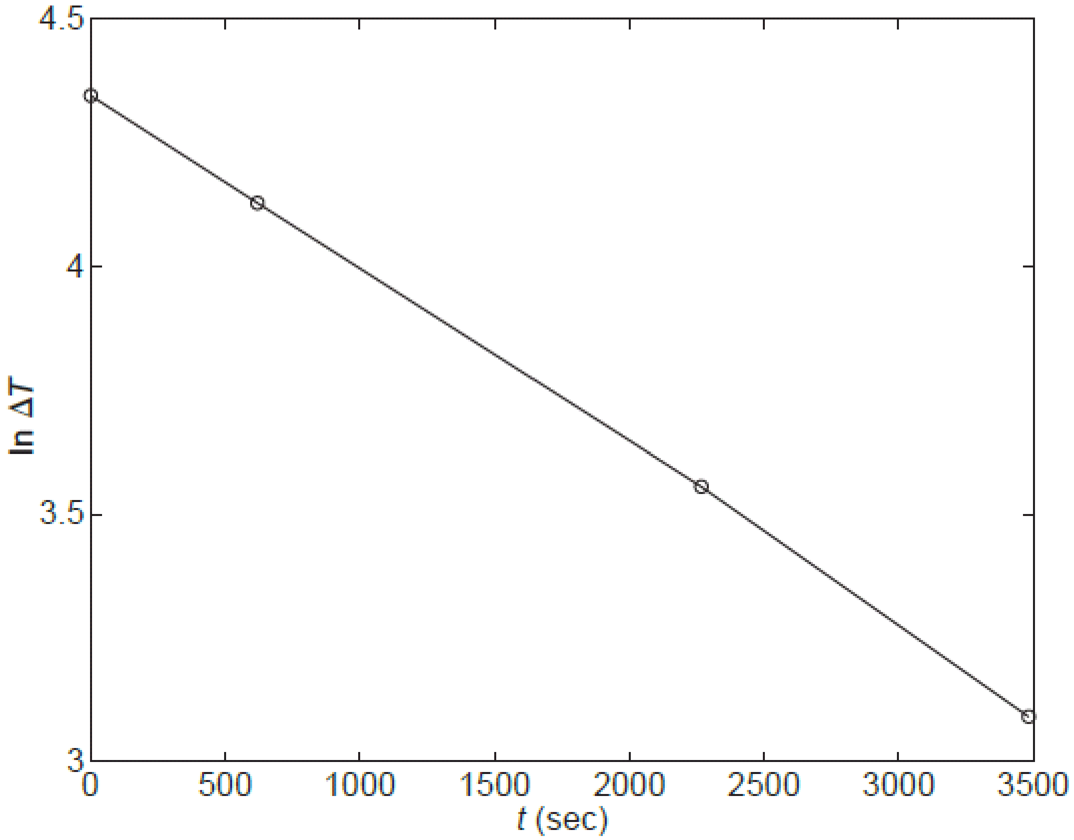
The transformed data $\ln \Delta T(t)$ are plotted in Figure a. Because they fall near a straight line, we can use equation (2) to fit the data. The values obtained are $\ln \Delta T(0) = 4.35$ and $\tau = 2792$ sec. This gives $\Delta T(0) = 77°$F. Thus the model is

