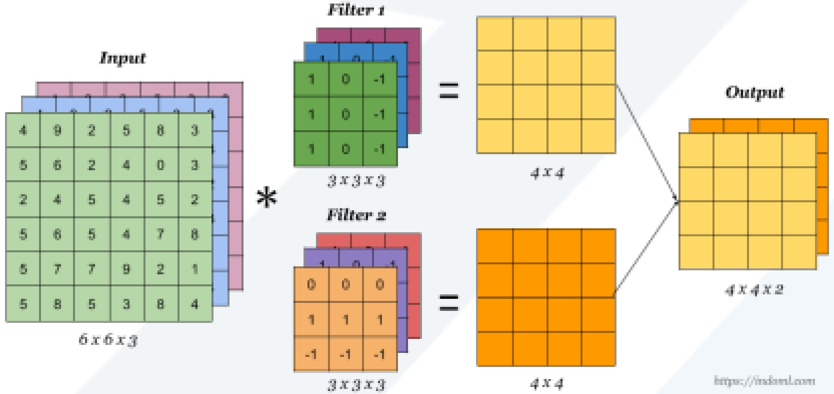
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 1 & 2: Linear Equations and Matrices



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Chapter 1

Linear Equations and Matrices

1. Systems of Linear Equations
2. Gaussian Elimination and Gauss-Jordan Elimination
3. Matrices and Matrix Operations
4. The Inverse of a Matrix
5. Elementary Matrices
6. Complex Matrices

1. Systems of Linear Equations

- Definition: An equation is where two mathematical expressions are defined as being equal. A linear equation is one where all variables (unknowns) such as x, y, z (x_1, x_2, x_3) have power of 1 only.
- The following are also linear equations:

$$x + 2y + z = 5; \qquad x + 2y = \sqrt{2};$$
$$3x_1 + x_2 + \pi x_3 + x_4 = -8; \qquad \sin(\pi/2)x - y = e^2.$$

- The following are not linear equations:

$$xy + 5z = 2; \qquad \sin x_1 + 2x_2 - 3x_3 = 0;$$
$$e^x - 2y = 3; \qquad 1/x + y^3 = 4.$$

- A linear equation in n variables: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$; $a_1, a_2, \ldots, a_n, b \in R$.
  a_1: leading coefficient; x_1: leading variable.

- Note: Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.
- Generally a finite number of linear equations with a finite number of unknowns x, y, z, w, ... is called a system of linear equations or just a linear system.
- For example, the following is a linear system of two equations with three unknowns x, y and z:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \end{array} \qquad (*)$$

- In general, a linear system of m equations in n unknowns x_1, x_2, ..., x_n is written as:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & \vdots & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where the coefficients a_{ij} and b_j represent real numbers.

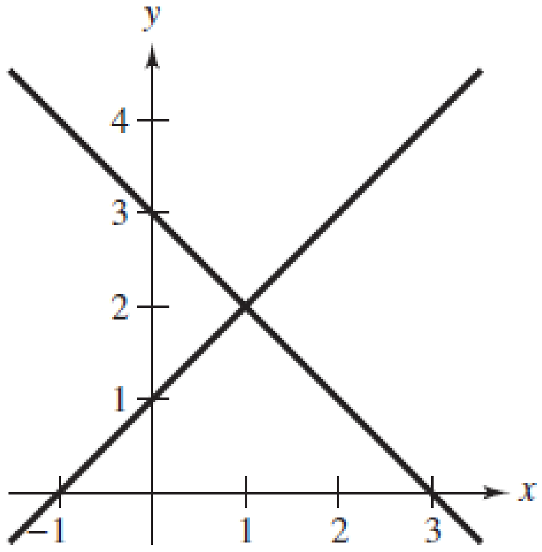
- A solution of a linear system in n unknowns x_1, x_2, ..., x_n is a sequence of n numbers s_1, s_2, ..., s_n for which the substitution $x_1 = s_1$, $x_2 = s_2$, ..., $x_n = s_n$ makes each equation a true statement.
- For example, the system in $(*)$ has the solution $x = 1$, $y = -1$ and $z = 2$, which can be written as (1, –1, 2).
- The set of all solutions of a linear equation is its solution set.
- A linear system that has no solution is called inconsistent.
- A linear system that has at least one solution is called consistent.
- Notes: Every system of linear equations has either:

  (1) exactly one solution, (2) infinitely many solutions, or

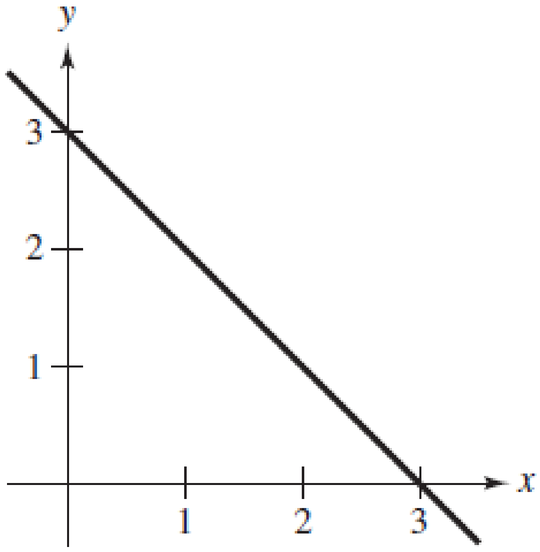
  (3) no solution.

- Example 1: (Systems of Two Equations in Two Variables)

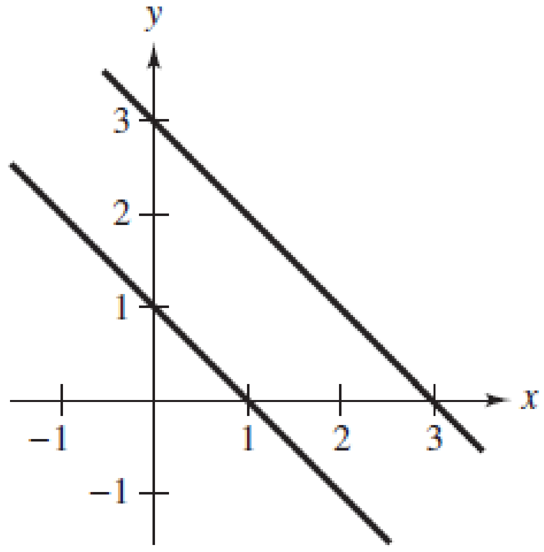
two intersecting lines	two coincident lines	two parallel lines
$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$	$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$	$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$



exactly one solution



infinite number



no solution

- Example 2: (Using back substitution to solve a system in row echelon form)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & (1) \\ & & y & + & 3z & = & 5 & (2) \\ & & & & z & = & 2 & (3) \end{array}$$

Substitute $z = 2$ into (2) $\Rightarrow$ $y + 3(2) = 5$

$$y = -1$$

and substitute $y = -1$ and $z = 2$ into (1) $\Rightarrow$ $x - 2(-1) + 3(2) = 9$

$$x = 1$$

The system has exactly one solution: $x = 1$, $y = -1$, $z = 2$

- Definition: Two systems of linear equations are called equivalent if they have precisely the same solution set. The aim here is to convert the given system into an equivalent simpler system that is in row-echelon form.

- Operations that Produce Equivalent Systems:

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

- Example 3: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & (1) \\ -x & + & 3y & & & = & -4 & (2) \\ 2x & - & 5y & + & 5z & = & 17 & (3) \end{array}$$

$(1)+(2) \rightarrow (2)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3) \rightarrow (3)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & (5) \end{array}$$

$(4)+(5)\rightarrow(5)$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \quad (6) \end{array}$$

$(6)\times\frac{1}{2}\rightarrow(6)$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is: $x=1,\ y=-1,\ z=2$

- Example 4: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & (1) \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 & (2) \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 & (3) \end{array}$$

$(1)\times(-2)+(2)\rightarrow(2) \qquad (1)\times(-1)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & \\ & & 5x_2 & - & 4x_3 & = & 0 & (4) \\ & & 5x_2 & - & 4x_3 & = & -2 & (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ \boxed{0 = -2} \end{aligned}$$

(a false statement)

