Finite Element Methods

Week No-01

Introduction to Matrix Structural Analysis

- Classical Methods vs. Matrix Methods.
- Planar Frame Member, Nodal Displacements & Nodal Forces in Local Coordinates
- Truss Analysis by Stiffness Matrix Method
  - Truss Member Stiffness Relations in Local Coordinates
  - Member Stiffness Relations in Global Coordinates
  - Structure Stiffness Relations
  - Procedure for Analysis
- Frame & Beam Analysis by Stiffness Matrix Method
  - Analytical Model for Planar Frame Structure
  - Global & Local Coordinate Systems
  - Member Stiffness Relations in Local Coordinates
  - Stiffness Matrix of 2D Frame Member in Global Coordinates
  - Member Stiffness Relations in Global Coordinates
  - Structure Stiffness Relations

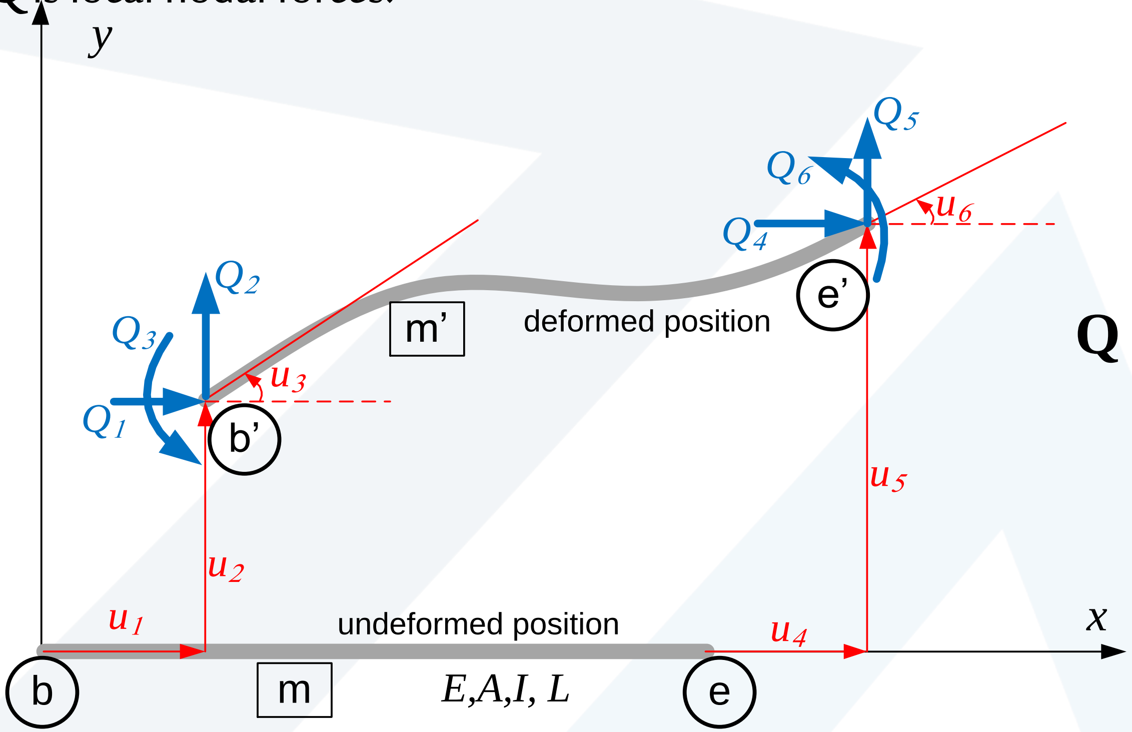
Classical Methods vs Matrix Methods

Classical Methods	Matrix Methods
Help to understand the structural behavior & the principles of structural Analysis	Simplify the overall picture of Structural Analysis
Time consuming for the analysis of large systems	Time saving as being computerized
Vary according to the structure type	Less varying