$$T(t) = 68 + 77e^{-t/2792} \tag{3}$$

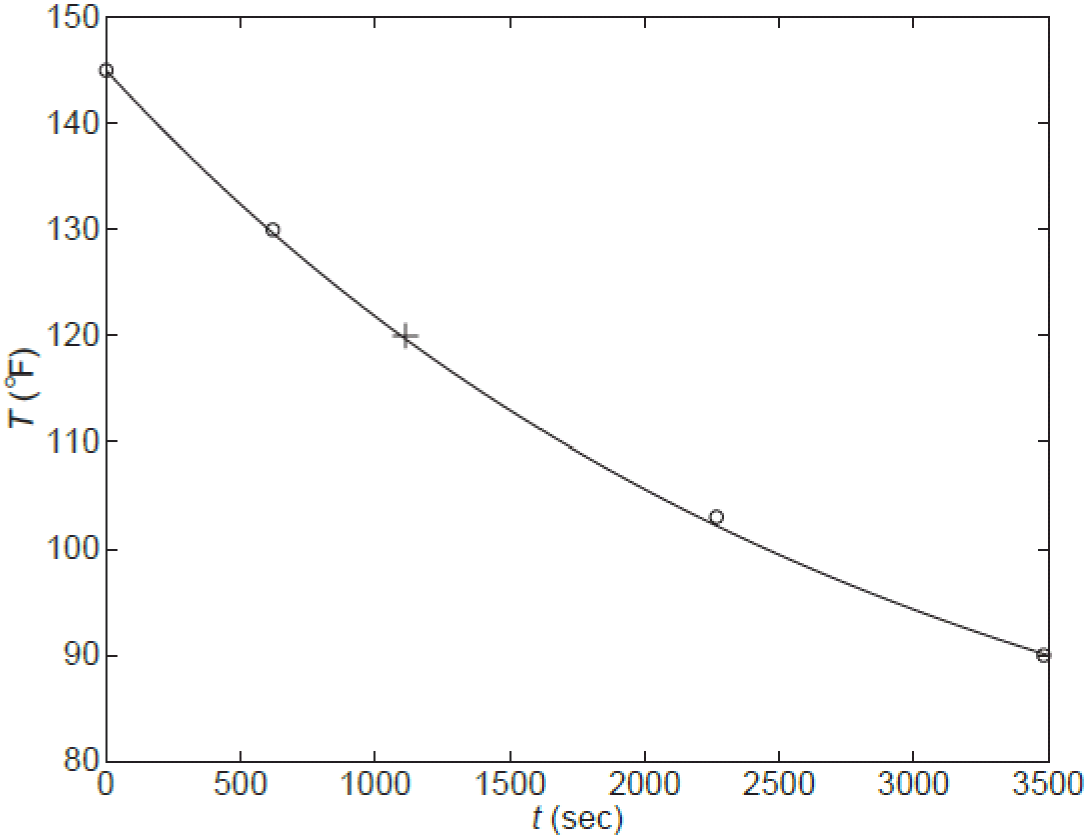
The computed time to reach 120°F is

$$t = -2792 \ln \frac{120 - 68}{77} = 1112 \text{ sec}$$

The plot of equation (3), along with the data and the estimated point (1112, 120) marked with a "+" sign, is shown in part (b) of Figure . Because the graph of our model lies near the data points, we can treat its prediction of 1112 sec with some confidence.


4.5
4
3.5
3
ln ΔT
0
500
1000
1500
2000
2500
3000
3500
t (sec)

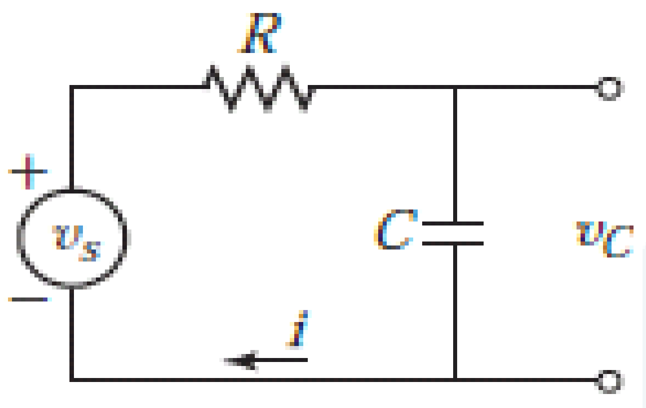
(a)


150
140
130
120
110
100
90
80
T (°F)
0
500
1000
1500
2000
2500
3000
3500
t (sec)

