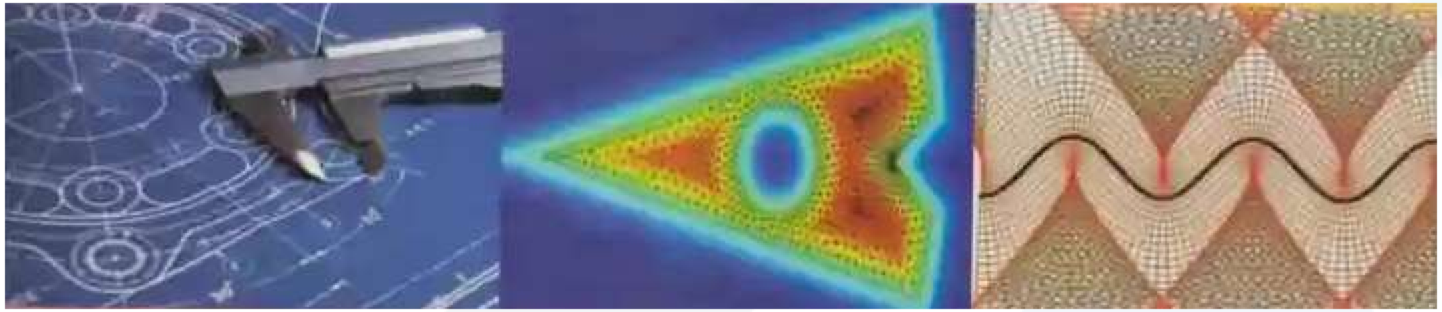
CEDC301: Engineering Mathematics

Lecture Notes 1 & 2: Functions of a Complex Variable



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Chapter 1

Functions of a Complex Variable

1. Complex Numbers
2. Powers and Roots
3. Sets in the Complex Plane
4. Functions of a Complex Variable
5. Cauchy-Riemann Equations
6. Exponential and Logarithmic Functions
7. Trigonometric and Hyperbolic Functions
8. Inverse Trigonometric and Hyperbolic Functions

1. Complex Numbers

- **Definition:** A number of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ (imaginary unit), is called a complex number.

  x is called the real part of z and is written as $Re(z)$ and y is called the imaginary part and is written as $Im(z)$.

  For example, if $z = 4 + 9i$, then $Re(z) = 4$ and $Im(z) = 9$

  A real constant multiple of the imaginary unit is called a pure imaginary number

- **Definition:** Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, $z_1 = z_2$, if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

  A complex number $z = x + iy = 0$ if $x = 0$ and $y = 0$.

Arithmetic Operations

- If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$

Commutative laws: $\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$

Associative laws: $\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

- If $z = x + iy$ is a complex number, then the complex number $\overline{z} = x - iy$ is called the complex conjugate or, simply, the conjugate of z.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

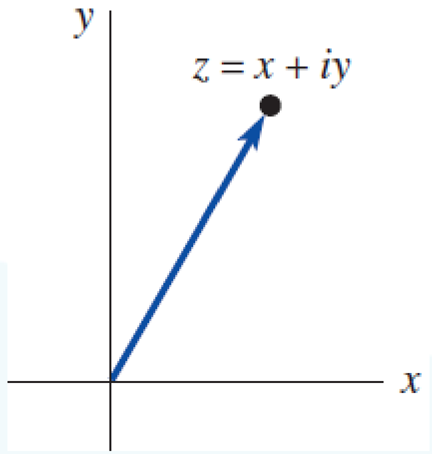
$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

For example, if $z = 4 + 9i$, then $\overline{z} = 4 - 9i$

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2Re(z)$$
$$z - \overline{z} = (x + iy) - (x - iy) = 2iy = 2Im(z)$$
$$\Rightarrow Re(z) = \frac{z + \overline{z}}{2}, \quad Im(z) = \frac{z - \overline{z}}{2i}$$

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$$

Geometric Interpretation



A complex number $z = x + iy$ can be viewed as a vector whose initial point is the origin and whose terminal point is (x, y). The coordinate plane is called the complex plane or simply the z-plane. The horizontal or x-axis is called the real axis and the vertical or y-axis is called the imaginary axis.

- Definition: The modulus or absolute value of $z = x + iy$, denoted by $|z|$, is the real number

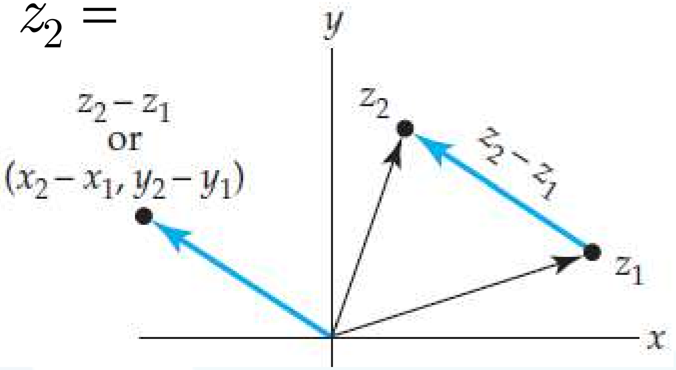
$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

For example, if $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$|z_1 + z_2| \leq |z_1| + |z_2|$ the triangle inequality $|z_1 + z_2| \geq |z_1| - |z_2|$

- Note: The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by $|z| = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)|$ or

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



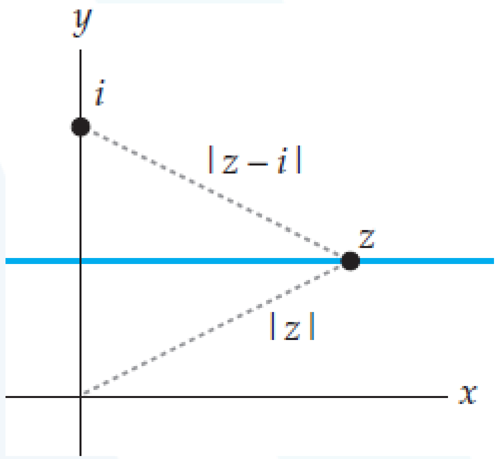
- Example 1: Set of Points in the Complex Plane

Describe the set of points z in the complex plane that satisfy $|z| = |z - i|$.

The distance from a point z to the origin equals the distance from z to the point i.

$$\sqrt{x^2 + y^2} = \sqrt{x^2 + (y-1)^2} \Rightarrow x^2 + y^2 = x^2 + (y-1)^2 \Rightarrow y = \frac{1}{2}$$

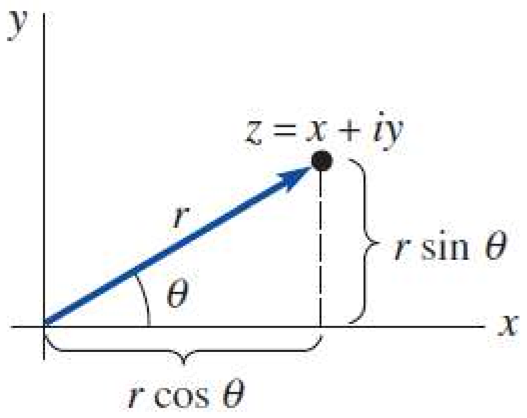
Complex numbers satisfying $|z| = |z - i|$ can then be written as $z = x + \frac{1}{2}i$.



2. Powers and Roots

Polar Form

- A nonzero complex number $z = x + iy$ can be written as
  $z = (r\cos\theta) + i(r\sin\theta)$ or $z = r(\cos\theta + i\sin\theta)$ polar form
  $r = |z|$ $\theta = \arg z = \tan^{-1}(y/x)$
  θ measured in radians is called an argument of z $(\arg z)$.

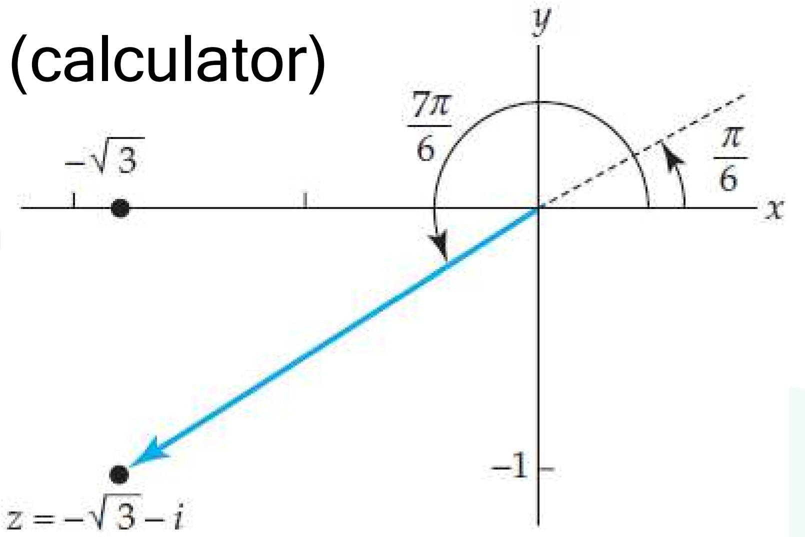


- If θ_0 is an argument of z, then the angles $\theta_0 \pm 2\pi k,\ k \in N$ are also arguments.

