Calculus 1

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Calculus 1

Lecture 1

Functions

Chapter 1

Functions

1.1 Functions and Their Graphs

1.2 Some Important Functions

Functions

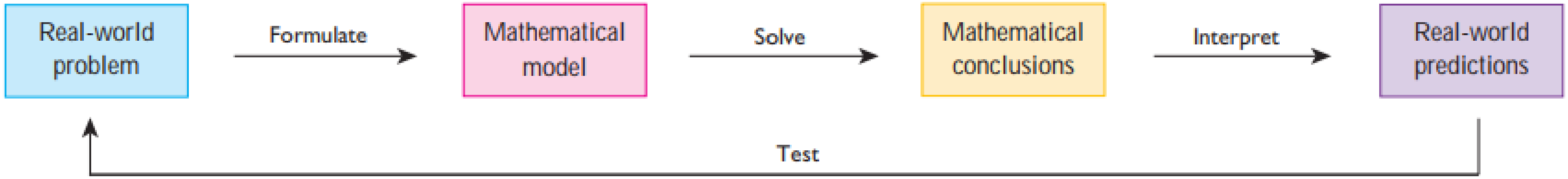
introduction

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say *y*, depends on the value of another variable quantity, which we often call *x*. We say that "*y* is a function of *x*" and write this symbolically as

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f, and y is the **dependent variable** or output value of f at x.



Functions

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D.

FIGURE 1.1 A diagram showing a function as a kind of machine.

It's helpful to think of a function as a **machine** (see Figure 1.1)

Functions

The **domain** of f is the set X.

The number y is the **image** of x under f and is denoted by f(x) , which is called the **value of at** The **range** of is a subset of Y and consists of all images of numbers in (s

f: is a function

x: is element in the domain

$f(x)$: is called the value of the function at x

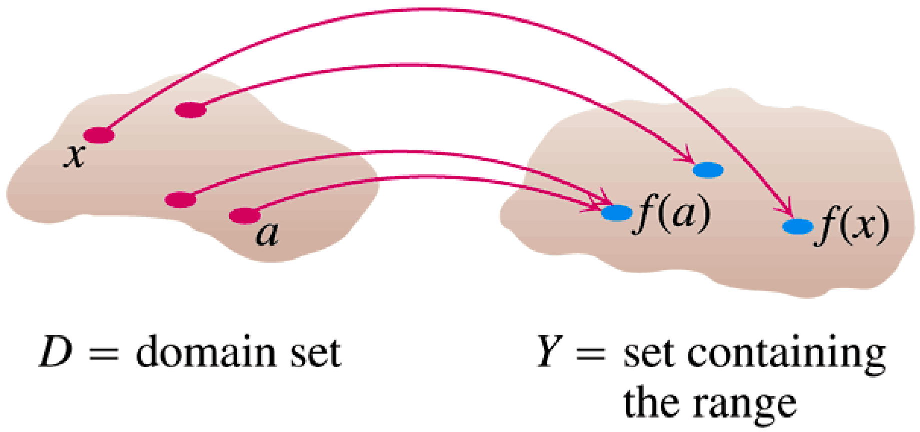


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

Functions

EXAMPLE 3

Evaluating a Function Let f be the function with domain all real numbers x and defined by the formula

$$f(x) = 3x^3 - 4x^2 - 3x + 7.$$

Find $f(2)$ and $f(-2)$.

SOLUTION To find $f(2)$, we substitute 2 for every occurrence of x in the formula for $f(x)$:

$$\begin{aligned} f(2) &= 3(2)^3 - 4(2)^2 - 3(2) + 7 && \text{Substitute 2 for } x. \\ &= 3(8) - 4(4) - 3(2) + 7 && \text{Evaluate exponents.} \\ &= 24 - 16 - 6 + 7 && \text{Multiply.} \\ &= 9. && \text{Add and subtract.} \end{aligned}$$

To find $f(-2)$, we substitute (-2) for each occurrence of x in the formula for $f(x)$. The parentheses ensure that the -2 is substituted correctly. For instance, x^2 must be replaced by $(-2)^2$, not -2^2:

$$\begin{aligned} f(-2) &= 3(-2)^3 - 4(-2)^2 - 3(-2) + 7 && \text{Substitute } (-2) \text{ for } x. \\ &= 3(-8) - 4(4) - 3(-2) + 7 && \text{Evaluate exponents.} \\ &= -24 - 16 + 6 + 7 && \text{Multiply.} \\ &= -27. && \text{Add and subtract.} \end{aligned}$$

Functions

EXAMPLE **Evaluating a Function** If $f(x) = (4 - x)/(x^2 + 3)$, what is (a) $f(a)$? (b) $f(a + 1)$?

SOLUTION (a) Here, a represents some number. To find $f(a)$, we substitute a for x wherever x appears in the formula defining $f(x)$:

$$f(a) = \frac{4 - a}{a^2 + 3}.$$

(b) To evaluate $f(a + 1)$, substitute $a + 1$ for each occurrence of x in the formula for $f(x)$:

$$f(a + 1) = \frac{4 - (a + 1)}{(a + 1)^2 + 3}.$$

We can simplify the expression for $f(a + 1)$ using the fact that $(a + 1)^2 = (a + 1)(a + 1) = a^2 + 2a + 1$:

Expand   Add and Subtract

$$f(a + 1) = \frac{4 - (a + 1)}{(a + 1)^2 + 3} = \frac{4 - a - 1}{a^2 + 2a + 1 + 3} = \frac{3 - a}{a^2 + 2a + 4}.$$

«

Functions

EXAMPLE 7 **Domains of Functions** Find the domains of the following functions:

(a) $f(x) = \sqrt{4 + x}$ (b) $g(x) = \dfrac{1}{\sqrt{1 + 2x}}$ (c) $h(x) = \sqrt{1 + x} - \sqrt{1 - x}$

SOLUTION (a) Since we cannot take the square root of a negative number, we must have $4 + x \geq 0$, or equivalently, $x \geq -4$. So the domain of f is $[-4, \infty)$.

