



Calculus 2

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Calculus 2

Lecture 2

Exercices 2
Improper Integrals



Improper Integrals

The integral $\int_{-1}^1 \frac{dx}{x^3}$ diverges.

Trying to compute that integral by the Fundamental Theorem of Calculus, one gets

$$\int_{-1}^1 \frac{dx}{x^3} = \frac{1}{-2} x^{-2} \Big|_{-1}^1 = \frac{1}{2} \left(\frac{1}{2} \right) = 0.$$

This is an incorrect computation.

Evaluating an integral on

$(-\infty, \infty)$

$$\int_{-\infty}^{\infty} e^{-|x|} dx$$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|x|} dx &= \int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx \\ &= \lim_{a \rightarrow -\infty} \int_a^0 e^x dx + \lim_{a \rightarrow \infty} \int_0^a e^{-x} dx \\ &= \lim_{a \rightarrow -\infty} (1 - e^a) + \lim_{a \rightarrow \infty} (-e^{-a} + 1) \\ &= 1 + 1 = 2\end{aligned}$$

Evaluating an integral on $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

Solution

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\eta \rightarrow 1} \int_0^{\eta} \frac{dx}{\sqrt{1-x^2}} = \lim_{\eta \rightarrow 1} \arcsin(\eta) = \frac{\pi}{2}$$

For what values of p is the integral $\int_1^{+\infty} \frac{dx}{x^p}$ convergent?

Solution

$$\int_1^{\infty} \frac{dx}{x^P} \quad P > 0$$

(P is a constant.)

$$\int_1^{\infty} x^{-P} dx$$

$$\lim_{b \rightarrow \infty} \int_1^b x^{-P} dx$$

$$\lim_{b \rightarrow \infty} \frac{1}{-P+1} x^{-P+1} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1}$$

What happens here?

If $P < 1$ then b^{-P+1} gets bigger and bigger as $b \rightarrow \infty$ therefore the integral diverges.

If $P > 1$ then b has a negative exponent and $b^{-P+1} \rightarrow 0$ therefore the integral converges.

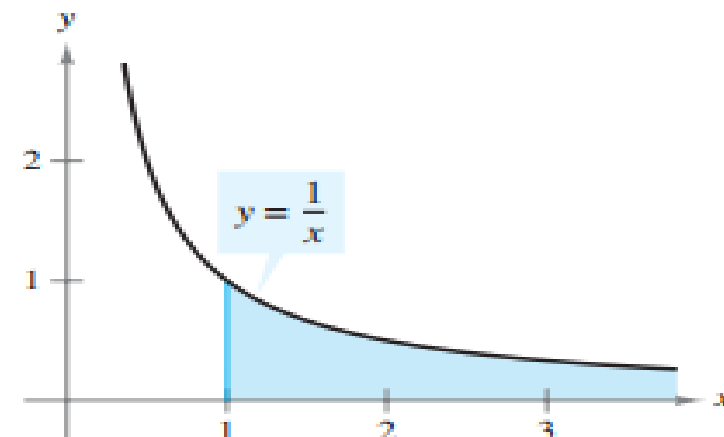
When $p \neq 1$,

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \rightarrow \infty} \frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \begin{cases} +\infty & p < 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

When $p = 1$,

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln x \Big|_1^b = \lim_{b \rightarrow \infty} (\ln b - \ln 1) = +\infty$$

Thus, the integral diverges.



Diverges (infinite area)

evaluate the integral or state that it diverges.

$$\int_0^{+\infty} \frac{dx}{e^{-x} + e^x}$$

Solution

$$\int_0^{+\infty} \frac{dx}{e^{-x} + e^x} = \int_1^{+\infty} \frac{dt}{t \left[\frac{1}{t} + t \right]} = \int_1^{+\infty} \frac{dt}{1+t^2} =$$

$$= \lim_{B \rightarrow +\infty} \int_1^B \frac{dt}{1+t^2} = \lim_{B \rightarrow +\infty} [\arctgt]_1^B =$$

$$= \lim_{B \rightarrow +\infty} [\arctg B - \arctg 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\int_0^{+\infty} \frac{dx}{e^{-x} + e^x} = \frac{\pi}{4} \text{ هذا يعني أن التكامل يتقارب ويكون:}$$

$$\frac{1}{e^{-x} + e^x} = \frac{e^x}{1 + e^{2x}}$$

وبفرض

$$e^x = t$$

يكون

evaluate the integral or state that it diverges.