- Note: We have to choose θ consistent with the quadrant in which z is located; since tan θ has period π, so that the arguments of z and $-z$ have the same tangent.

  For example, if $z = -\sqrt{3} - i$, then $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$ and

$\theta = \tan^{-1}(y/x) = \tan^{-1}(-1/-\sqrt{3}) = \tan^{-1}(1/\sqrt{3}) = \pi/6$ (calculator)



which is an angle whose terminal side is in the first quadrant. But since the point $(-\sqrt{3}, -1)$ lies in the third quadrant ⇒ $\theta = \pi/6 + \pi = 7\pi/6$.

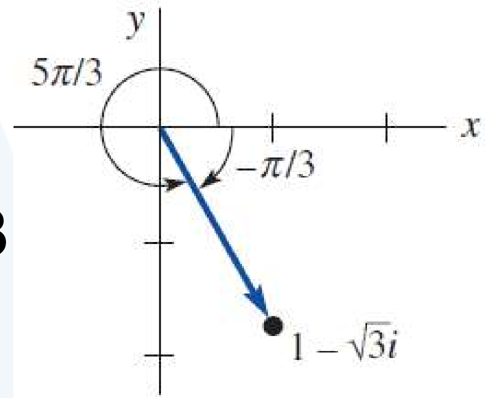
$$z = 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right]$$

- The argument of a complex number in the interval $-\pi < \theta \leq \pi$ is called the principal argument of z and is denoted by $\operatorname{Arg} z$.

For example, if $z = 1 - \sqrt{3}i$, then $z = 2\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$

in the interval $(\pi, \pi]$, the principal argument of z, is Arg $z = -\pi/3$

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$



- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

- Note: It is not true, in general, that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ and $\operatorname{Arg}(z_1/z_2) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$ (although it may be true for some complex numbers).

  For example, if $z_1 = -1$ and $z_2 = 5i$, then

  $$\operatorname{Arg}(z_1) = \pi,\ \operatorname{Arg}(z_2) = \pi/2,\ \operatorname{Arg}(z_1 z_2) = -\pi/2,\ \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = 3\pi/2 \neq \operatorname{Arg}(z_1 z_2)$$

  If $z_1 = -1$ and $z_2 = -5i$, then

  $$\operatorname{Arg}(z_1) = \pi,\ \operatorname{Arg}(z_2) = -\pi/2,\ \operatorname{Arg}(z_1/z_2) = -\pi/2,\ \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = 3\pi/2 \neq \operatorname{Arg}(z_1/z_2)$$

Integer Powers of z

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

For example, if $z = 1 - \sqrt{3}i$, then $z^3 = 2^3\left[\cos\left(3\left(-\pi/3\right)\right) + i\sin\left(3\left(-\pi/3\right)\right)\right]$

$$z^3 = 2^3\left[\cos\left(-\pi\right) + i\sin\left(-\pi\right)\right] = -8$$

DeMoivre's Formula

$$(\cos\,\theta + i\sin\,\theta)^n = \cos\,n\theta + i\sin\,n\theta$$

Roots

- A number w is said to be an n^{th} root of a nonzero complex number z if $w^n = z$.

$z = r(\cos\theta + i\sin\theta) \Rightarrow$

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$

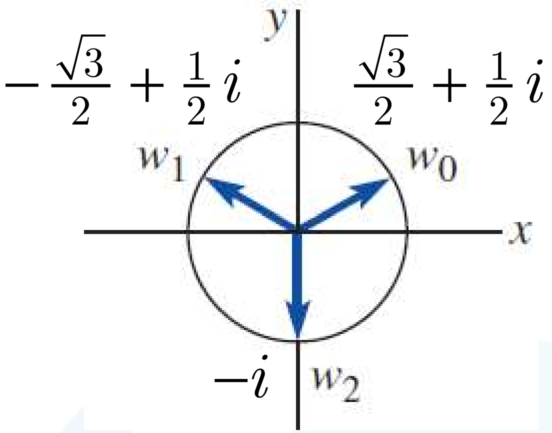
For example, the three cube roots of $z = i$ are:

$$w_k = 1^{1/3}\left[\cos\left(\frac{\pi/2 + 2\pi k}{3}\right) + i\sin\left(\frac{\pi/2 + 2\pi k}{3}\right)\right],\ k = 0, 1, 2$$

$$k=0,\quad w_0=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$k=1,\quad w_1=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$k=2,\quad w_2=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i$$

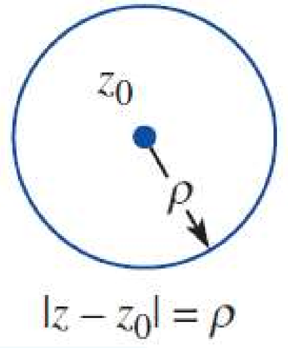


Principal n^{th} Root

- The root w of a complex number z obtained by using the principal argument of z with $k=0$ is sometimes called the principal n^{th} root of z.
  In previous example we see that $w_0=\frac{\sqrt{3}}{2}+\frac{1}{2}i$ is the principal cube root of i.
  The choice of Arg(z) and $k=0$ guarantees us that when z is a positive real number r, the principal nth root is $\sqrt[n]{r}$.

3. Sets in the Complex Plane

- Circles Suppose $z_0 = x_0 + iy_0$. Since $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation $|z - z_0| = \rho$, $\rho > 0$, lie on a circle of radius ρ centered at the point z_0.

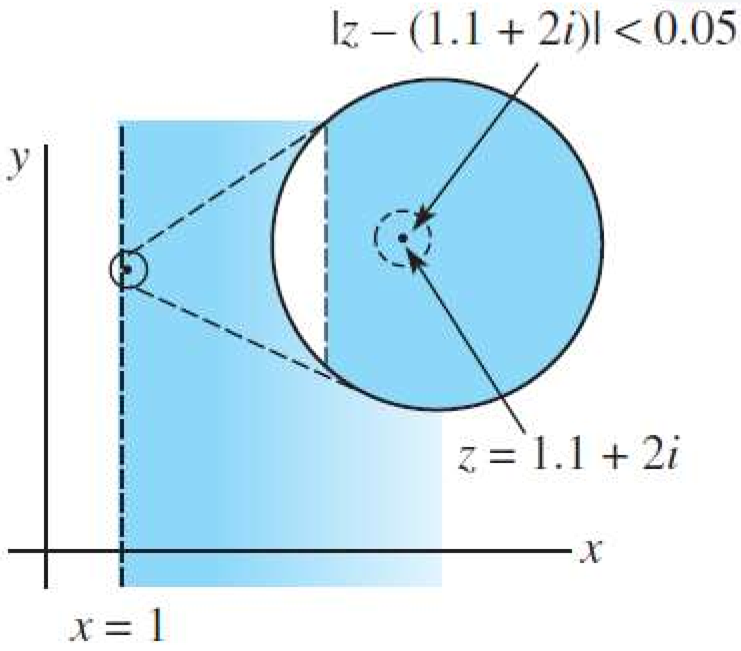


- The points z satisfying the inequality $|z - z_0| < \rho$, $\rho > 0$, lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a neighborhood of z_0 or an open disk.

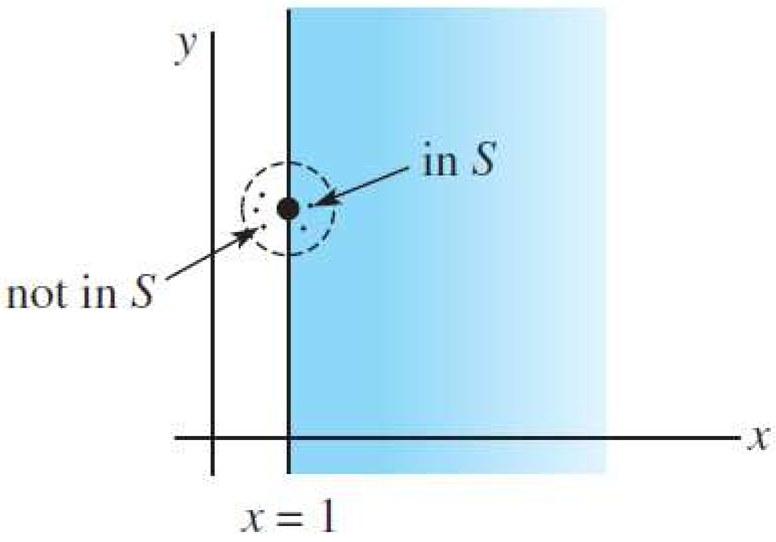
- A point z_0 is said to be an interior point of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an open set.