(b) Here, the domain consists of all x for which

$$1 + 2x > 0$$

$$2x > -1 \qquad \text{Subtract 1 from both sides.}$$

$$x > -\frac{1}{2} \qquad \text{Divide both sides by 2.}$$

The domain is the open interval $\left(-\frac{1}{2}, \infty\right)$.

(c) In order to be able to evaluate both square roots that appear in the expression of $h(x)$, we must have

$$1 + x \geq 0 \quad \text{and} \quad 1 - x \geq 0.$$

The first inequality is equivalent to $x \geq -1$, and the second inequality to $x \leq 1$. Since x must satisfy both inequalities, it follows that the domain of h consists of the closed interval $[-1, 1]$.

Functions; Domain and Range

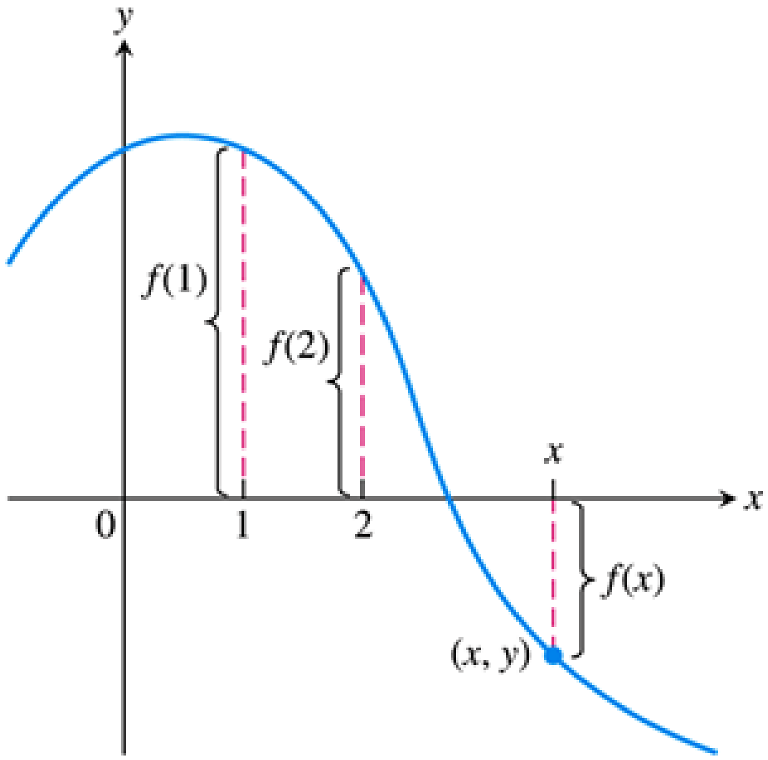
EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Graphs of Functions

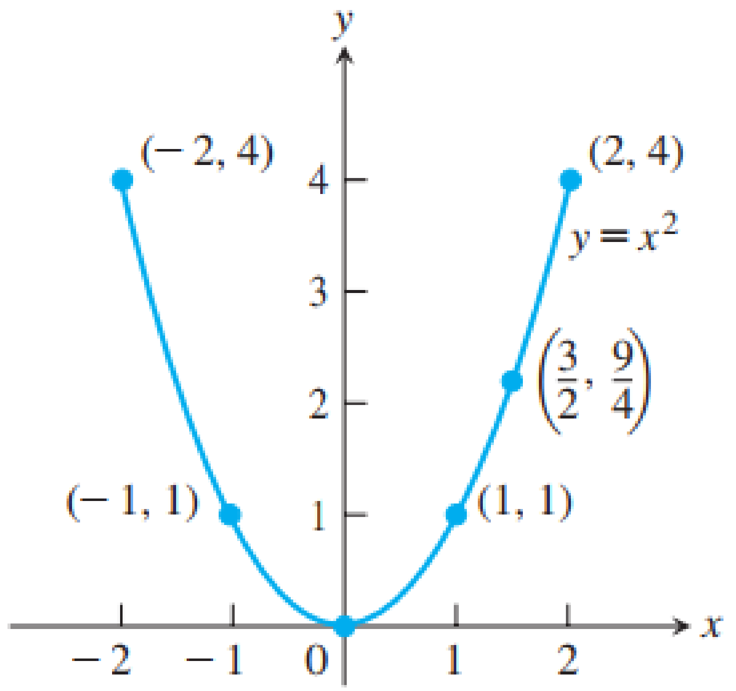
If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$



EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



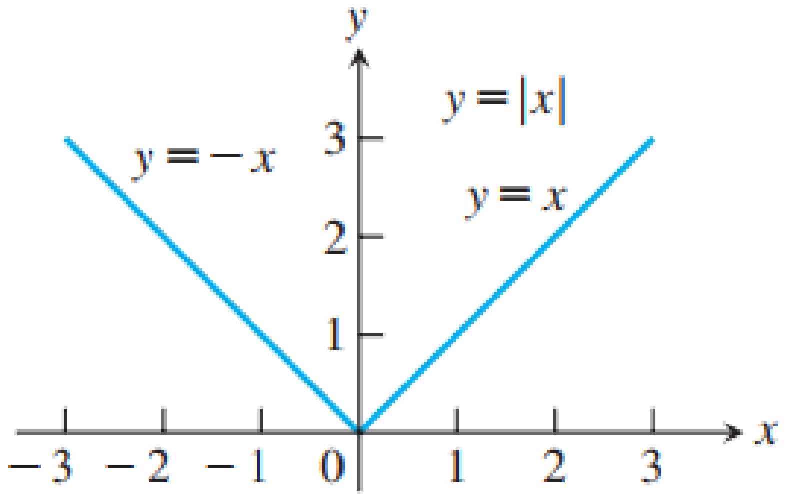
Graphs of Functions

One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

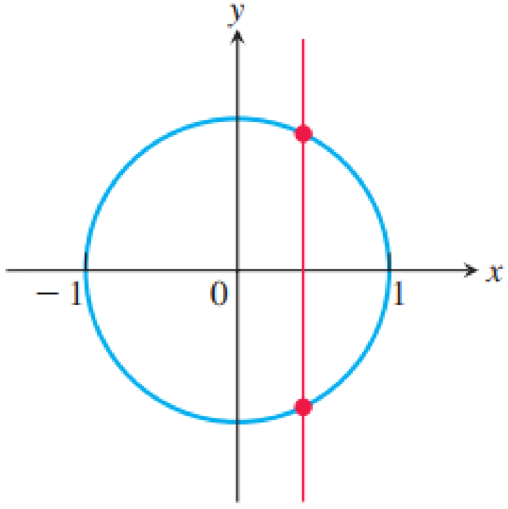
First formula

Second formula

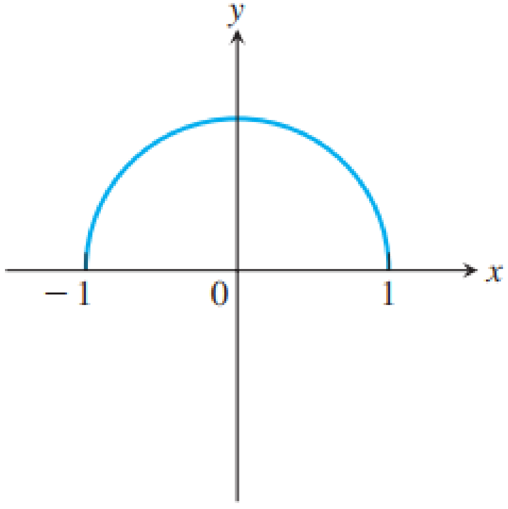


Graphs of Functions

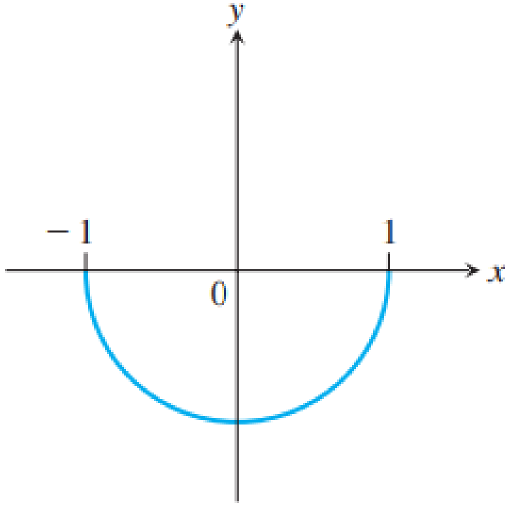
(a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.



(a) $x^2 + y^2 = 1$



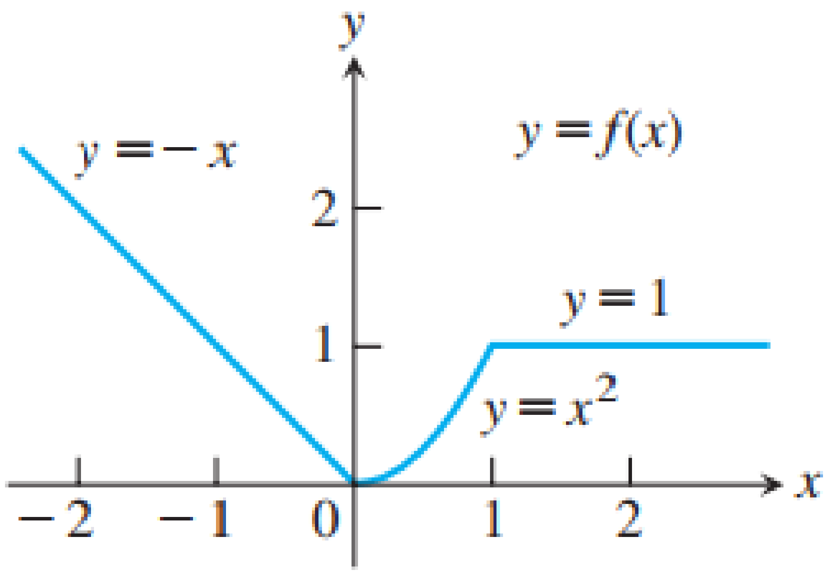
(b) $y = \sqrt{1 - x^2}$



(c) $y = -\sqrt{1 - x^2}$

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 \quad \text{First formula} \\ x^2, & 0 \le x \le 1 \quad \text{Second formula} \\ 1, & x > 1 \quad \text{Third formula} \end{cases}$$



Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is ***increasing***. If the graph descends or falls as you move from left to right, the function is ***decreasing*****.**

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.

