Calculus 2

Dr. Yamar Hamwi

Al-Manara University

2024-2025

Calculus 2

Lecture 2

Improper Integrals

Improper Integrals

- **Evaluate an improper integral that has an infinite limit of integration.**
- **Evaluate an improper integral that has an infinite discontinuity.**
- **Tests for Convergence and Divergence**

Quick Review

If f is continuous on the interval $[a,b]$ and F is any function that satisfies $F'(x) = f(x)$ throughout this interval then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

REMEMBER: [a,b] is a closed interval

introduction

Up to now we have focused on **definite integrals** with **continuous integrands** and **finite intervals of integration**.

we extend the concept of a definite integral to the cases where:

- The interval is infinite
- f has an infinite discontinuity in $[a, b]$

These are called **improper integrals**

INTEGRALS OF TYPE 1: INFINITE INTERVALS

Recall in the definition of $\int_a^b f(x)dx,$

the interval [*a*, *b*] was finite. If *a* or *b* (or both) are ∞ or −∞, we call the integral an **improper integral of type 1 with an infinite interval**. **For example**:

- $\int_0^\infty \frac{dx}{1+x^2}$
- $\int_{-\infty}^{-1} xe^{-x^2} dx$
- $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$

Improper integral: type I:

1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2) If $f(x)$ is continuous on $[-\infty, b)$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

If the Limit is finite, then the Improper Integral converges.

If the Limit fails, then it diverges

3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

If both Improper Integrals converge,

then so does

Evaluating an Improper Integral on [1,∞)

Does the improper integral $\int_0^\infty e^{-x}\,dx$ converge or diverge?

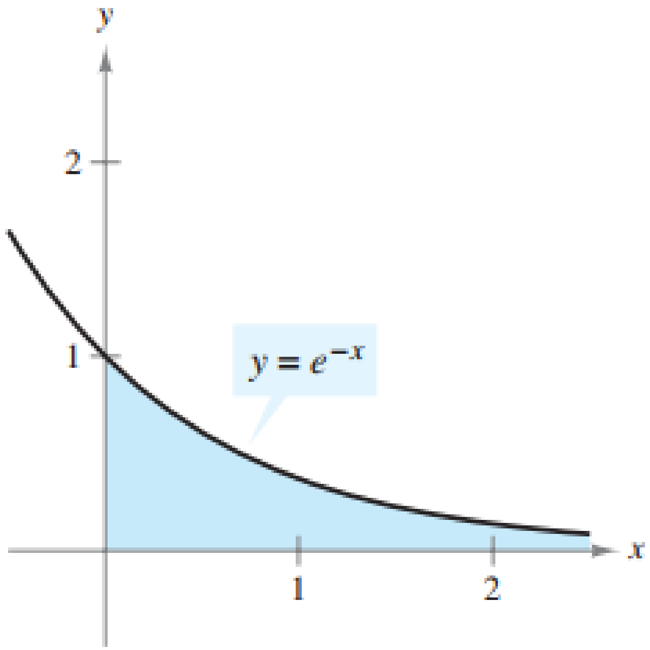
$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left(-e^{-x}\Big|_0^b\right)$$

$$= \lim_{b\to\infty}\left(-e^{-b} + e^{-0}\right)$$

$$= \lim_{b\to\infty}\left(1 - e^{-b}\right)$$

$$= 1$$



Example Using L'Hôpital's Rule with Improper Integrals

Evaluate $\int_1^{\infty} xe^{-x}dx$.

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_1^{b} xe^{-x}\,dx$$

$u = x$	$dv = e^{-x}\,dx$
$du = 1\,dx$	$v = -e^{-x}$

$$= \lim_{b\to\infty}\left(-xe^{-x}\Big|_1^b - \int_1^b -e^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(\left(-xe^{-x} - e^{-x}\right)\Big|_1^b\right)$$

$$= \lim_{b\to\infty}\left(\left(-be^{-b} - e^{-b}\right) - \left(-1\cdot e^{-1} - e^{-1}\right)\right)$$

$$= \lim_{b\to\infty}\left(-e^{-b}(b+1) + 2e^{-1}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{(b+1)}{e^b}\right) + \frac{2}{e}$$