Planar Frame Member, Nodal Displacements & Nodal Forces in Local Coordinates

- x & y are the Local Coordinates system of the undeformed frame member be
- Under external actions the member move to the deformed position $b'e'$
- Nodal (adj.) of node = joint
- The Column vector **u** is the local nodal displacements
- The Column vector **Q** is local nodal forces.

$$\mathbf{u} \equiv \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$



$$\mathbf{Q} \equiv \{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Typing $\mathbf{Q}_{6\times1} = \mathbf{k}_{6\times6}\mathbf{u}_{6\times1}$ *Hand writing* $\{Q\}_{6\times1} = [k]_{6\times6}\{u\}_{6\times1}$

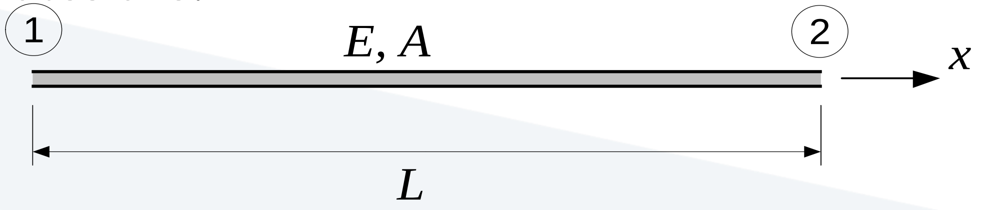
Truss Analysis by stiffness matrix method

A truss is a structural system that satisfies the following requirements:

1. Members are straight, slender, and prismatic.
2. Cross-sectional dimensions are small compared with lengths.
3. Weights of the members are small and can be neglected.
4. Joints are assumed to be frictionless pins (or internal hinges).
5. Loads are applied only at the joints in the form of concentrated forces.

Truss Analysis by stiffness matrix method

The parameters and variables describing the behavior of a typical truss element in local coordinates are:



Geometric & Material Properties: A, L & E

Kinematic response: The local displacements u_1 of joint 1 , & u_4 of joint 2, & δ the elongation of the element.

Static response: The local forces Q_1 *at* node 1, & Q_4 at node 2 , & the internal axial forces N of the element.

When this elastic bar is subjected to an internal axial force N by Hooke's Law it deforms by an amount δ suc that

$$\delta = \varepsilon L = \frac{\sigma L}{E} = \frac{N\ L}{EA}$$

Truss Analysis by stiffness matrix method

From compatibility

$$\delta = u_4 - u_1$$

From equilibrium

$$N = Q_4 = -Q_1$$

Substituting in Hook's law, we find the two equations

$$Q_1 = -\frac{EA}{L}(u_4 - u_1) \quad \& \quad Q_4 = \frac{EA}{L}(u_4 - u_1)$$

Grouping in a matrix form

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_4 \end{Bmatrix}$$

Local Element Eq.