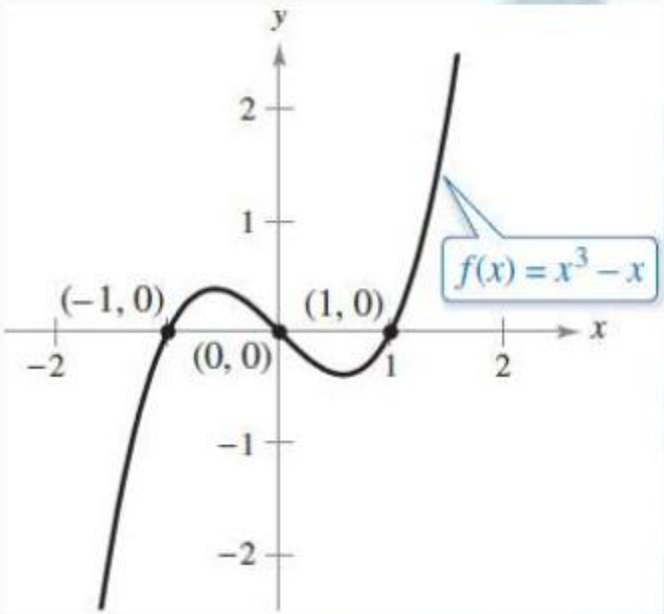
1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Even Functions and Odd Functions: Symmetry

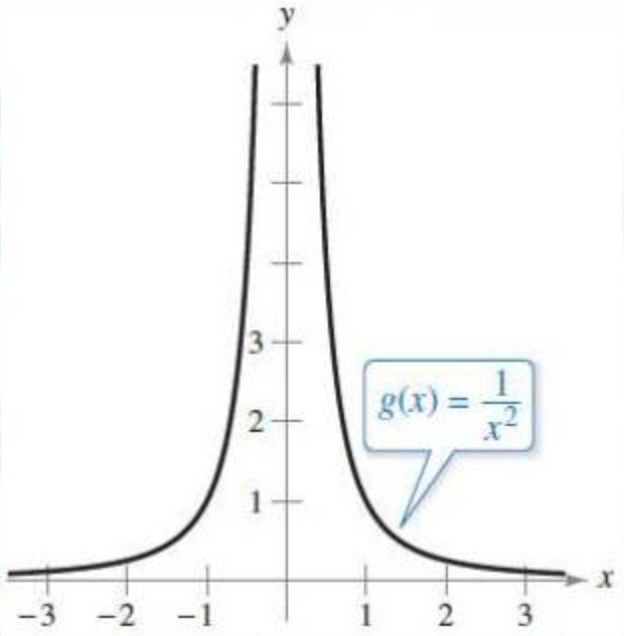
DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

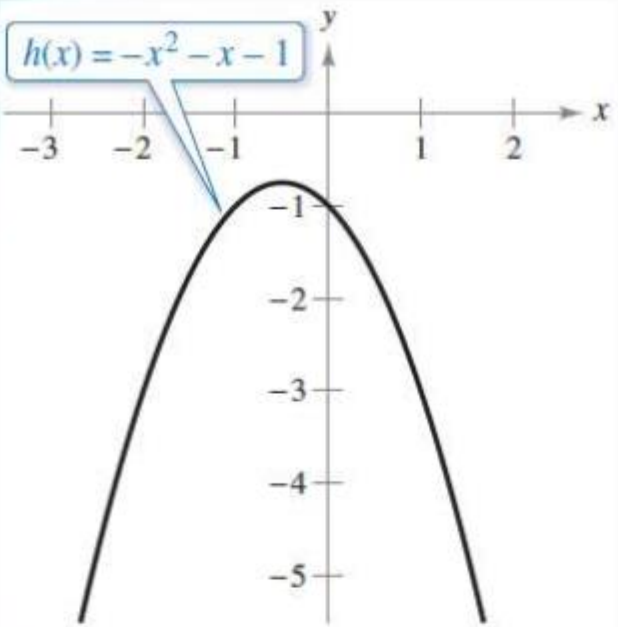
for every x in the function's domain.



Odd function



Even function



Neither even nor odd

Even Functions and Odd Functions: Symmetry

EXAMPLE 8 Here are several functions illustrating the definitions.

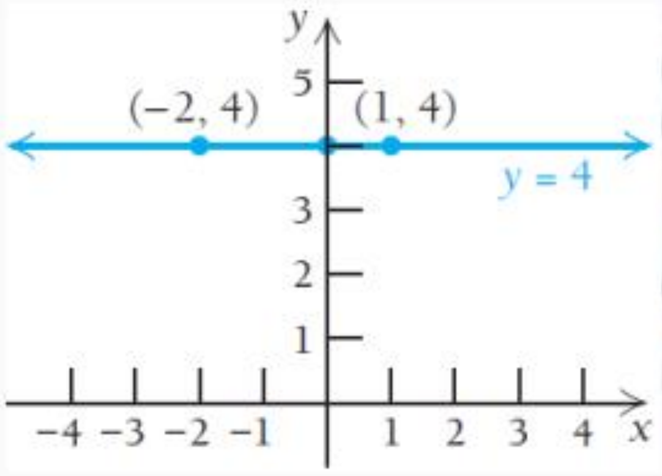
$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis (Figure 1.13a).
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of an odd function.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.
	Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