For example, the inequality $Re(z) > 1$ is an open set.

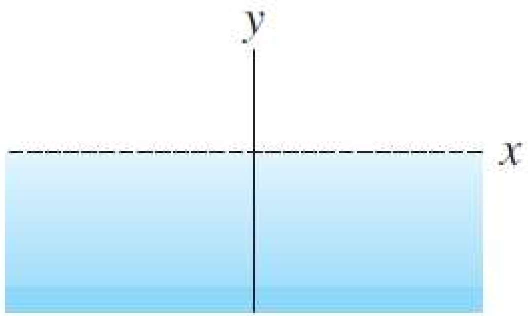
- The set S of points in the complex plane defined by $Re(z) \geq 1$ is not an open set.



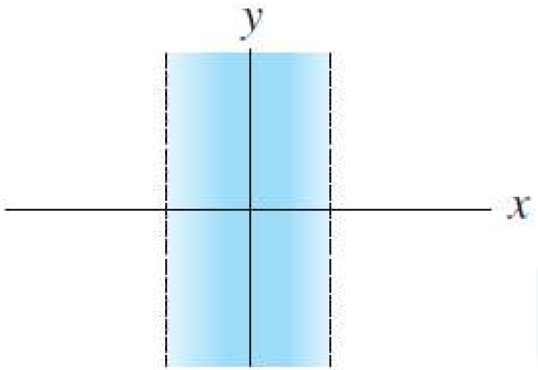
Open set magnified view of a point near $x = 1$



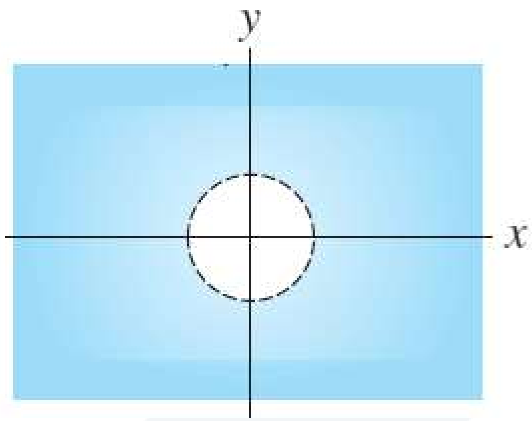
Set S is not open



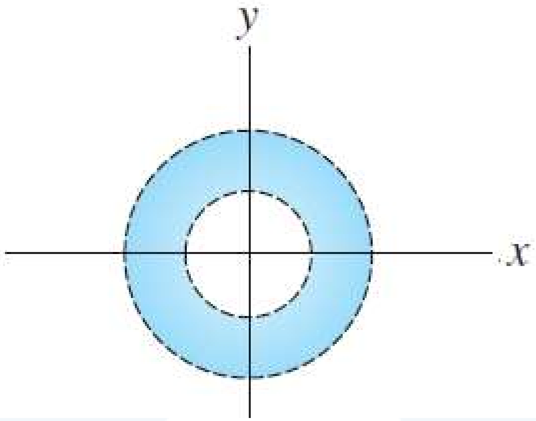
(a) $Im(z) < 0$; lower half-plane



(b) $-1 < Re(z) < 1$; infinite strip



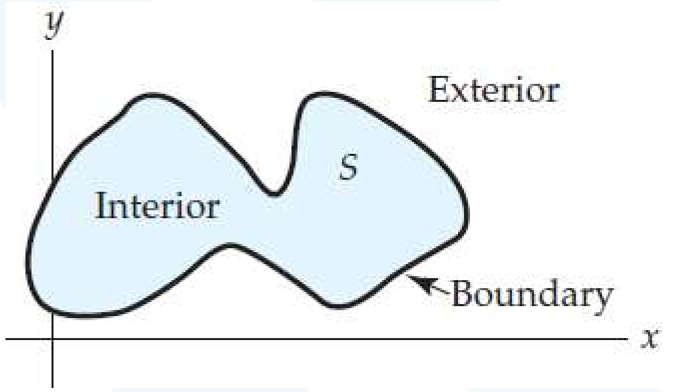
(c) $|z| > 1$; exterior of unit circle



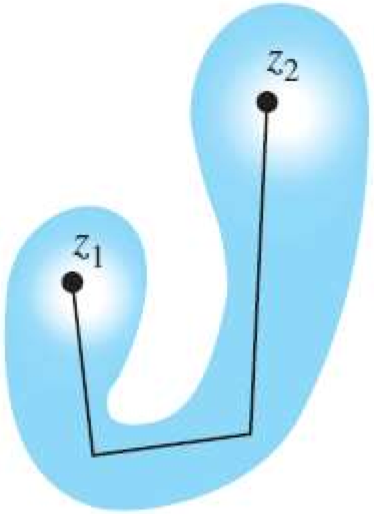
(d) $1 < |z| < 2$; circular ring

Figure 1 Four examples of open sets

- If every neighborhood of a point z_0 contains at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a boundary point of S.
- The boundary of a set S is the set of all boundary points of S.
- For the set of points defined by $Re(z) \geq 1$, the points on the line $x = 1$ are boundary points.
- The points on the circle $|z - i| = 2$ are boundary points for the disk $|z - i| \leq 2$.
- point z that is neither an interior point nor a boundary point of a set S is said to be an exterior point of S; in other words, z_0 is an exterior point of a set S if there exists some neighborhood of z_0 that contains no points of S.



- Annulus: If $0 < \rho_1 < \rho_2$, the set of points satisfying $\rho_1 < |z - z_0| < \rho_2$ is called an open circular annulus. For $\rho_1 = 0$, we obtain a deleted neighborhood of z_0.
- Domain: If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that lies entirely in the set, then the open set S is said to be connected.

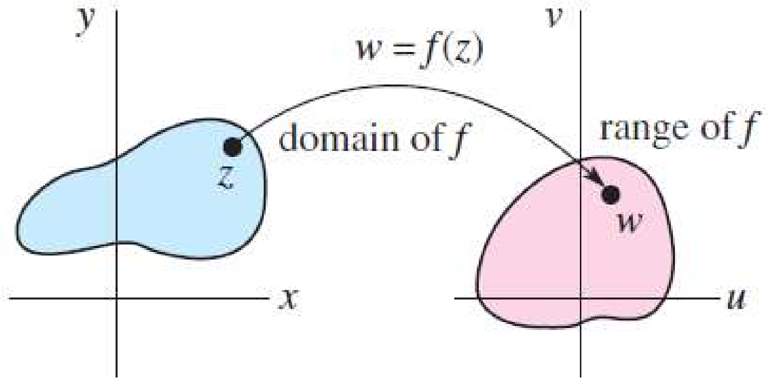


- An open connected set is called a domain. Each of the open sets in Figure 1 is connected and so are domains.
- The set of points satisfying $Re(z) \neq 4$ is an open set but is not connected.
- Regions: A region is a domain in the complex plane with all, some, or none of its boundary points.

- Since an open connected set does not contain any boundary points, it is automatically a region.
- A region containing all its boundary points is said to be closed. The disk defined by $|z - i| \leq 2$ is an example of a closed region and is referred to as a closed disk.
- A region may be neither open nor closed; the annular region defined by $1 \leq |z - 5| < 3$ contains only some of its boundary points and so is neither open nor closed.
- Note: Do not confuse the concept of "domain" defined here as open connected set with the concept of the "domain of a function."

4. Functions of a Complex Variable

- Definition: A complex function is a function f whose domain and range are subsets of the set C of complex numbers.
- A complex function is also called a complex-valued function of a complex variable.
- The image w of a complex number $z = x + iy$ will be some complex number $w = u + iv$; that is, $w = u(x, y) + iv(x, y) = f(z)$, where u, v are real functions of x and y.
- If to each value of z, there corresponds one and only one value of w, then w is said to be a single-valued function of z otherwise a multi-valued function.



For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

Some examples of functions of a complex variable are:

$$f(z) = z^2 - 4z = (x^2 - y^2 - 4x) + i(2xy - 4y), \quad z \in C$$

$$f(z) = \frac{z}{z^2 + 1}, \quad z \in C \setminus \{i, -i\} \qquad f(z) = z + Re(z), \quad z \in C$$

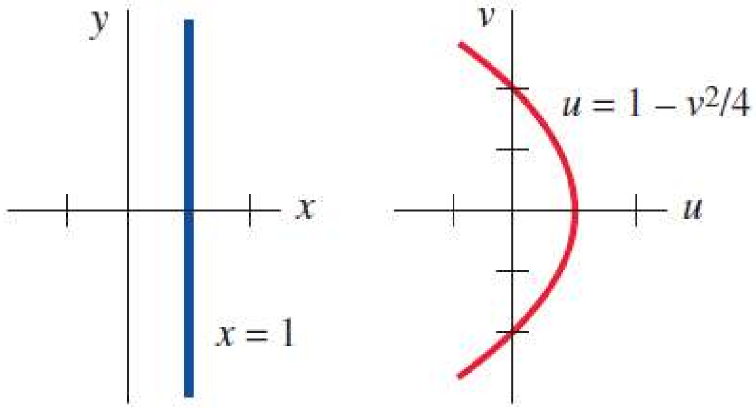
- Note: we cannot draw a graph of a complex function $w = f(z)$. We, say that a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the function $w = f(z)$ which defines a mapping or transformation of the z-plane into the w-plane.