Or symbolically

$$\mathbf{k}_{2\times2}\mathbf{u}_{2\times1} = \mathbf{Q}_{2\times1}$$

$\mathbf{k}_{2\times2}$ Local element stiffness matrix

$\mathbf{u}_{2\times1}$ Local element nodal displacements

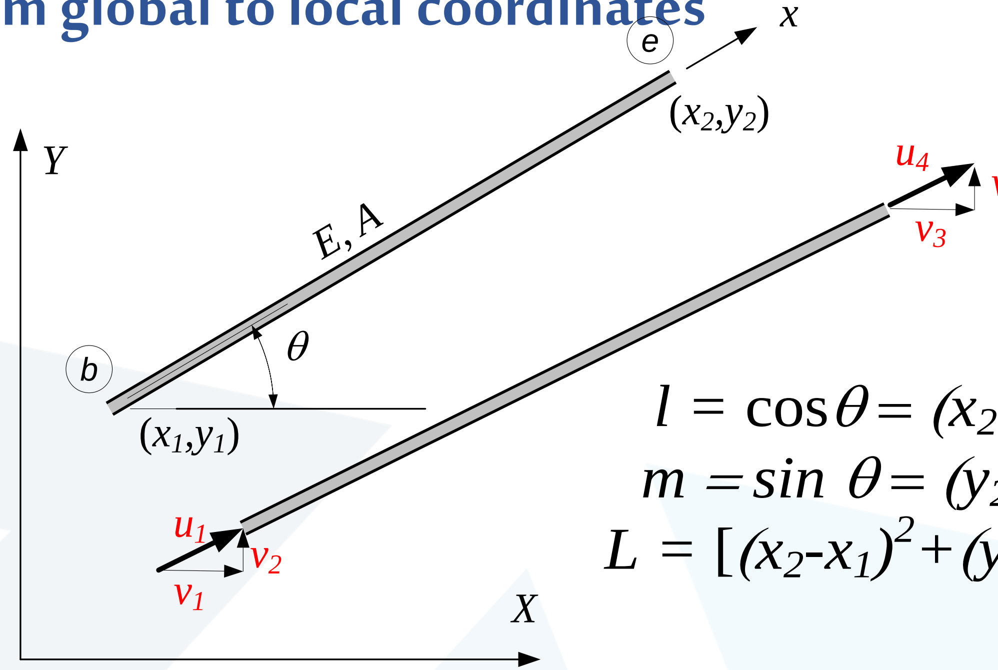
$\mathbf{Q}_{2\times1}$ Local element nodal forces