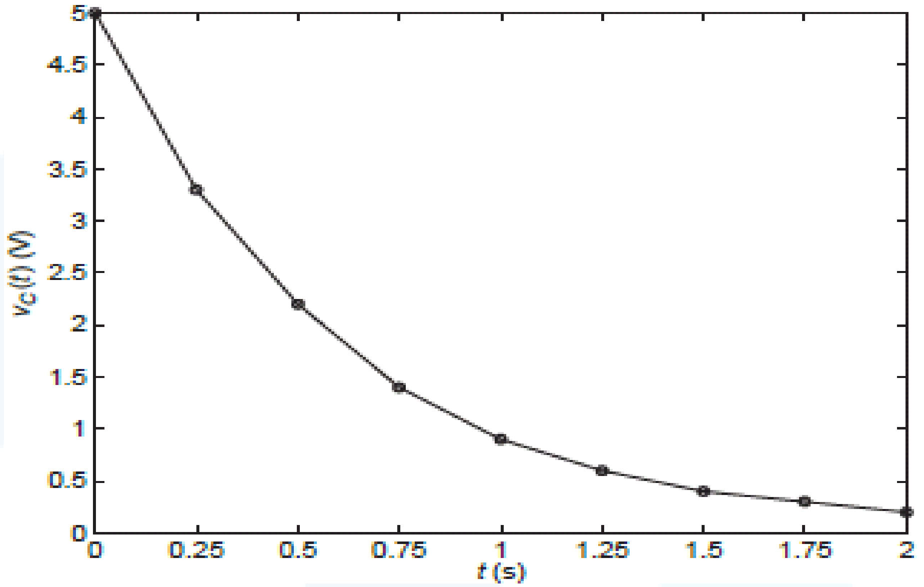
(b)

Example

Commercially available resistors are marked with a color code that indicates their resistance value. Suppose that the resistance in the circuit of Figure is $10^5\ \Omega$. A voltage is applied to the circuit for $t < 0$ and then is suddenly removed at time $t = 0$. The voltage across the capacitor as measured by a data acquisition system is plotted and given in the following table. Use the data to estimate the value of the capacitance C.



Time t (s)	Voltage v_C (V)
0	5
0.25	3.3
0.5	2.2
0.75	1.4
1	0.9
1.25	0.6
1.5	0.4
1.75	0.3
2	0.2



Solution

The circuit model may be derived from Kirchhoff's voltage law, which gives

$$v_s = Ri + v_C \qquad \text{or} \qquad i = \frac{v_s - v_C}{R}$$

For the capacitor we have

$$v_C = \frac{1}{C}\int i\,dt \qquad \text{or} \qquad \frac{dv_C}{dt} = \frac{i}{C}$$

$$\frac{dv_C}{dt} = \frac{v_s - v_C}{RC} \qquad \text{or} \qquad RC\frac{dv_C}{dt} + v_C = v_s$$

The free response has the form

$$v_C(t) = v_C(0)e^{-t/RC} = v_C(0)e^{-t/\tau}$$

where the time constant is $\tau = RC$. Taking the natural logarithm of both sides gives

$$\ln v_C(t) = \ln v_C(0) - \frac{t}{\tau}$$

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When the logarithmic transformation is applied to the original data, we obtain the following table.

$$m\sum_{i=1}^{n} t_i^{\,2} + b\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \ln(vc)_i * t_i$$

$$m\sum_{i=1}^{n} t_i + b*9 = \sum_{i=1}^{n} \ln(vc)_i$$

$$\sum_{i=1}^{n} t_i^{\,2} = 12.75 \quad \sum_{i=1}^{n} t_i = 9 \quad \sum_{i=1}^{n} \ln(vc)_i * t_i = -6.4991 \quad \sum_{i=1}^{n} \ln(vc)_i = -0.4176$$

m= -1.6217 b= 1.575

Time t (s)	$\ln v_C$
0	1.609
0.25	1.194
0.5	0.789
0.75	0.337
1	−0.105
1.25	−0.511
1.5	−0.916
1.75	−1.204
2	−1.609