$$\int_0^2 \frac{dx}{1-x^2}$$

Solution

$\int_0^2 \frac{dx}{1-x^2}$ has an infinite discontinuity at $x = 1$

$$= \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \rightarrow 1^-} \int_0^c \frac{dx}{1-x^2} + \lim_{c \rightarrow 1^+} \int_c^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \rightarrow 1^-} \left(-\frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) \right)_0^c + \lim_{c \rightarrow 1^+} \left(-\frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) \right)_c^2$$

$= \infty$, the integral diverges

Use the comparison test to determine if the following integral are convergent or divergent

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} dx .$$

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$

converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right], \quad 0 \leq \cos x \leq 1 \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \rightarrow 0^+} \left[\sqrt{4x} \right]_a^{\pi/2} \quad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \rightarrow 0^+} \left(\sqrt{2\pi} - \sqrt{4a} \right) = \sqrt{2\pi} \quad \text{converges.} \end{aligned}$$

Evaluate the improper integral or state that it diverges.

$$\int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

$$\int_0^1 (\ln x)^2 dx$$

$$\int_1^{\infty} \frac{dx}{e^x + e^{-x}}$$

$$\int_{-1}^1 \frac{dx}{x^4}$$

Evaluate the improper integral

$$\int_0^1 (\ln x)^2 dx$$

Solution

يفرض أن $u = (\ln x)^2$, $dv = dx$, $du = \frac{2}{x} \ln x \cdot dx$, $v = x$

$$\int (\ln x)^2 dx = (\ln x)^2 \cdot x - \int x \cdot 2 \ln x \cdot \frac{1}{x} dx = x (\ln x)^2 - 2 \int \ln x dx$$

نفرض $v = dx, u = \ln x$ ونكامل مرة ثانية بالتجزئة فيكون:

$$I = (\ln x)^2 x - 2 \left[\ln x x - \int x \cdot \frac{1}{x} dx \right] =$$

$$= x (\ln x)^2 - 2x \cdot \ln x + 2x \quad (*)$$

$$\lim_{x \rightarrow 0^+} x (\ln x)^2 = \lim_{x \rightarrow 0} \frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x \rightarrow 0} \frac{2 \ln x}{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{2 \cdot \frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \rightarrow 0} 2x = 0$$

بالطريقة نفسها نجد أن $\lim_{x \rightarrow 0^+} \int_1^x (\ln x)^2 dx$

ومن العلاقة (11) واستناداً إلى العلاقة (*) يكون:

$$I_2 = \int_0^1 (\ln x)^2 dx = \lim_{\eta' \rightarrow 0^+} \left[x (\ln x)^2 - 2x \ln x + 2x \right]_{\eta'}^1 = 2$$

هذا يعني أن التكامل المعطى يتقارب و يكون : $I_2 = \int_0^1 (\ln x)^2 dx = 2$

Evaluate the improper integral or state that it diverges.

i. $\int_1^{\infty} \frac{1}{x^3} dx$

ii. $\int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx$

iii. $\int_{-\infty}^0 xe^{-4x} dx$

iv. $\int_0^{\infty} x^2 e^{-x} dx$

v. $\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$

i. $\int_0^{\infty} \frac{e^x}{1+e^x} dx$

ii. $\int_0^{\infty} \sin \frac{x}{2} dx$

iii. $\int_0^{\infty} e^{-x} \cos x dx$

TABLE 8.1 Basic integration formulas

$$1. \int k \, dx = kx + C \quad (\text{any number } k)$$

$$2. \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$3. \int \frac{dx}{x} = \ln |x| + C$$

$$4. \int e^x \, dx = e^x + C$$

$$5. \int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

$$6. \int \sin x \, dx = -\cos x + C$$

$$7. \int \cos x \, dx = \sin x + C$$

$$8. \int \sec^2 x \, dx = \tan x + C$$

$$8. \int \csc^2 x \, dx = -\cot x + C$$

$$10. \int \sec x \tan x \, dx = \sec x + C$$

$$11. \int \csc x \cot x \, dx = -\csc x + C$$

$$12. \int \tan x \, dx = \ln |\sec x| + C$$

$$13. \int \cot x \, dx = \ln |\sin x| + C$$

$$14. \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$15. \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

$$16. \int \sinh x \, dx = \cosh x + C$$

$$17. \int \cosh x \, dx = \sinh x + C$$

$$18. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$19. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$20. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$$

$$21. \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$$

$$22. \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$$



Thank you for your attention