- Example 2: Image of the line $Re(z) = 1$ under the mapping $f(z) = z^2$

$$f(z) = z^2 \Rightarrow u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy$$

$$Re(z) = x = 1 \Rightarrow u(x, y) = 1 - y^2 \text{ and } v(x, y) = 2y$$

$$\Rightarrow u = 1 - v^2/4$$



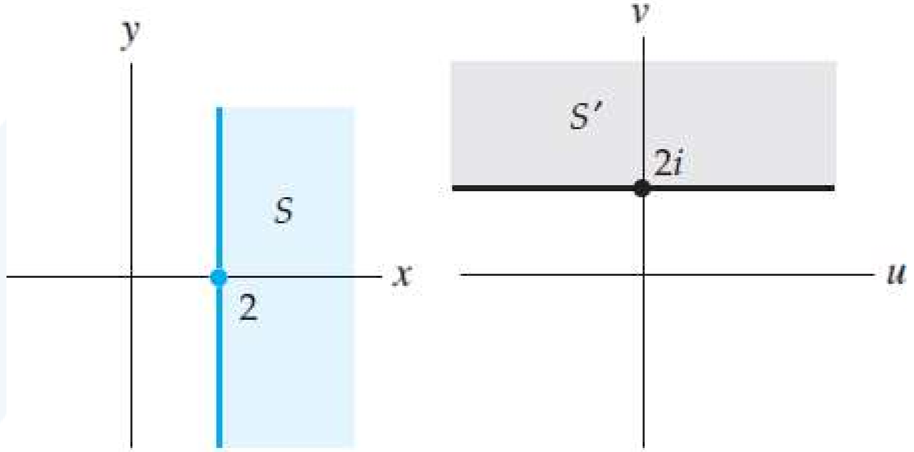
(a) z-plane (b) w-plane

- Example 3: Image of the Half-Plane $Re(z) \geq 2$ under the mapping $f(z) = iz$

$$f(z) = iz \Rightarrow u(x, y) = -y \text{ and } v(x, y) = x$$

$$x \geq 2, -\infty < y < \infty \Rightarrow v \geq 2, -\infty < u < \infty$$

$$\Rightarrow Im(z) \geq 2$$



(a) z-plane (b) w-plane

Principal Square Root Function $z^{1/2}$

The square root of a nonzero complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is given by:

$$\sqrt{r}\left[\cos\left(\frac{\theta+2\pi k}{2}\right) + i\sin\left(\frac{\theta+2\pi k}{2}\right)\right] = \sqrt{r}e^{i(\theta+2k\pi)/2},\ k = 0, 1$$

By setting $\theta = \text{Arg}(z)$ and $k = 0$, we can define a function that assigns to z the unique principal square root.

- Definition: The function $z^{1/2}$ defined by: $z^{1/2} = \sqrt{|z|}e^{i\text{Arg}(z)/2}$ is called the principal square root function.

- Example 4: Values of $z^{1/2}$ for $z = -2i$

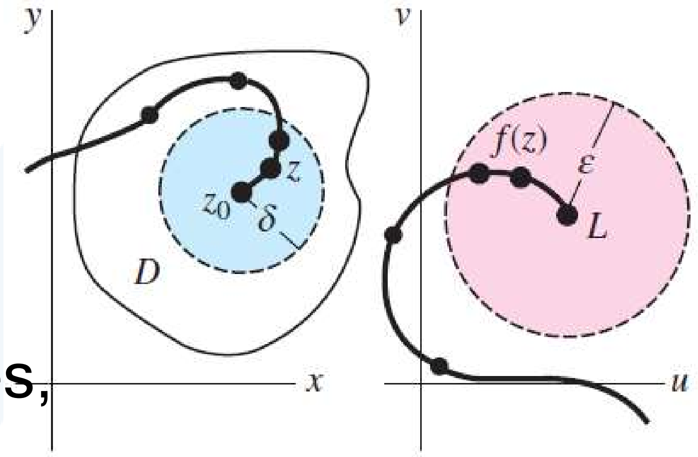
$$(-2i)^{1/2} = \sqrt{2}e^{i(-\pi/2+2k\pi)/2}_{k=0,\,1} = \begin{cases} \sqrt{2}e^{i(-\pi/4)} = 1 - i & \text{principal square root} \\ \sqrt{2}e^{i(3\pi/4)} = -1 + i \end{cases}$$

Limits and Continuity

- **Definition:** Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a limit at z_0, written:

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

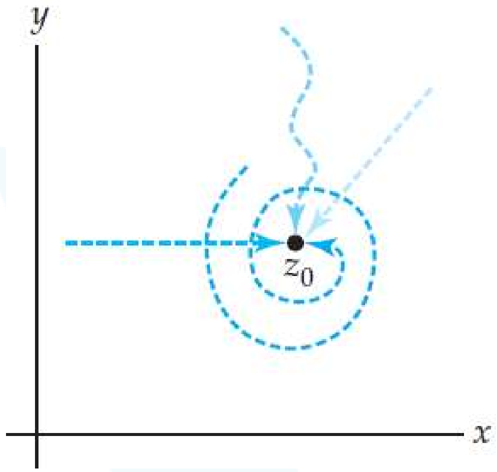


(a) δ-neighborhood (b) ε-neighborhood

- Complex and real limits have many common properties, but there is at least one very important difference.
- For real functions, $\lim_{x \to x_0} f(x) = L$ iff: $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$

  There are two directions from which x can approach x_0 on the real line.

- For limits of complex functions, z is allowed to approach z_0 from any direction in the complex plane, that is, along any path through z_0.



- In order that $\lim_{z \to z_0} f(z)$ exists and equals L, we require that $f(z)$ approach the same complex number L along every possible path through z_0.

Criterion for the Nonexistence of a Limit

- If f approaches two complex numbers $L_1 \neq L_2$ for two different paths or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

- Example 5: An Epsilon-Delta Proof of a Limit

  Using the epsilon-delta definition, Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$

$\lim_{z\to 1+i}(2+i)z = 1+3i \Leftrightarrow$ if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|(2+i)z-(1+3i)| < \varepsilon$ whenever $0 < |z-(1+i)| < \delta$

$$\left|(2+i)z-(1+3i)\right| < \varepsilon \Rightarrow \left|2+i\right|\left|z-\frac{1+3i}{2+i}\right| < \varepsilon$$

$$\sqrt{5}\left|z-(1+i)\right| < \varepsilon \Rightarrow \left|z-(1+i)\right| < \frac{\varepsilon}{\sqrt{5}} = \delta$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/\sqrt{5}$. If $0 < |z-(1+i)| < \delta$ then $|(2+i)z-(1+3i)| < \varepsilon$.
So, according to Definition $\lim_{z\to 1+i}(2+i)z = 1+3i$

- Example 6: A Limit That Does Not Exist

Show that $\lim_{z\to 0}\frac{z}{\bar{z}}$ does not exist.

z approach 0 along the real axis $\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{x\to 0}\frac{x+0i}{x-0i} = 1$

z approach 0 along the imaginary axis $\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{y\to 0}\frac{0+iy}{0-iy} = -1$

- **Theorem 1:** Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ if and only if:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

- **Example 7:** Using Theorem 1 to Compute the Limit $\lim_{z\to 1+i}(z^2+i)$

$$f(z) = z^2 + i = x^2 - y^2 + (2xy+1)i$$

$u_0 = \lim_{(x,y)\to(1,1)}(x^2 - y^2) = 0$ and $v_0 = \lim_{(x,y)\to(1,1)}(2xy+1) = 3 \Rightarrow \lim_{z\to 1+i}(z^2+i) = 3i$

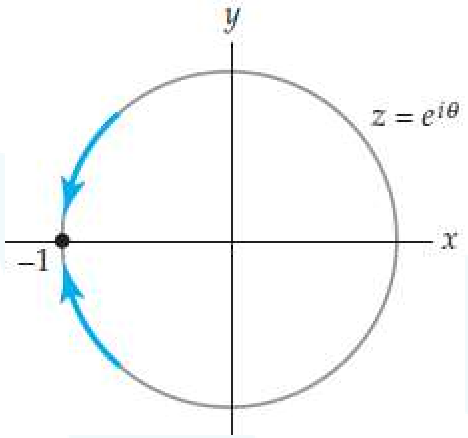
- Theorem 2: Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

$$\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2 \qquad \lim_{z \to z_0} [f(z)g(z)] = L_1 L_2 \qquad \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2},\ L_2 \neq 0$$

- Definition: A function f is continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$
- As a consequence, if two functions f and g are continuous at a point z_0, then their sum and product are continuous at z_0. The quotient of the two functions is continuous at z_0 provided $g(z_0) \neq 0$.
- A polynomial of degree $n > 0$ $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 z$ where, $a_n \neq 0$, $a_i \in C$, $i = 0, 1, \ldots, n$ is continuous everywhere.
- A rational function $f(z) = g(z)/h(z)$, where g and h are polynomial functions, is continuous except at those points at which $h(z)$ is zero.