- Example 5: Solve a system of linear equations (infinitely many solutions)

$$\begin{aligned} x_2 - x_3 &= 0 \quad (1) \\ x_1 - 3x_3 &= -1 \quad (2) \\ -x_1 + 3x_2 &= 1 \quad (3) \end{aligned}$$

$$(1)\leftrightarrow(2)$$

$$\begin{aligned} x_1 - 3x_3 &= -1 \quad (1) \\ x_2 - x_3 &= 0 \quad (2) \\ -x_1 + 3x_2 &= 1 \quad (3) \end{aligned}$$

$$(1)+(3)\rightarrow(3)$$

$$\begin{aligned} x_1 - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \quad (4) \end{aligned}$$

$(2)\times(-3)+(4)\rightarrow(4)$

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned} \qquad \Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

(a True statement)

letting $x_3 = t$, then the solutions are: $\{(3t-1,\ t,\ t) \mid t \in R\}$

2. Gaussian Elimination and Gauss-Jordan Elimination

Matrices

- **Definition:** A matrix with m rows and n columns ($m\times n$ matrix) is a rectangular array.

$$\begin{array}{c} \\ \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{cccc} \text{Column } 1 & \text{Column } 2 & \cdots & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

- Notes: Every system of linear equations has either:

(1) Every entry a_{ij} in a matrix is a number.

(2) A matrix with m rows and n columns is said to be of size $m \times n$.

(3) If $m = n$, then the matrix is called square of order n.

(4) For a square matrix, $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal entries.

$$\begin{bmatrix} 2 \end{bmatrix}$$ Size 1x1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ Size 2x2

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$ Size 1x4

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$ Size 3x2

- Note: One common use of matrices is to represent systems of linear equations. For A system of m equations in n variables:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $A\boldsymbol{x} = \boldsymbol{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] = [\,A|\boldsymbol{b}\,]$$

Coefficient matrix   Augmented matrix

- Elementary row operation:

  (1) Interchange two rows. $r_{ij}: R_i \leftrightarrow R_j$

  (2) Multiply a row by a nonzero constant. $r_i^{(k)}: (k)R_i \rightarrow R_i$

  (3) Add a multiple of a row to another row. $r_{ij}^{(k)}: (k)R_i + R_j \rightarrow R_j$

- Two matrices are said to be row equivalent if one can be obtained from the other by a finite sequence of elementary row operations.

- Example 6: (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- **Example 7: Using elementary row operations to solve a system**

Linear System	Augmented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned} \quad \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad r_{12}^{(1)}: (1)R_1 + R_2 \to R_2$$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned} \quad \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad r_{13}^{(-2)}: (-2)R_1 + R_3 \to R_3$$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned} \quad \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad r_{23}^{(1)}: (1)R_2 + R_3 \to R_3$$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned} \quad \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad r_3^{(\frac{1}{2})}: (\tfrac{1}{2})R_3 \to R_3$$

Use back-substitution to find the solution, as in Example 2. The solution is $x = 1$, $y = -1$, $z = 2$.

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Notes:

  (1) Each column of the coefficient matrix corresponds to a variable in the system of equations, we call each variable associated to a leading 1 in the RREF a leading variable.

  (2) A variable (if any) that is not a leading variable is called a free variable.

- Example 8: leading and free variables

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_2 - 4x_3 + x_4 &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3 are leading variables and x_2, x_4 are free variables.

- Example 9: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination with Back-substitution: The procedure for reducing a matrix to a row-echelon form, and use back-substitution to find the solution.
- Gauss-Jordan elimination: The procedure for reducing a matrix to a reduced row-echelon form.
- Notes:

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.
- Example 10: Solve a system by Gauss-Jordan elimination method (one solution)

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\ r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_{3}^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-3)},\ r_{32}^{(-3)},\ r_{21}^{(2)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{matrix} x & & & = & 1 \\ & y & & = & -1 \\ & & z & = & 2 \end{matrix}$$

row-echelon form

reduced row-echelon form

- Example 11: Solve a system by G.-J. elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \quad\quad &= 1 \end{aligned}$$

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\ r_{12}^{(-3)},\ r_2^{(-1)},\ r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

augmented matrix — reduced row-echelon form

$$\longrightarrow \begin{aligned} x_1 \quad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

leading variables: x_1, x_2

free variable: x_3

$$\begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

$x_3 = t$, then the solutions are: $\{(2 - 5t, -1 + 3t, t) | t \in R\}$

So the system has infinitely many solutions.