Use L'Hôpital's Rule

$$= \lim_{b\to\infty}\left(-\frac{1}{e^b}\right) + \frac{2}{e}$$

$$= \frac{2}{e}$$

Express the Example Evaluating an Integral on (- ∞,∞)

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

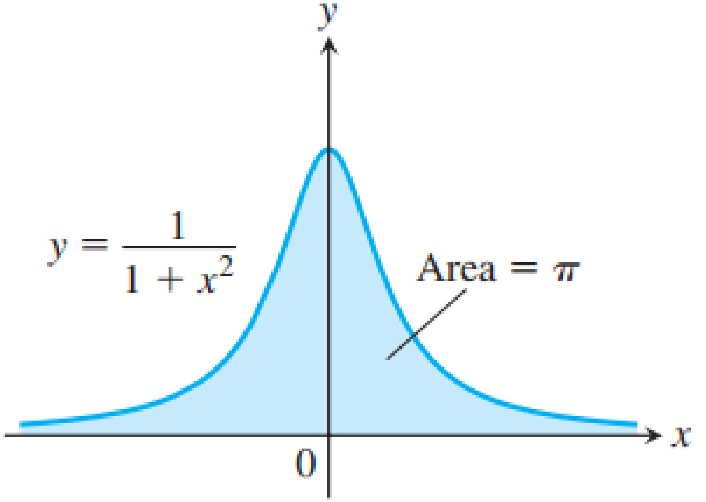
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \left(\arctan x\Big|_a^0\right) + \lim_{b\to\infty} \left(\arctan x\Big|_0^b\right)$$

$$= \lim_{a\to-\infty} \left(\arctan 0 - \arctan a\right) + \lim_{b\to\infty} \left(\arctan b - \arctan 0\right)$$

$$= \left(0 - -\frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$



IMPROPER INTEGRALS OF TYPE 2: INFINITE INTEGRANDS

Recall in the definition of $\int_a^b f(x)dx,$

the functions f was bounded on [a, b]. If f is <u>not bounded</u> on [a, b] (that is, has an x-value, a ≤ x ≤ b, where the limit is ∞ or −∞, we call the integral an **improper integral of type 2 with infinite integrand.** For example

$$\int_0^1 \frac{dx}{x^2}, \quad \int_{-2}^0 \frac{dx}{x^2}, \quad \int_{-2}^1 \frac{dx}{x^2}$$

Improper integral: type 2:

1) If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

2) If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$

If the Limit is finite, then the Improper Integral converges.

If the Limit fails, then it diverges

3) If $f(x)$ is continuous on $[a, c)\cup(c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If both Improper Integrals converge, then so does $\int_a^b f(x)\,dx$.

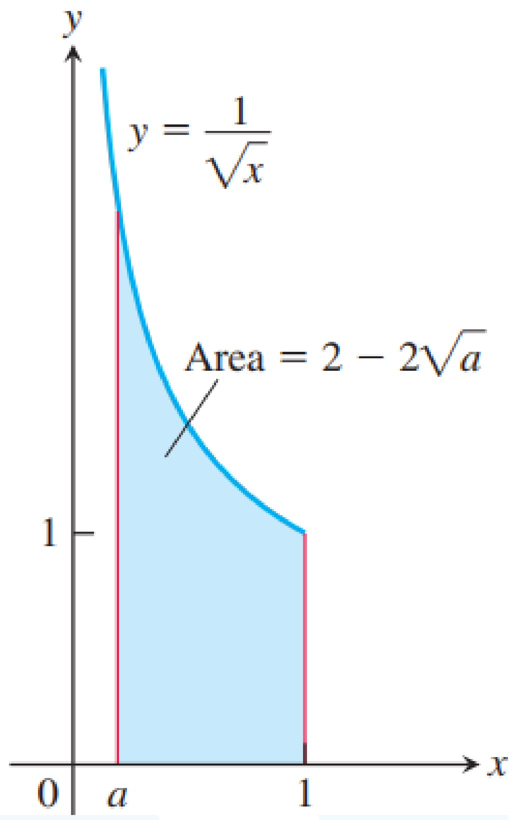
Example Infinite Discontinuity at an Interior Point

Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{c\to 0^+}\left(2\sqrt{x}\Big|_c^1\right)$$

$$= \lim_{c\to 0^+}\left(2\sqrt{1}-2\sqrt{c}\right) = 2$$



Example Infinite Discontinuity at an Interior Point

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$.