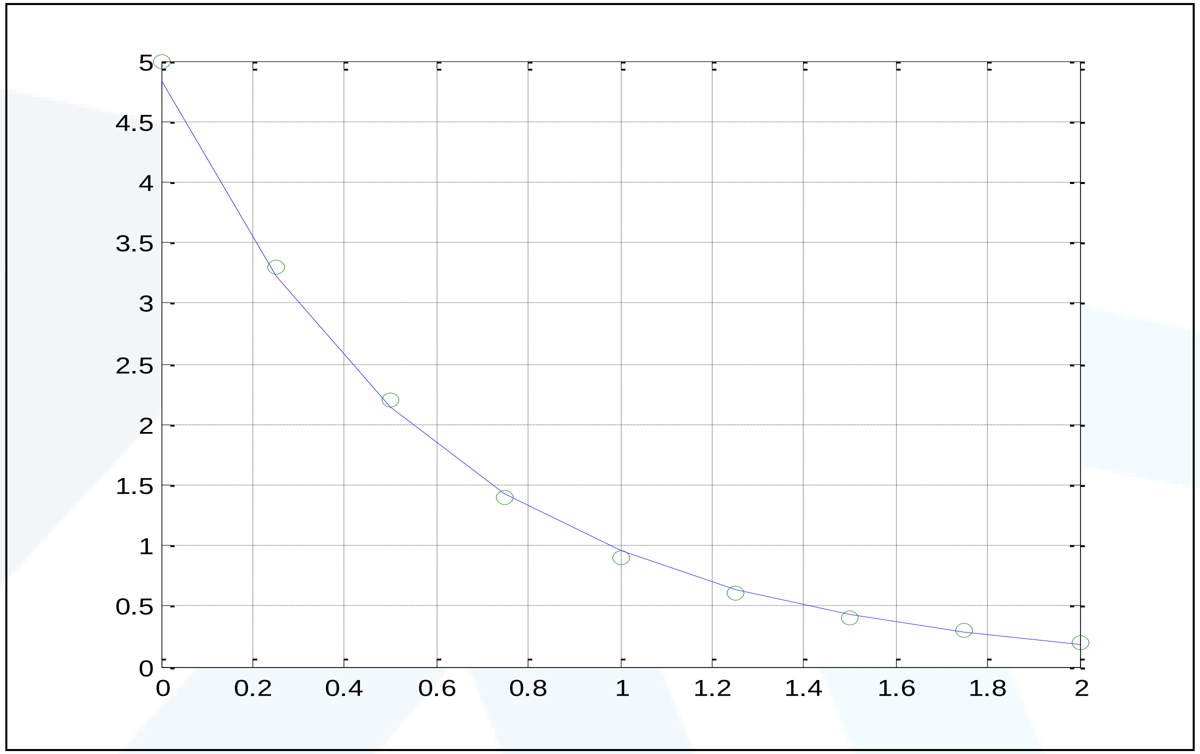
$$vc = \exp(1.575).e^{-1.6217t} = 4.832e^{-1.6127t}$$

$$\tau = 1/1.621 = 0.6166$$

Because we know that $R = 10^5\ \Omega$, we obtain $C = \tau / R = 0.617/10^5 = 6.17 \times 10^{-6}$ F.

```
time = [0:0.25:2];
voltage = [5 3.3 2.2 1.4 0.9 0.6 0.4 0.3 0.2];
logvoltage = log(voltage);
p = polyfit(time, logvoltage , 1);
m = p(1),b = exp(p(2))
T=-1/m;
C=T/(10^5)
vc = b*exp(time*m);
plot(time, vc, time , voltage ,'o'),
grid
J = sum((vc - voltage).^2)
S = sum((voltage - mean(voltage)).^2)
r2 = 1 - J/S
```

```
m =
  -1.6217
b =
   4.8323
C =
  6.1662e-06
J =
   0.0433
S =
  21.4289
r2 =
   0.9980
```



Parameter Estimation using the Step Response of the First Order Model

The free response of the model $m\dot{v} + cv = f(t)$ enables us to estimate τ, but does not give enough information to find both m and c separately. However, we may use the step response if available. The step response for a step input of magnitude F is

$$v(t) = \left[v(0) - \frac{F}{c}\right] e^{-t/\tau} + \frac{F}{c}$$

Assume that we know F and $v(0)$ accurately and that we can measure the step response long enough to estimate accurately the steady-state response v_{ss}. Then we can compute c from the steady-state response $v_{ss} = F/c$; that is, $c = F/v_{ss}$. To estimate m we rearrange the step response as follows, using the fact that $F/c = v_{ss}$.

$$\ln\left[\frac{v(t) - v_{ss}}{v(0) - v_{ss}}\right] = -\frac{t}{\tau}$$

Transform the data $v(t)$ using

$$z(t) = \ln\left[\frac{v(t) - v_{ss}}{v(0) - v_{ss}}\right]$$

This gives the zero-intercept, linear model: $z(t) = -t/\tau$, and we can find τ.
Assuming we have calculated c from the steady-state response, we can find m from $m = c\tau$.

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