Kinematic Transformation from global to local coordinates

$$(u_1)^2 = (v_1)^2 + (v_2)^2$$

$$u_1 = \frac{v_1}{u_1} v_1 + \frac{v_2}{u_1} v_2$$

$$u_1 = l v_1 + m v_2$$



$l = \cos\theta = (x_2 - x_1)/L$

$m = \sin\theta = (y_2 - y_1)/L$

$L = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{½}$

Similarly

$$u_4 = l v_3 + m v_4$$

Grouping in a matrix form

$$\begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

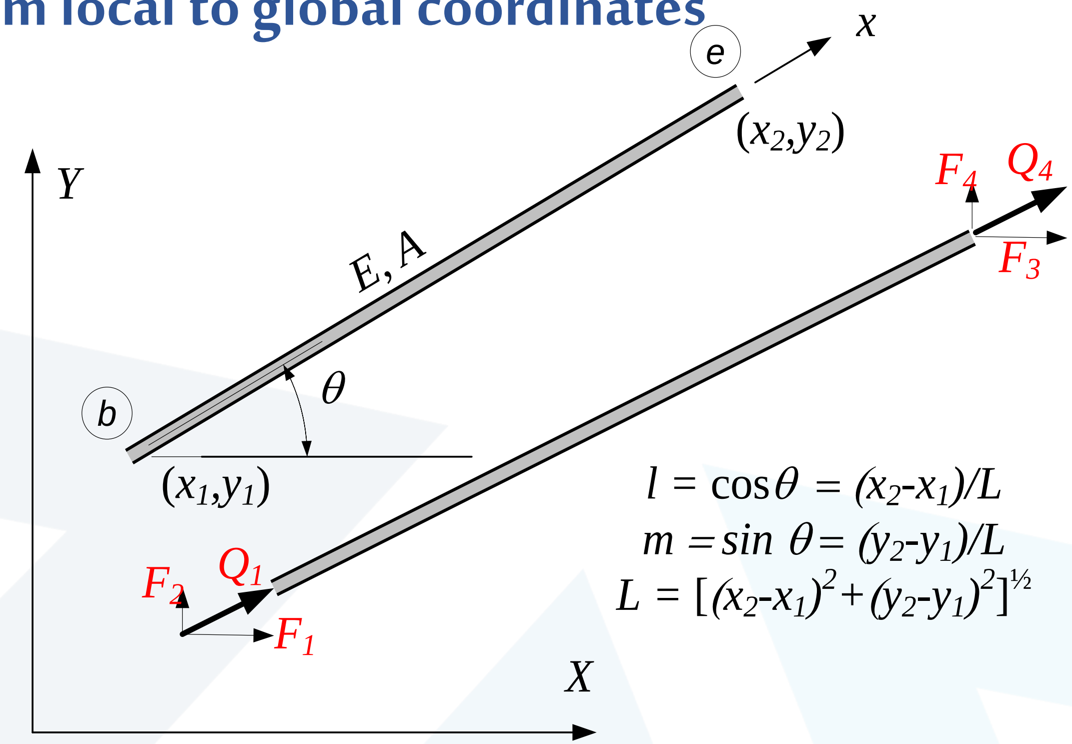
Symbolically

$$\mathbf{u}_{2\times1} = \mathbf{T}_{2\times4} \mathbf{v}_{4\times1}$$

$\mathbf{v}_{4\times1}$ Global element nodal displacements

$\mathbf{T}_{2\times4}$ Kinematic G-to-L transformation matrix

Static Transformation from local to global coordinates



Equilibrium at node *b*

$$F_1 = lQ_1 \ \& \ F_2 = mQ_1$$

Similarly at node *e*

$$F_3 = lQ_4 \ \& \ F_4 = mQ_4$$

Grouping in a matrix form

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_4 \end{Bmatrix}$$

Symbolically $\mathbf{F}_{4\times1} = \mathbf{T}^{\mathrm{T}}_{4\times2}\mathbf{Q}_{2\times1}$

$\mathbf{T}^{\mathrm{T}}_{4\times2}$ Static L-to-G transformation matrix

$\mathbf{F}_{4\times1}$ Global element nodal forces

Global Element Equations

Combining Static & kinematic transformations with the Local Element Eq.

$$\mathbf{F}_{4\times1} = \mathbf{T}^{T}_{4\times2}\mathbf{Q}_{2\times1} \qquad \mathbf{k}_{2\times2}\mathbf{u}_{2\times1} = \mathbf{Q}_{2\times1} \qquad \mathbf{u}_{2\times1} = \mathbf{T}_{2\times4}\mathbf{v}_{4\times1}$$

$$\mathbf{F}_{4\times1} = \mathbf{T}^{T}_{4\times2}\mathbf{k}_{2\times2}\mathbf{T}_{2\times4}\mathbf{v}_{4\times1} \Rightarrow \mathbf{K}_{4\times4}\mathbf{v}_{4\times1} = \mathbf{F}_{4\times1}$$

Global Planar Truss Element Eq.

$$\mathbf{K}_{4\times4} = \mathbf{T}^{T}_{4\times2}\mathbf{k}_{2\times2}\mathbf{T}_{2\times4}$$ the global planar truss element matrix

$$\mathbf{K}_{4\times4} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

Global Element Equations

Detailed form of the global planar truss element stiffness matrix

$$\mathbf{K}_{4\times4} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Like the local matrix it is symmetric and singular. It relates the nodal global displacements to the global nodal forces

$$\frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Compatibility –Equilibrium Equation