- Example 8: Discontinuity of Principal Square Root Function $f(z) = z^{1/2}$ at $z_0 = -1$

  z approaching -1 along the second quadrant. That is, $z = e^{i\theta}$, $\pi/2 < \theta < \pi$, with θ approaching π



$$\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|} e^{i\text{Arg}(z)/2} = \lim_{\theta\to\pi} e^{i\theta/2} = \lim_{\theta\to\pi} (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) = i$$

z approaching -1 along the third quadrant. That is, $z = e^{i\theta}$, $-\pi < \theta < -\pi/2$, with θ approaching $-\pi$

$$\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|} e^{i\text{Arg}(z)/2} = \lim_{\theta\to -\pi} e^{i\theta/2} = \lim_{\theta\to -\pi} (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) = -i$$

$\Rightarrow \lim_{z\to -1} z^{1/2}$ does not exist

$\Rightarrow$ The principal SQRT function $f(z) = z^{1/2}$ is discontinuous at $z_0 = -1$

- **Theorem 3 (Real and Imaginary Parts of a Continuous Function):** Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point (x_0, y_0).

  The function $f(z) = \overline{z}$ is continuous on C.

- **Theorem 4 (Properties of Continuous Functions):** If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

  (i) cf, c a complex constant,

  (ii) $f \pm g$,

  (iii) $f.g$, and

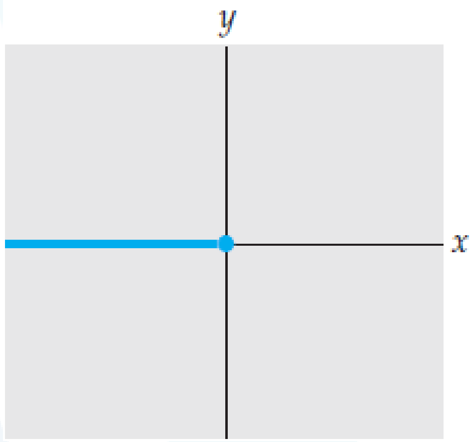
  (iv) f/g provided $g(z_0) \neq 0$.

- Branches: A branch of a multiple-valued function F is a function f_1 that is continuous on some domain and that assigns exactly one of the multiple values of F to each point z in that domain.
- The requirement that a branch be continuous means that the domain of a branch is different from the domain of the multiple-valued function.
- Note: For the multiple-valued function $F(z) = z^{1/2}$, and even though the principal SQRT function $f(z) = z^{1/2}$ does assign exactly one value of F to each input z, f is not a branch of F (Example 8: $f(z)$ is not continuous at $z_0 = -1$).
- Note: The principal SQRT function $f(z) = z^{1/2}$ is discontinuous at every point on the negative real axis.
- To obtain a branch of $F(z) = z^{1/2}$ that agrees with the principal square root function, we must restrict the domain to exclude points on the negative real axis.

$$f_1(z) = \sqrt{r}e^{i\theta/2}, \quad r > 0, \quad -\pi < \theta < \pi$$

$$f_1(z) = \sqrt[4]{x^2 + y^2}\, e^{i\tan^{-1}(y/x)/2} = \sqrt[4]{x^2 + y^2}\cos\left(\frac{\tan^{-1}(y/x)}{2}\right) + i\sqrt[4]{x^2 + y^2}\sin\left(\frac{\tan^{-1}(y/x)}{2}\right)$$

We call the function f_1 the principal branch of $F(z) = z^{1/2}$.



Branch Cuts and Points

- A branch cut for a branch f_1 of a multiple-valued function F is a portion of a curve that is excluded from the domain of F so that f_1 is continuous on the remaining points.
- The nonpositive real axis, shown in color in figure above, is a branch cut for the principal branch f_1 given above of the multiple-valued function $F(z) = z^{1/2}$.
- A different branch of F with the same branch cut: $f_2(z) = \sqrt{r}e^{i\theta/2}, \quad \pi < \theta < 3\pi$

- These branches are distinct because for, say, $z = i$ we have $f_1(i) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$, but $f_2(i) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$.
- Note: If we set $\phi = \theta - 2\pi$, then the branch f_2 can be expressed as:

$$f_2(z) = \sqrt{r} e^{i(\phi+2\pi)/2} = \sqrt{r} e^{i\phi/2} e^{i\pi} = -\sqrt{r} e^{i\phi/2}, \quad -\pi < \theta < \pi$$

Thus, we have shown that $f_2 = -f_1$. We can think of these two branches of $F(z) = z^{1/2}$ as being analogous to the positive and negative square roots of a positive real number.

- A point with the property that it is on the branch cut of every branch is called a branch point of F.
- The point $z = 0$ is on the branch cut for f_1, f_2 and on the branch cut of every branch of the multiple-valued function $F(z) = z^{1/2}$.

Derivative

- Definition: Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

  provided this limit exists.

- If the limit exists, the function f is said to be differentiable at z_0.
- As in real variables, If f is differentiable at z_0, then f is continuous at z_0.

  Moreover, the rules of differentiation are the same as in the calculus of real variables.

- If f and g are differentiable at a point z, and c is a complex constant, then:

Constant Rules: $\frac{d}{dz}c = 0, \quad \frac{d}{dz}cf(z) = cf'(z)$

Sum Rule: $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$

Product Rule: $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$

Quotient Rule: $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$

Chain Rule: $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$

Power Rule: $\frac{d}{dz}z^n = nz^{n-1}$, n an integer

- Note: In order for a complex function f to be differentiable at a point z_0,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must approach the same complex number from any direction.

- Example 9: A Function That Is Nowhere Differentiable.

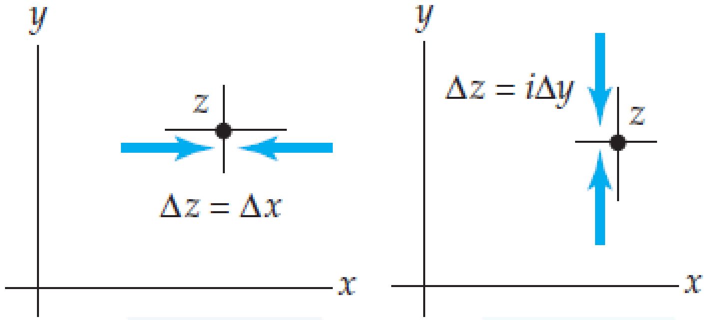
Show that the function $f(z) = x + 4iy$ is nowhere differentiable

$$\Delta z = \Delta x + i\Delta y \;\Rightarrow\; f(z+\Delta z) - f(z) = (x+\Delta x) + 4i(y+\Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

$\Delta z \to 0$ along a line // to x-axis ($\Delta y = 0$) ⇒ limit is 1.

$\Delta z \to 0$ along a line // to y-axis ($\Delta x = 0$) ⇒ limit is 4.



Analytic Functions

- Definition: A complex function $w = f(z)$ is said to be analytic (holomorphic) at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.
- A function f is analytic in a domain D if it is analytic at every point in D.

$f(z) = |z|^2$ is differentiable at $z = 0$ but is differentiable nowhere else. Hence, $f(z) = |z|^2$ is nowhere analytic.

In contrast, the simple polynomial $f(z) = z^2$ is differentiable at every point z in the complex plane. Hence, $f(z) = z^2$ is analytic everywhere.

- A function that is analytic at every point z is said to be an entire function.
- Polynomial functions are differentiable at every point z and so are entire functions.