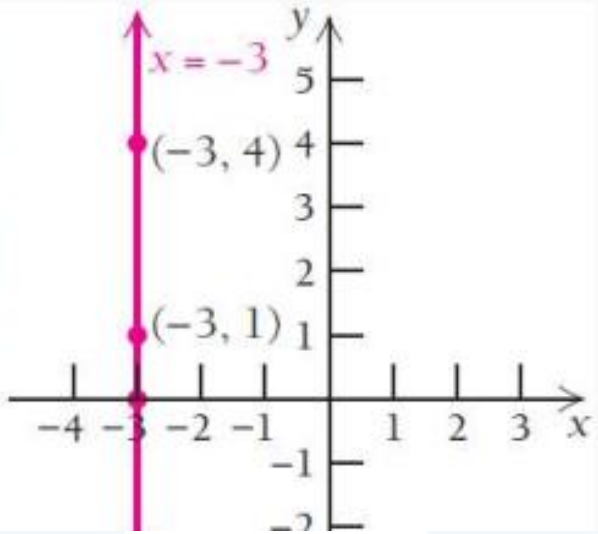
Common Functions

Linear Functions

Horizontal and Vertical Lines

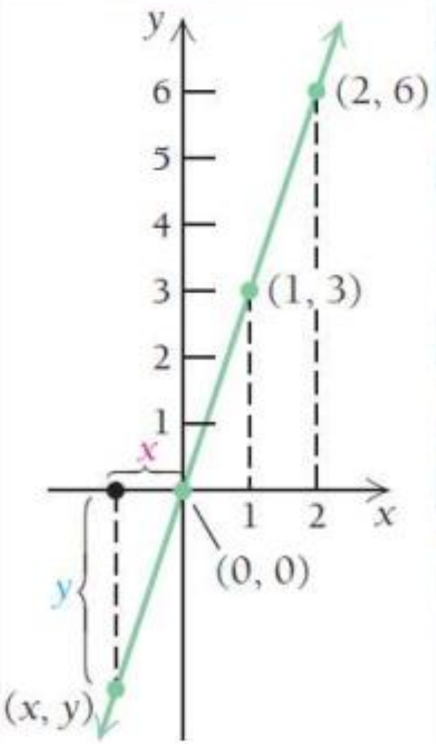
The graph of $y = c$, or $f(x) = c$, a horizontal line, is the graph of a function. Such a function is referred to as a constant function. The graph of $x = a$ is a vertical line, and $x = a$ is not a function

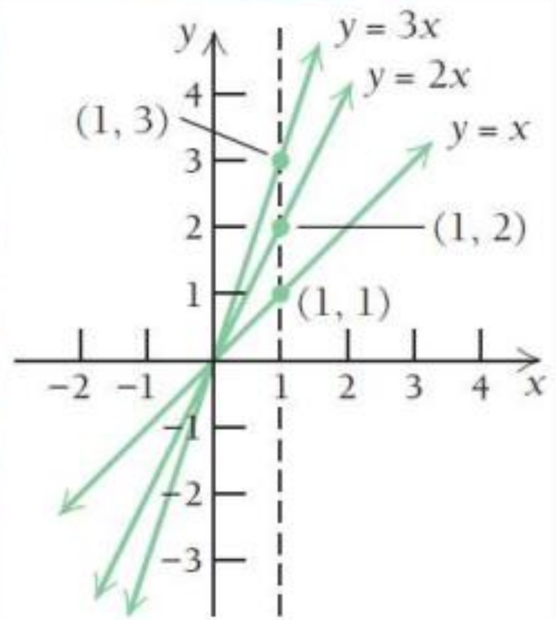




The Equation $y = mx$

The graph of the function given by $y = mx$ or $f(x) = mx$ is the straight line through the origin (0, 0) and the point (1, m). The constant m is called the **slope** of the line. We also say that y is directly proportional to x





Common Functions

A **linear function** is given by $y = mx + b$ or $f(x) = mx + b$

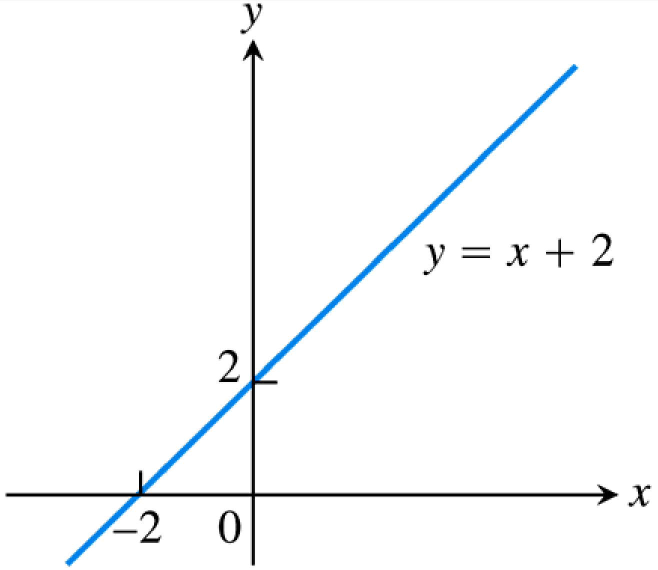
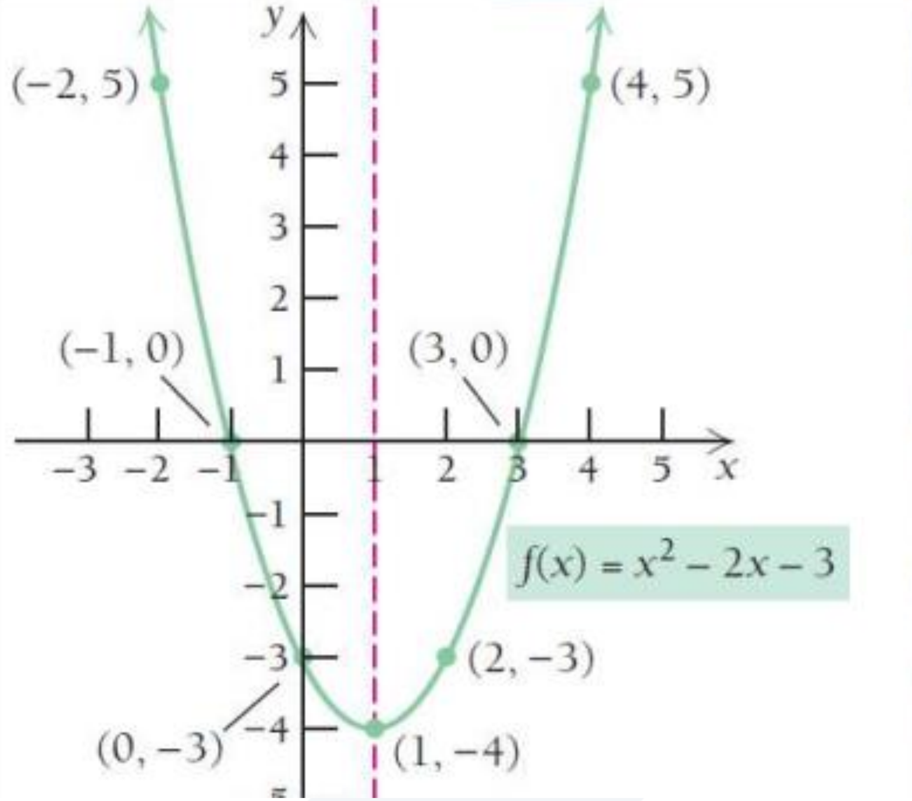