EXAMPLE . Element Stiffness Matrix

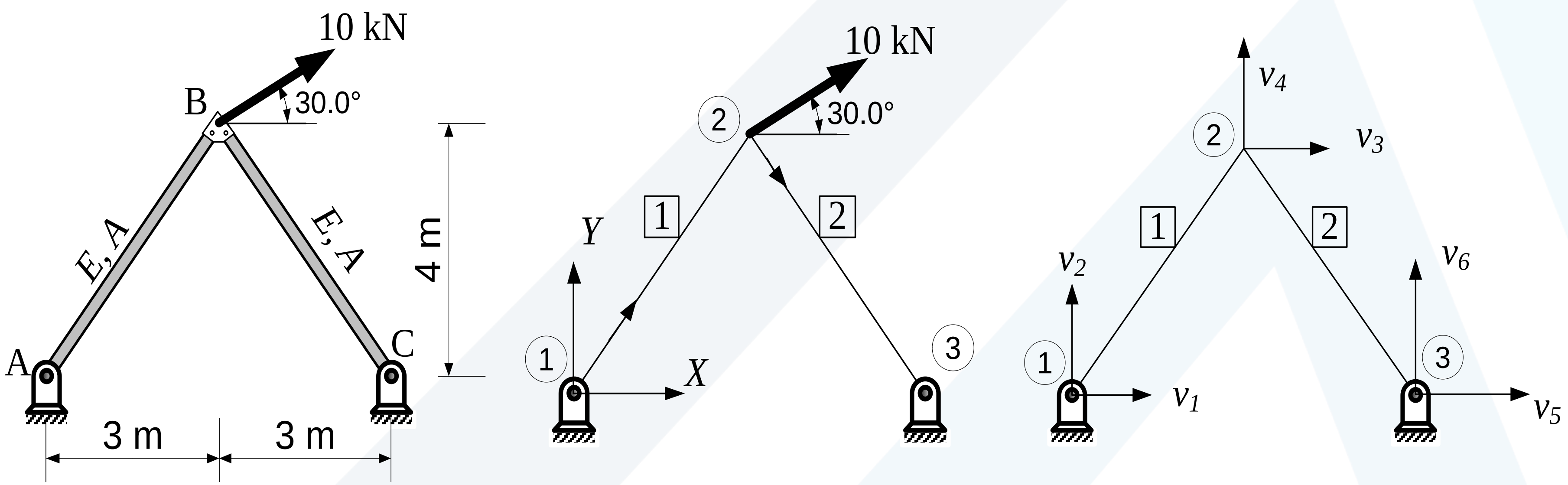
B. Haidar

Figure shows a planar truss. The material is steel, $E = 2(10)^{11}$Pa and the cross-sectional area of member AB and BC are respectively 0.01 m^2 and 0.02 m^2.

1. Construct the global stiffness matrix for each member & the truss stiffness matrix, then find the displacement of joint B.
2. Find the member internal forces and the support reactions.

SOLUTION

Step 1: We select kN, m as the problem units. The origin of the global coordinate system will be located at point A. The labeled model is shown in Figs. and (c).



It helps to create the following table that contains the terms in the global element stiffness matrix.

EXAMPLE . Element Stiffness Matrix

Member	(X_b,Y_b)	(X_e,Y_e)	L	l	m	AE/L
1	(0,0)	(3,4)	5	0.6	0.8	4×10^5
2	(3,4)	(6,0)	5	0.6	-0.8	8×10^5

Step 2& 3: We now create the two global element stiffness matrices.

For element 1, we have:

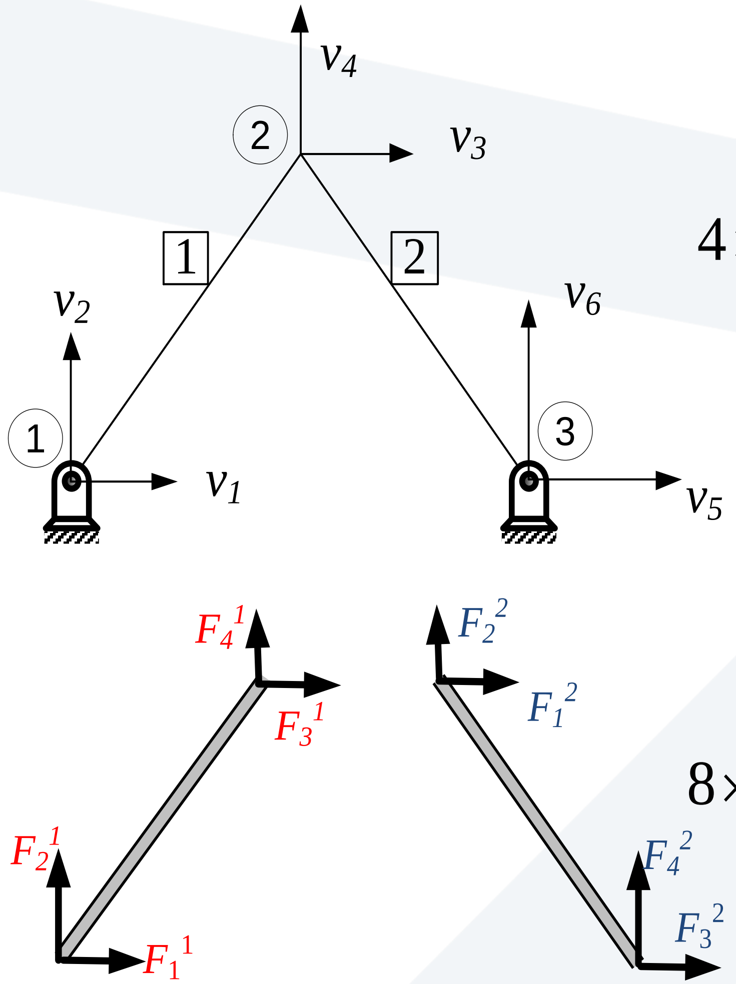
$$\mathbf{K}^1_{4\times4}=\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}=4\times10^5\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