- A rational function $f(z) = p(z)/q(z)$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Analyticity of Sum, Product, and Quotient

- The sum $f(z) + g(z)$, difference $f(z) - g(z)$, and product $f(z)g(z)$ are analytic. The quotient $f(z)/g(z)$ is analytic provided $g(z) = 0$ in D.
- A rational function $f(z) = p(z)/q(z)$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

An Alternative Definition of $f(z)$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

- Theorem 5 (L'Hˆopital's Rule): Suppose f and g are analytic at the point z_0 and $f(z_0) = 0$, $g(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

5. Cauchy-Riemann Equations

- Theorem 6 (Cauchy-Riemann Equations): Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This result is a necessary condition for analyticity

For example the polynomial $f(z) = z^2 + z$ is analytic for all z

$$f(z) = x^2 - y^2 + x + i(2xy + y)$$

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equations are satisfied

Criterion for Non-analyticity

- If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x,\ y) + iv(x,\ y)$ cannot be analytic in D.

- Example 10: Using the Cauchy-Riemann Equations

  Show that the function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

$$\frac{\partial u}{\partial x} = 4x \quad \text{and} \quad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$ is satisfied only on the line $y = 2x$

However, for any point z on the line, there is no neighborhood or open disk about z in which f is differentiable. We conclude that f is nowhere analytic.

A Sufficient Condition for Analyticity

- **Theorem 7: (Criterion for Analyticity)** Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$ is analytic in any domain not containing $z = 0$.

- **Note:** Analyticity implies differentiability but not vice versa.

Sufficient Conditions for Differentiability

- If the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first order partial derivatives in a neighborhood of z, and if u and v satisfy the Cauchy-Riemann equations at the point z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f'(z)$ is given by:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

- Example 11: In Example 10 we saw that $f(z) = 2x^2 + y + i(y^2 - x)$ was nowhere analytic, but yet the Cauchy-Riemann equations were satisfied on the line $y = 2x$. Since the functions $u(x, y) = 2x^2 + y$, $\partial u/\partial x = 4x$, $\partial u/\partial y = 1$, $v(x, y) = y^2 - x$, $\partial v/\partial x = -1$ and $\partial v/\partial y = 2y$ are continuous at every point, it follows that f is differentiable on the line $y = 2x$. On this line f' is given by $f'(z) = 4x - i = 2y - i$.

- **Theorem 8: (Constant Functions)** Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D.
  (i) If $|f(z)|$ is constant in D, then so is $f(z)$.
  (ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

Polar Coordinates

- In polar coordinates, $f(z) = u(r, \theta) + iv(r, \theta)$, Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The polar version of $f'(z)$ at a point z is

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

Harmonic Functions

- Definition: A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation ($\partial^2\phi/\partial^2 x + \partial^2\phi/\partial^2 y = 0$) is said to be harmonic in D.

- Theorem 9 (Harmonic Functions): Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

Harmonic Conjugate Functions If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D. Now suppose $u(x, y)$ is a given function that is harmonic in D. It is then sometimes possible to find another function $v(x, y)$ that is harmonic in D so that $u(x, y) + iv(x, y)$ is an analytic function in D. The function v is called a harmonic conjugate function of u.

- Example 12: Harmonic Function/Harmonic Conjugate Function

Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. Find the harmonic conjugate function of u.

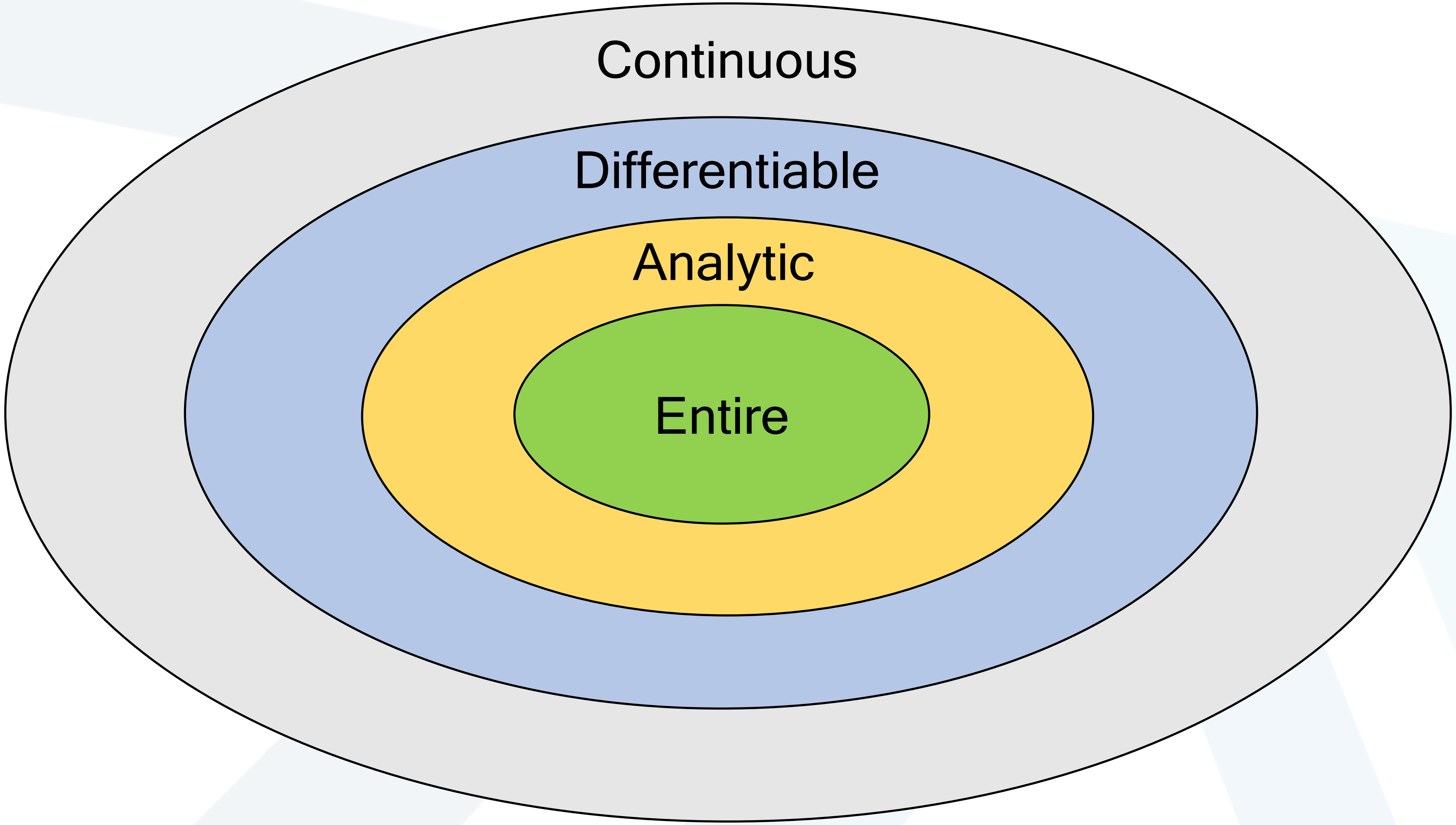
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial u}{\partial y} = -6xy - 5, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 5$$

$$v(x, y) = 3x^2y - y^3 + h(x) \Rightarrow \frac{\partial v}{\partial x} = 6xy + h'(x) \Rightarrow h'(x) = 5 \Rightarrow h(x) = 5x + C$$

$$f(z) = x^3 - 3xy^2 - 5y + i(3x^2y - y^3 + 5x + C)$$


Continuous
Differentiable
Analytic
Entire

6. Exponential and Logarithmic Functions

Exponential Function

We want the definition of the complex function $f(z) = e^z$, where $z = x + iy$, to reduce e^x for $y = 0$ and to possess the properties $f'(z) = f(z)$ and $f(z_1 + z_2) = f(z_1)f(z_2)$.

- **Definition:** The complex exponential function is defined as:

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

- The real and imaginary parts of e^z are continuous and have continuous first partial derivatives at every point z of the complex plane. Moreover, the Cauchy-Riemann equations are satisfied at all points of the complex plane:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$f(z) = e^z$ is analytic for all z

f is an entire function

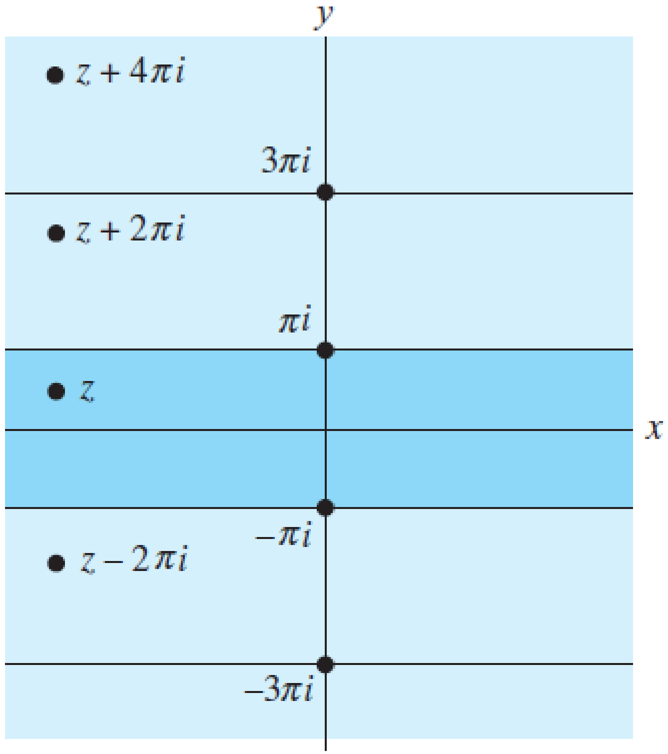
Properties

$$\frac{d}{dz}e^z = e^z, \quad e^0 = 1, \quad e^{z_1}e^{z_2} = e^{z_1+z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}, \quad \overline{e^z} = e^{\bar{z}}$$

Periodicity

Unlike the real function e^x, the complex function $f(z) = e^z$ is periodic with the complex period $2\pi i$. $f(z+2\pi i) = f(z)$

If we divide the complex plane into horizontal strips defined by $(2n-1)\pi < y \le (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$, then, for any point z in the strip $-\pi < y \le \pi$, the values $f(z)$, $f(z+2\pi i)$, $f(z-2\pi i)$, $f(z+4\pi i)$, and so on, are the same. The strip $-\infty < x < \infty$, $-\pi < y \le \pi$ is called the fundamental region for the exponential function $f(z) = e^z$.