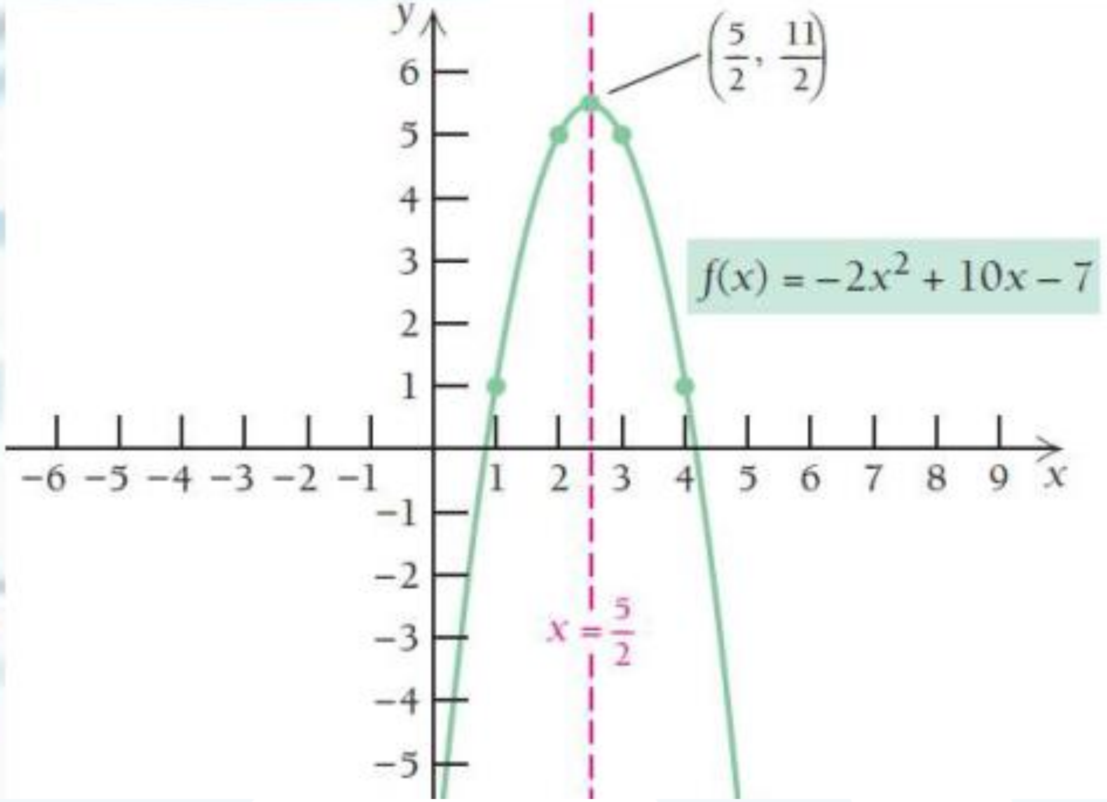
FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

Non Linear Functions

Quadratic Functions

Definition: A quadratic function f is given by $f(x) = ax^2 + bx + c, \quad \text{where } a \neq 0$





Polynomial Functions

Definition: A polynomial function f is given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer and $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, called the **coefficients**

$f(x) = -5$	(Constant function)
$f(x) = 4x + 3$	(Linear function)
$f(x) = -x^2 + 2x + 3$	(Quadratic function)
$f(x) = 2x^3 - 4x^2 + x + 1$	(Cubic function)