For element 2, we have

$$\mathbf{K}^2_{4\times4}=\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}=8\times10^5\begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix}$$

EXAMPLE . Element Stiffness Matrix

Finally, the global element equilibrium equations for the two elements can be written as



For Element 1

$$4\times10^5\begin{bmatrix}0.36 & 0.48 & -0.36 & -0.48\\ 0.48 & 0.64 & -0.48 & -0.64\\ -0.36 & -0.48 & 0.36 & 0.48\\ -0.48 & -0.64 & 0.48 & 0.64\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{Bmatrix}=\begin{Bmatrix}F_1^1\\ F_2^1\\ F_3^1\\ F_4^1\end{Bmatrix}$$

For Element 2

$$8\times10^5\begin{bmatrix}0.36 & -0.48 & -0.36 & 0.48\\ -0.48 & 0.64 & 0.48 & -0.64\\ -0.36 & 0.48 & 0.36 & -0.48\\ 0.48 & -0.64 & -0.48 & 0.64\end{bmatrix}\begin{Bmatrix}v_3\\ v_4\\ v_5\\ v_6\end{Bmatrix}=\begin{Bmatrix}F_1^2\\ F_2^2\\ F_3^2\\ F_4^2\end{Bmatrix}$$

Useless equations: the two matrices are singular & the two nodal load columns are unknown

EXAMPLE . Element Stiffness Matrix

Step4. From the 8 equations of the 2 separated members, we need to construct the 6 equations describing the equilibrium of the entire truss. Starting with member 1.

$$4\times10^{5}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 0.64 & -0.48 & -0.64 & 0 & 0 \\ -0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\ -0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}=\begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 \\ F_4^1 \\ 0 \\ 0 \end{Bmatrix}$$