Logarithmic Function

The logarithm of a complex number $z = x + iy$, $z \neq 0$, is defined as the inverse of the exponential function, $w = \log z$ if $z = e^{w}$.

- Definition: The multiple-valued function Logarithm of a Complex Number $z = x + iy$, $z \neq 0$, is defined as:

$$\log z = \ln|z| + i \arg z = \ln|z| + i(\operatorname{Arg} z + 2\pi n), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\log(-2) = \ln 2 + i(\pi + 2\pi n)$$

$$\log(i) = i(\tfrac{\pi}{2} + 2\pi n)$$

$$\log(-1 - i) = \ln\sqrt{2} + i(\tfrac{5\pi}{4} + 2\pi n)$$

Principal Value

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad z \neq 0, \quad -\pi < \operatorname{Arg} z \leq \pi$$

$f(z) = \text{Log}\, z$ is called the principal branch of $\log z$, or the principal logarithmic function.

$$\text{Log}\,(-2) = \ln 2 + \pi i$$

$$\text{Log}\,(i) = \frac{\pi}{2} i$$

$$\text{Log}\,(-1-i) = \ln\sqrt{2} - \frac{3\pi}{4} i$$

Properties

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log \frac{z_1}{z_2} = \log z_1 - \log z_2$$

$$\log z^n = n \log z$$

- Note: The identities above are not necessarily satisfied by the principal value. For example, it is not true that $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ for all complex numbers z_1 and z_2 (although it may be true for some complex numbers).

- Example 13: $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

  If $z_1 = i$ and $z_2 = -1 + i$, then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(-1 - i) = \ln\sqrt{2} - \frac{3\pi}{4} i$$

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \frac{\pi}{2} i + \left(\ln\sqrt{2} + \frac{3\pi}{4} i\right) = \ln\sqrt{2} + \frac{5\pi}{4} i \neq \operatorname{Log}(z_1 z_2)$$

$\operatorname{Log} z$ as an Inverse Function

$$e^{\operatorname{Log} z} = z,\ z \neq 0$$

$$\operatorname{Log} e^{z} = z \text{ if } -\infty < x < \infty \text{ and } -\pi < y \leq \pi$$

- If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, then f is one-to-one and the inverse function of f is the principal value of the complex logarithm $f^{-1}(z) = \text{Log } z$.

For example, for the point $z = 1 + 3\pi i/2$, which is not in the fundamental region, we have:

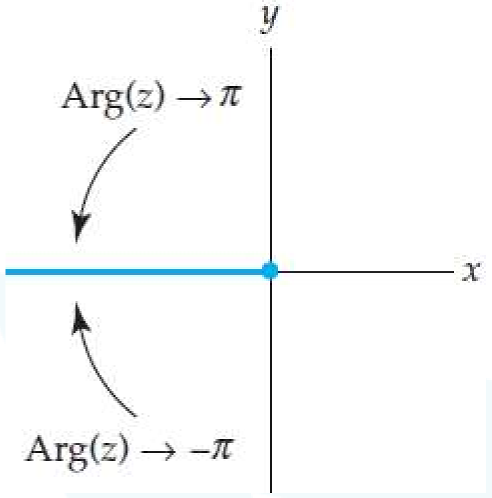
$$\text{Log } e^{1+3\pi i/2} = 1 - \pi i/2 \neq 1 + 3\pi i/2$$

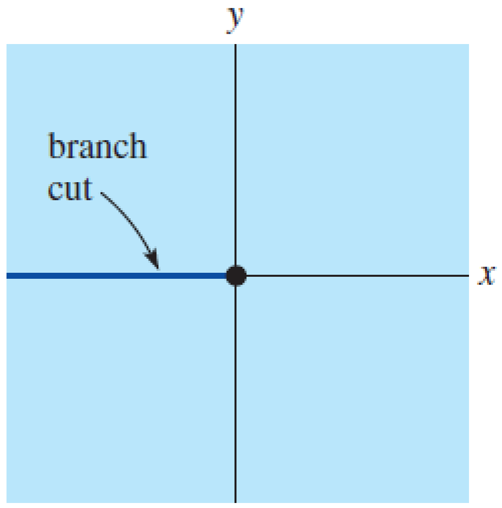
Analyticity

- The logarithmic function $f(z) = \text{Log } z$ is not continuous at $z = 0$ since $f(0)$ is not defined.
- The logarithmic function $f(z) = \text{Log } z$ is discontinuous at all points of the negative real axis.

- This is because the imaginary part of the function, $v = \text{Arg } z$, is discontinuous only at these points.
- Suppose x_0 is a point on the negative real axis. As $z \to x_0$ from the upper half-plane, Arg $z \to \pi$, whereas if $z \to x_0$ from the lower half-plane, then Arg $z \to -\pi$.
- This means that $f(z) = \text{Log } z$ is not analytic on the nonpositive real axis.
- However, $f(z) = \text{Log } z$ is analytic throughout the domain D consisting of all the points in the complex plane except those on the nonpositive real axis.

$$|z| > 0, -\pi < \arg(z) < \pi$$





- It is convenient to think of D as the complex plane from which the nonpositive real axis has been cut out.
- Since $f(z) = \text{Log } z$ is the principal branch of $\log z$, the nonpositive real axis is referred to as a branch cut for the function.
- The Cauchy-Riemann equations are satisfied throughout this cut plane and that the derivative of $\text{Log } z$ is given by:

$$\frac{d}{dz}\text{Log } z = \frac{1}{z} \quad \text{for all } z \text{ in } D$$

- Example 14: Derivatives of Logarithmic Functions

  Find the derivatives of the following functions in an appropriate domain:

  (a) $z \,\text{Log } z$ (b) $\text{Log}(z+1)$

(a) $z\text{Log}\, z$ is differentiable at all points where both of the functions z and $\text{Log}\ z$ are differentiable. z is entire and $\text{Log}\ z$ is differentiable on the domain: $|z| > 0$, $-\pi < \arg\ z < \pi$.

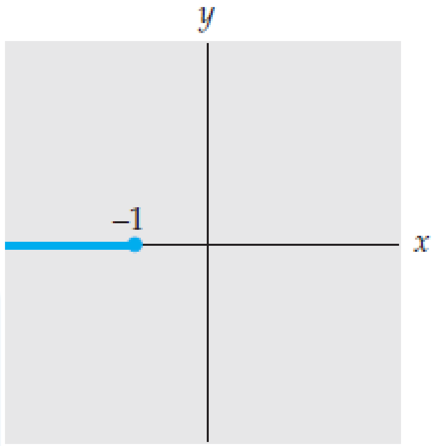
So $z \log z$ is differentiable on the domain defined by:

$$|z| > 0,\ -\pi < \arg\ z < \pi$$

$$\frac{d}{dz}[z\,\text{Log}\ z] = z \cdot \frac{1}{z} + 1 \cdot \text{Log}\ z = 1 + \text{Log}\ z$$

(b) The function $\text{Log}(z+1)$ is a composition of the functions $\text{Log}\ z$ and $z+1$. $z+1$ is entire and $\text{Log}(z+1)$ is differentiable at all points $w = z+1$ such that $|w| > 0$ and $-\pi < \arg(w) < \pi$.

$$\frac{d}{dz}\text{Log}\,(z+1) = \frac{1}{z+1} \cdot 1 = \frac{1}{z+1}$$



Complex Powers

- If α is a complex number and $z = x + iy$, then z^α is defined by:

$$z^\alpha = e^{\alpha \log z}, \quad z \neq 0$$

- In general, z^α is multiple-valued since $\log z$ is multiple-valued. However, in the special case when $\alpha = n =$ 0, ±1, ± 2, ... z^α is single-valued.
- Note: If we use $\mathrm{Log}\, z$ in place of $\log z$, then z^α gives the principal value.
- Example 15: Complex Power