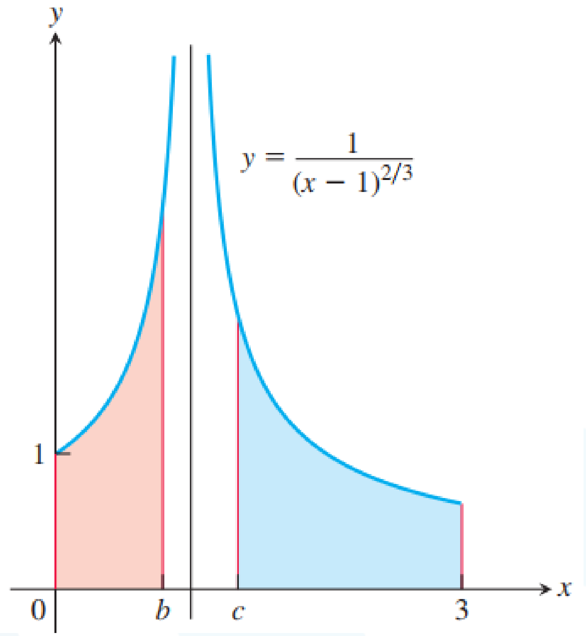
$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{c\to 1^-}\int_0^c \frac{1}{(x-1)^{2/3}}\,dx + \lim_{c\to 1^+}\int_c^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{c\to 1^-}\left(3(x-1)^{1/3}\Big|_0^c\right) + \lim_{c\to 1^+}\left(3(x-1)^{1/3}\Big|_c^3\right)$$

$$= \lim_{c\to 1^-}\left(3(c-1)^{1/3} - 3(0-1)^{1/3}\right) + \lim_{c\to 1^+}\left(3(3-1)^{1/3} - 3(c-1)^{1/3}\right)$$

$$= (0+3) + \left(3\sqrt[3]{2} - 0\right) = 3 + 3\sqrt[3]{2}$$



Example : Improper integral

$$\int_1^{\infty} e^{-x}dx$$ ← **Converges**

$$\lim_{b\to\infty}\int_1^{b} e^{-x}dx$$

$$\lim_{b\to\infty} -e^{-x}\Big|_1^{b}$$

$$\lim_{b\to\infty} -e^{-b}-\left(-e^{-1}\right)$$

$$\lim_{b\to\infty} -\frac{1}{e^{b}}+\frac{1}{e} = \frac{1}{e}$$

($\frac{1}{e^b} \to 0$)

Example : Improper integral

Does $\int_1^{\infty} e^{-x^2}\,dx$ converge?

Compare:

$$\frac{1}{e^{x^2}} \text{ to } \frac{1}{e^x}$$ for positive values of x.

For $x > 1,\ e^{x^2} > e^x \quad \therefore \quad \frac{1}{e^{x^2}} < \frac{1}{e^x}$

Since $\frac{1}{e^x}$ converges to a finite number, $\frac{1}{e^{x^2}}$ must also converge!

Comparison Test (Using estimation to show convergence or divergence)

Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$, then:

① $\int_a^{\infty} f(x)\, dx$ converges if $\int_a^{\infty} g(x)\, dx$ converges.

② $\int_a^{\infty} g(x)\, dx$ diverges if $\int_a^{\infty} f(x)\, dx$ diverges.

Using estimation to show convergence or divergence

Example $$\int_1^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

Solution:

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty)$$

Since $\frac{1}{x^2}$ converges, $\frac{\sin^2 x}{x^2}$ converges.

Using estimation to show convergence or divergence

Example $$\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}}\, dx$$

Solution:

$\sqrt{x^2 - 0.1} < x$ for positive values of x, so:

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on} \quad [1, \infty)$$

Since $\frac{1}{x}$ diverges, $\frac{1}{\sqrt{x^2 - 0.1}}$ diverges.

Thank you for your attention