Now using the equations from member 2, we get

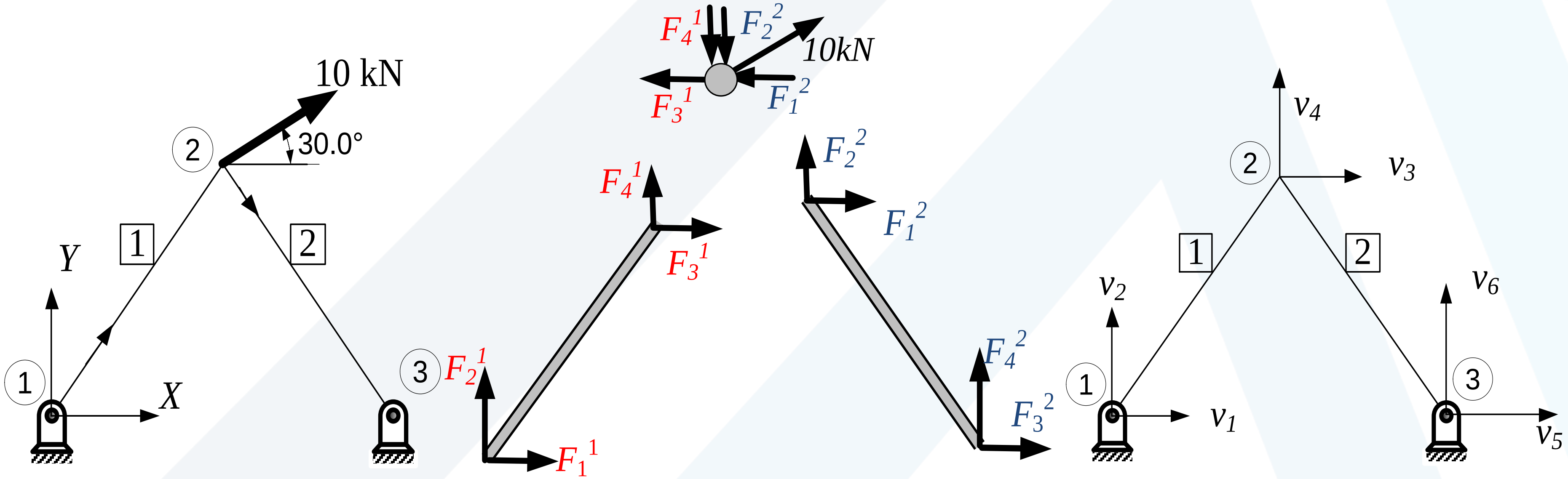
$$4\times10^{5}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.72 & -0.96 & -0.72 & 0.96 \\ 0 & 0 & -0.96 & 1.28 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & 0 & 0.96 & -1.28 & -0.96 & 1.28 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix}$$

EXAMPLE . Element Stiffness Matrix

$$4\times10^5\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 0.64 & -0.48 & -0.64 & 0 & 0 \\ -0.36 & -0.48 & 0.36+0.72 & 0.48-0.96 & -0.72 & 0.96 \\ -0.48 & -0.64 & 0.48-0.96 & 0.64+1.28 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & 0 & 0.96 & -1.28 & -0.96 & 1.28 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix}$$

Elements within the rectangle show the coupling between the two truss members.

The component of the applied load at node 2 are *10cos(30°)* & *10sin(30°)*, we have the system equations as.



$$
4\times10^{5}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 0.64 & -0.48 & -0.64 & 0 & 0 \\ -0.36 & -0.48 & 1.08 & -0.48 & -0.72 & 0.96 \\ -0.48 & -0.64 & -0.48 & 1.92 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & 0 & 0.96 & -1.28 & -0.96 & 1.28 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ 8.66 \\ 5.00 \\ F_5 \\ F_6 \end{Bmatrix}
$$

Symbolically

$$
\mathbf{K}_{6\times6}\mathbf{v}_{6\times1} = \mathbf{F}_{6\times1}
$$

We can carry out a few checks to ensure that the results are acceptable. First, the structural stiffness matrix **K** should be symmetric. Second, **K** is *usually* diagonally dominant, meaning that the diagonal element has the largest magnitude in that row and column. As we can see in the equations above, this condition is met by most but not all diagonal elements. However, the largest number is a diagonal element, K_{44}. Note also that all diagonal elements are positive.

Step 5: The boundary conditions for this problem are $v_1 = v_2 = v_5 = v_6 = 0$. Imposing these conditions on the system equations yields the modified system equations:

$$
4\times10^{5}\begin{bmatrix} 1.08 & -0.48 \\ -0.48 & 1.92 \end{bmatrix}\begin{Bmatrix} v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 8.66 \\ 5.00 \end{Bmatrix}
$$

or symbolically $\mathbf{K}_{2\times2}\mathbf{v}_{2\times1} = \mathbf{F}_{2\times1}$

Step 6: Solve the system equations **Kv = F** for the nodal displacements **v.**

$v_3 = 2.581\times10^{-5} m$ & $v_4 = 1.296\times10^{-5} m$,

$v=(v_3^2+v_4^2)^{1/2}=2.888\times10^{-5} m$