Common Functions

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

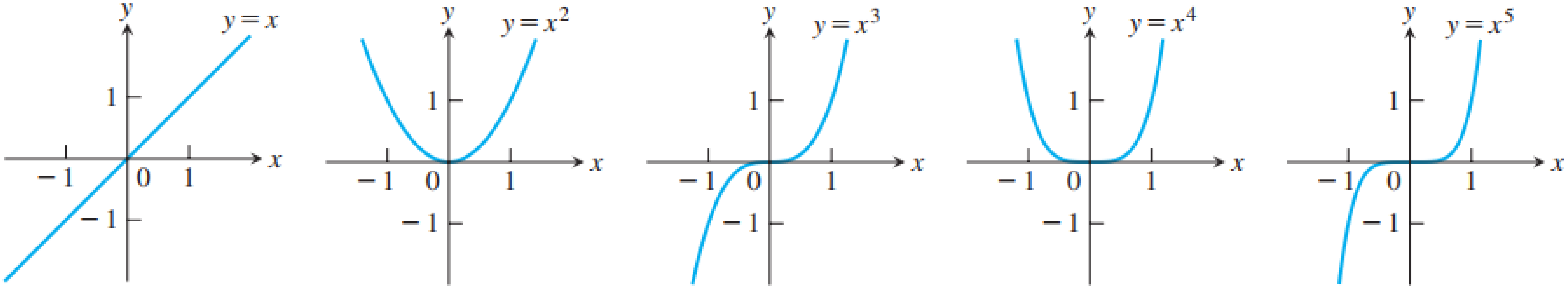


FIGURE Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

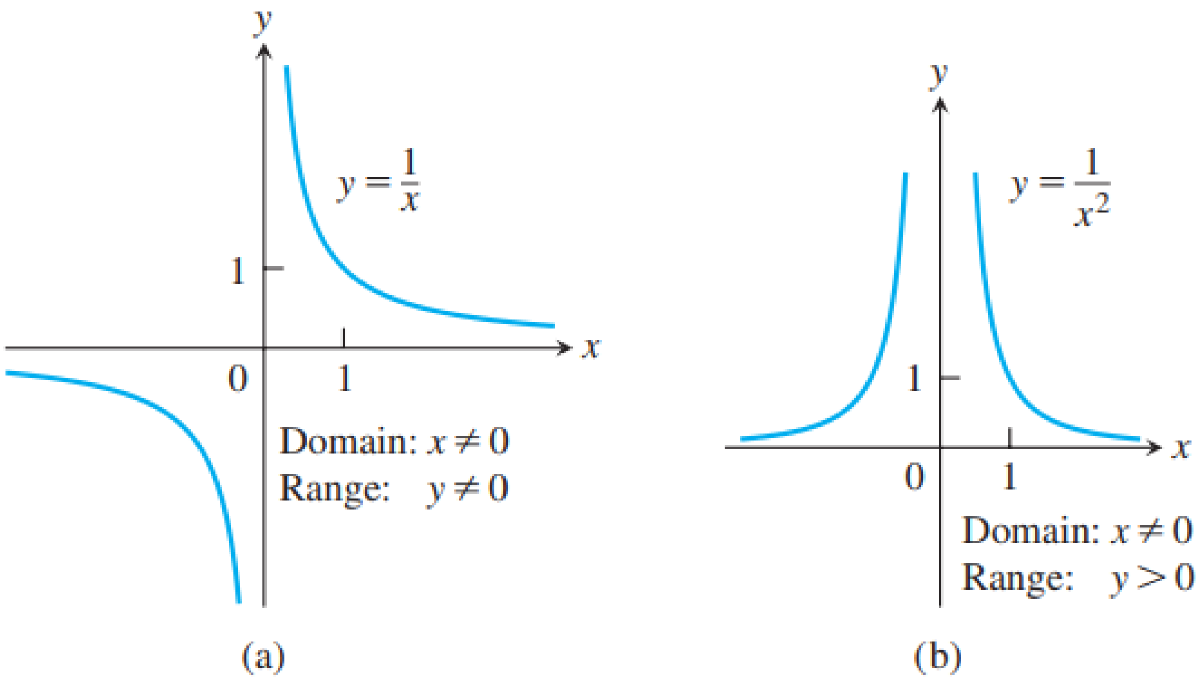


FIGURE Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

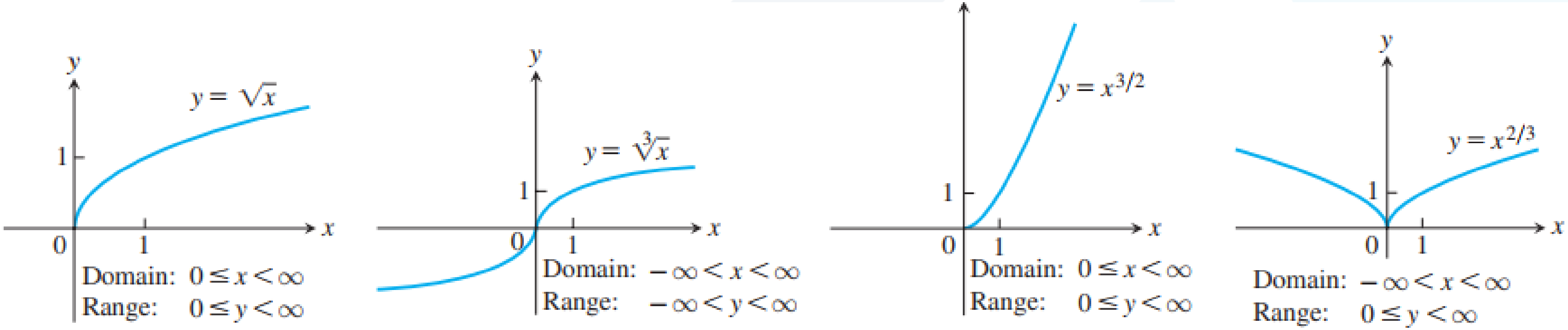
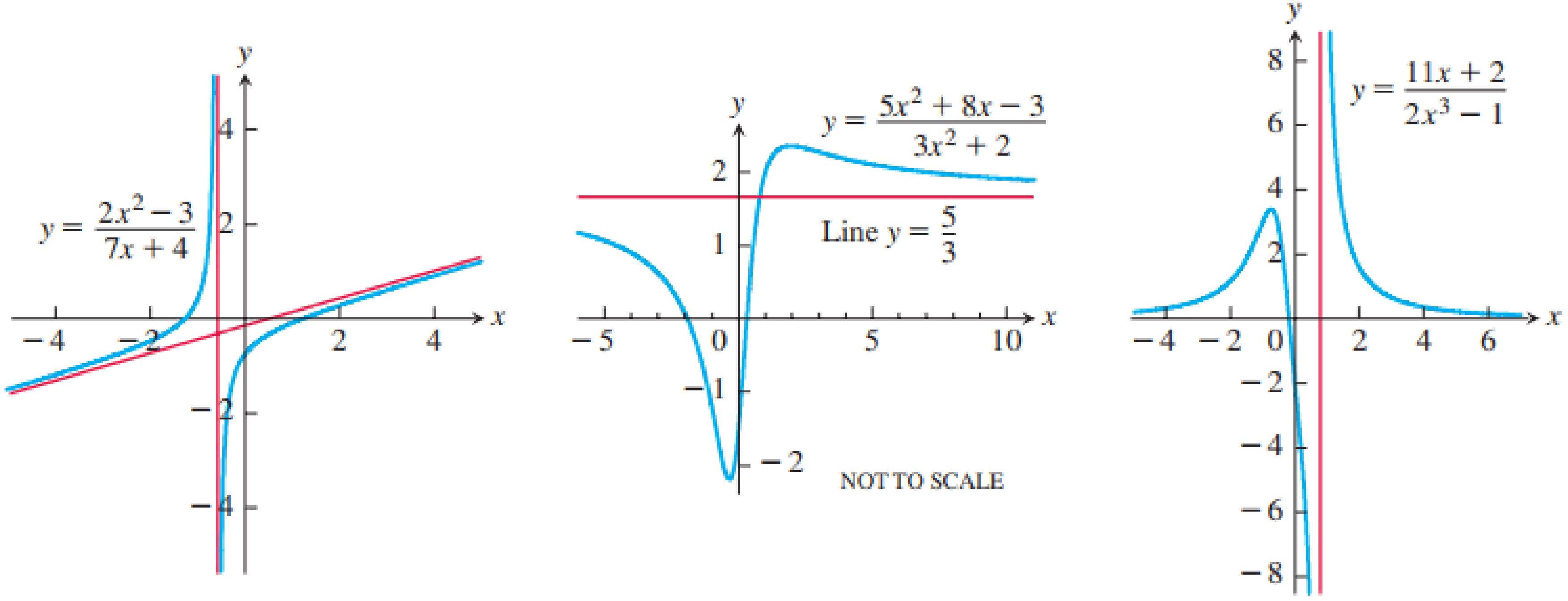


FIGURE Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

Common Functions

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are shown in Figure



(a) (b) (c)

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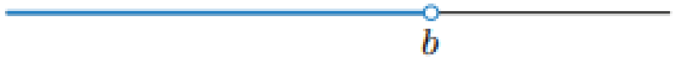
Inequality	Geometric Description	Interval Notation
$a \le x \le b$		$[a, b]$
$a < x < b$		(a, b)
$a \le x < b$		$[a, b)$
$a < x \le b$		$(a, b]$
$a \le x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \le b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

Exercices

Find the domain of each function

$$f(x)=\frac{7}{2x-10} \qquad g(x)=\sqrt{x+6}$$

$$h(x)=\sqrt{x^2+x+10} \qquad f(x)=\frac{x+3}{4-\sqrt{x^2-9}}$$

Find the domain and range of each function

$$f(x)=x^2+3 \qquad f(x)=\sqrt{6-x} \qquad g(x)=2+\sqrt{9+x}$$

$$f(x)=-|x+1| \qquad h(x)=\frac{2}{x+1}$$

Consider the function given by $f(x)=\begin{cases}-x^2+2, & \text{for } x<1,\\ 4, & \text{for } 1<x\le 2,\\ \frac{1}{2}x, & \text{for } x\ge 2\end{cases}$

a) Find *f*(–1), *f*(1.5), and *f*(6)

b) Graph the function