  Find the value of: (a) i^{2i} (b) $(1 + i)^i$

  (a) $i^{2i} = e^{2i[\ln 1 + i(\pi/2 + 2\pi n)]} = e^{-(1+4n)\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$

  The principal value of i^{2i} for $n = 0$: $i^{2i} = e^{-\pi}$

(b) $(1+i)^i = e^{i[\frac{1}{2}\ln 2 + i(\pi/4 + 2\pi n)]}, \quad n = 0, \pm 1, \pm 2, \ldots$

The principal value of $(1+i)^i$ for $n = 0$: $(1+i)^i = e^{-\frac{\pi}{4} + i\frac{\ln 2}{2}}$

Complex powers satisfy the following properties

$$z^\alpha z^\beta = z^{\alpha+\beta}, \qquad \frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}; \alpha, \beta \in C$$

$$(z^\alpha)^n = z^{n\alpha}; \alpha \in C, n \in Z$$

Analyticity

- The principal value of the complex power $z^\alpha = e^{\alpha \operatorname{Log} z}$ is differentiable and:

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}$$

- Example 16: Derivative of a Power Function

Find the derivative of the principal value z^i at the point $z = 1 + i$

$z = 1 + i$ is in the domain $|z| > 0, -\pi < \arg\ z < \pi$, $\frac{d}{dz} z^i = iz^{i-1}$

$$\left.\frac{d}{dz} z^i\right|_{z=1+i} = \left. iz^{i-1}\right|_{z=1+i} = i(1+i)^{i-1} = i(1+i)^i \frac{1}{1+i} = \frac{1+i}{2}(1+i)^i$$

the principal value of $(1 + i)^i$: $(1+i)^i = e^{-\pi/4+i(\ln 2)/2}$

$$\left.\frac{d}{dz} z^i\right|_{z=1+i} = \frac{1+i}{2} e^{-\pi/4+i(\ln 2)/2}$$

7. Trigonometric and Hyperbolic Functions

Trigonometric Functions

- **Definition:** For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

additional trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z},$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

For example $\cos i = \dfrac{e^{-1} + e}{2}$ $\qquad \tan(\pi - 2i) = -\dfrac{e^{2} - e^{-2}}{e^{2} + e^{-2}}\, i$

Periodicity

- The complex exponential function e^z is periodic with a pure imaginary period of $2\pi i$.
- e^{iz} and e^{-iz} are periodic functions with real period 2π.
- So, the complex sine and cosine functions are periodic functions with a real period of 2π. $\sin(z+2\pi)=\sin z$ and $\cos(z+2\pi)=\cos z$
- The complex tangent and cotangent functions are periodic with a real period of π. $\tan(z+\pi)=\tan z$ and $\cot(z+\pi)=\cot z$

Analyticity

- Since the exponential functions e^{iz} and e^{-iz} are entire functions, it follows that $\sin z$ and $\cos z$ are entire functions.

- $\sin z = 0$ only for the real numbers $z = n\pi$, n an integer, and $\cos z = 0$ only for the real numbers $z = (2n + 1)\pi/2$, n an integer.
- Thus, $\tan z$ and $\sec z$ are analytic except at the points $z = (2n + 1)\pi/2$, and $\cot z$ and $\csc z$ are analytic except at the points $z = n\pi$.

Derivatives

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z\tan z \qquad \frac{d}{dz}\csc z = -\csc z\cot z$$

Identities

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(2z) = 2\sin z \cos z \qquad \cos(2z) = \cos^2 z - \sin^2 z$$

Zeros

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin x \frac{e^{y} + e^{-y}}{2} + i\cos x \frac{e^{y} - e^{-y}}{2}$$

$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

$$\cos z = \cos x \cosh y - i\sin x \sinh y$$

$$\cosh^2 y = 1 + \sinh^2 y \Rightarrow |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y = 0 \Rightarrow \begin{cases} \sin x = 0 \\ \sinh y = 0 \end{cases} \Rightarrow \begin{cases} x = n\pi \\ y = 0 \end{cases}$$

$$\sin z = 0 \Rightarrow z = n\pi, \; n = 0, \; \pm 1, \; \pm 2, \; \ldots$$

$$\cos z = 0 \Rightarrow z = (2n+1)\pi/2, \; n = 0, \; \pm 1, \; \pm 2, \; \ldots$$

- Note: $|\sin x| \leq 1, \; |\cos x| \leq 1$ do not hold for the complex sine and cosine.
- Example 17: Solving a Trigonometric Equation

  Solve the equation $\cos z = 10$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 10 \Rightarrow e^{2iz} - 20e^{iz} + 1 = 0 \Rightarrow e^{iz} = 10 \pm 3\sqrt{11}$$

$$iz = \ln(10 \pm 3\sqrt{11}) + 2\pi n i, \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

$$\ln(10 - 3\sqrt{11}) = -\ln(10 + 3\sqrt{11})$$

$$z = 2\pi n \pm i\ln(10 + 3\sqrt{11}), \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

Hyperbolic Functions

- **Definition:** For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{1}{\tanh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}$$

Analyticity

- $\sinh z$ and $\cosh z$ are entire functions.
- $\tanh z$, $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are analytic except where the denominators are 0.

Derivatives

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z$$

$\sinh(iz) = i\sin z$ and $\cosh(iz) = \cos z$

$\sin z = -i\sinh(iz)$, $\cos z = \cosh(iz)$

$\sinh z = -i\sin(iz)$, $\cosh z = \cos(iz)$

Identities

$$\sinh(-z) = -\sinh z \qquad \cosh(-z) = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cos z_2 \pm \sinh z_1 \sin z_2$$

$$\sin(2z) = 2\sinh z \cosh z \qquad \cos(2z) = \cosh^2 z + \sinh^2 z$$

Zeros

$$\sinh z = \sinh x \cos y + i\cosh x \sin y$$

$$\cosh z = \cosh x \cos y + i\sinh x \sin y$$

$$\sinh z = 0 \Rightarrow z = n\pi i,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$\cosh z = 0 \Rightarrow z = (2n+1)\pi i/2,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

Periodicity

$\sin z$ and $\cos z$ are also periodic with the same real period 2π.

$\sinh z$ and $\cosh z$ have the imaginary period $2\pi i$.

8. Inverse Trigonometric and Hyperbolic Functions

Inverse Trigonometric Functions

The inverse multiple-valued sine function, $\sin^{-1} z$ or $\arcsin z$, is defined by:

$$w = \sin^{-1} z \text{ if } z = \sin w.$$

$$\frac{e^{iw} - e^{-iw}}{2i} = z \Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0 \Rightarrow e^{iw} = iz + (1 - z^2)^{1/2}$$

$$\sin^{-1} z = -i\log[iz + (1 - z^2)^{1/2}]$$

$$\cos^{-1} z = -i\log[z + i(1 - z^2)^{1/2}]$$

$$\tan^{-1} z = \frac{i}{2}\log\frac{i + z}{i - z}$$

- Example 18: Values of an Inverse of $\sin^{-1}\sqrt{5}$

$$\sin^{-1}\sqrt{5} = -i\log[\sqrt{5}i + (1-5)^{1/2}] = -i\log[(\sqrt{5}\pm 2)i] \qquad \left((1-5)^{1/2} = \pm 2i\right)$$

$$= -i[\ln(\sqrt{5}\pm 2) + (\pi/2 + 2\pi n)i],\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$\ln(\sqrt{5}-2) = -\ln(\sqrt{5}+2) \Rightarrow \sin^{-1}\sqrt{5} = \pi/2 + 2\pi \mathrm{n} \pm i\ln(\sqrt{5}+2),\ n = 0, \pm 1, \pm 2, \ldots$$

To obtain particular values of, $\sin^{-1}z$, we must choose a specific root of $1 - z^2$ and a specific branch of the logarithm. For example, if we choose $(-4)^{1/2} = 2i$ and the principal branch of the logarithm, then $\sin^{-1}\sqrt{5} = \pi/2 - i\ln(\sqrt{5}+2)$

Derivatives

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \qquad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}, \qquad \frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

Inverse Hyperbolic Functions

$$\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}] \qquad \frac{d}{dz}\sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}$$

$$\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}] \qquad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}}$$

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1 + z}{1 - z} \qquad \frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$$

- Example 19: Values of an Inverse Hyperbolic Cosine

  Find all values of $\cosh^{-1}(-1)$

$$\cosh^{-1}(-1) = \log(-1) = \ln 1 + (\pi + 2\pi n)i = (2n+1)\pi i,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$