Graph the following functions

$$f(x) = -\tfrac{1}{2}x + 3 \qquad g(x) = \sqrt{x} + 1$$

$$f(x) = -x^2 + 4 \qquad h(x) = |x - 4| - 4$$

Graph the following functions

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases} \qquad f(x) = \begin{cases} 4 - x^2, & x < 0 \\ 2 - x, & 0 \le x \end{cases}$$

Determine whether the function is even, odd, or neither

$$f(x) = x^4 - x^2, \qquad g(x) = \sqrt{x^3 + 1}$$

$$h(x) = x^4 + 3x^2 - 1, \qquad g(t) = 2|t| + 1$$

$$f(x) = \frac{1}{x^2 - 1}, \qquad f(x) = \frac{x}{x^2 - 1}$$

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \dfrac{4}{3 - t}$
6. $G(t) = \dfrac{2}{t^2 - 16}$

15. $f(x) = 5 - 2x$
16. $f(x) = 1 - 2x - x^2$
17. $g(x) = \sqrt{|x|}$
18. $g(x) = \sqrt{-x}$
19. $F(t) = t/|t|$
20. $G(t) = 1/|t|$

Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

26. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ x^2 + 2x, & x > 1 \end{cases}$

28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$

48. $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$

50. $f(x) = x^2 + x$

51. $g(x) = x^3 + x$

52. $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$

54. $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$

56. $h(t) = |t^3|$

57. $h(t) = 2t + 1$

58. $h(t) = 2|t| + 1$

59. $\sin 2x$

60. $\sin x^2$

61. $\cos 3x$

62. $1 + \cos x$

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Thank you for your attention