- Example 12: Solve a system by Gauss elimination method (no solution)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 \qquad + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{r_{12}^{(-1)},\ r_{13}^{(-2)},\ r_{14}^{(-3)},\ r_{23}^{(1)}} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

$$\longrightarrow \begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third equation is not possible, the system has no solution.

Homogeneous systems of linear equations

- A system of linear equations is said to be homogeneous if all the constant terms are zero.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & \vdots & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$
- Nontrivial solution: other solutions

- Theorem 1: (The Number of Solutions of a Homogeneous System)

  Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

- Example 13: Solve the following homogeneous system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 3x_3 & = & 0 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 0 \end{array}$$

augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\ r_2^{(-\frac{1}{3})},\ r_{21}^{(-1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{array}{rcrcl} x_1 & & + 2x_3 & = & 0 \\ & x_2 & + x_3 & = & 0 \end{array}$$

reduced row-echelon form

leading variables: $x_1,\ x_2$

free variable: x_3

letting $x_3 = t$, then the solutions are:

$$\{(-2t,\ -t,\ t) | t \in R\}$$

when $t = 0,\ x_1 = x_2 = x_3 = 0$ (trivial solution)

3. Matrices and Matrix Operations

$$A = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in M_{m\times n}(R \text{ or } C)$$

Matrix of size mxn

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

$$A = \operatorname{diag}(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Diagonal matrix

- If $A = \left[a_{ij}\right]_{n\times n}$, then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$ Trace of a Matrix

$$\begin{bmatrix} 1 & -1 & i \\ 3 & 2i & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & i & -i & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix}, \quad \begin{bmatrix} 1+i & 1 \\ i & 1-i \end{bmatrix}$$

Complex matrices

- Equal matrix: If $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$, then

  $A = B$ if and only if $a_{ij} = b_{ij} \ \forall\, 1 \le i \le m, 1 \le j \le n$

  $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  If $A = B$ Then $a = 1$, $b = 2$, $c = 3$, $d = 4$

- Matrix addition:

  If $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$, then $A + B = \left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n} = \left[a_{ij} + b_{ij}\right]_{m\times n}$

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Scalar multiplication: If $A = \left[a_{ij}\right]_{m\times n}$, c: scalar, then $cA = \left[ca_{ij}\right]_{m\times n}$
- Matrix subtraction: $A - B = A + (-1)B$
- Example 14: (Scalar multiplication and matrix subtraction)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find $(a)\ 3A$, $(b)\ -B$, $(c)\ 3A - B$

$$(a)\quad 3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

$$(b)\quad -B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

$$(c)\quad 3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- **Matrix multiplication:** If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$, then $AB = [a_{ij}]_{m\times n}[b_{ij}]_{n\times p} = [c_{ij}]_{m\times p}$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

- **Notes:** (1) $A + B = B + A$, (2) $AB \neq BA$
- **Example 15: (Find AB)**

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Properties of Matrix Operations

- Zero matrix: $O_{m\times n}$
- Identity matrix of order n: I_n
- Properties of matrix addition and scalar multiplication: If A, B, $C \in M_{m\times n}$, then

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd)A = c(dA)$

(4) $1A = A$

(5) $c(A + B) = cA + cB$

(6) $(c + d)A = cA + dA$

- Properties of zero matrices: If $A \in M_{m\times n}$, c scalar, then

(1) $A + O_{m\times n} = A$

(2) $A + (-A) = O_{m\times n}$

(3) $cA = O_{m\times n} \Rightarrow c = 0$ or $A = O_{m\times n}$

- Notes:

(1) $O_{m\times n}$: the additive identity for the set of all $m\times n$ matrices.

(2) $-A$: the additive inverse of A.

- Properties of matrix multiplication:

(1) $A(BC) = (AB)C$ (2) $A(B + C) = AB + AC$

(3) $(A + B)C = AC + BC$ (4) $c(AB) = (cA)B = A(cB)$

- Properties of identity matrix: If $A \in M_{m\times n}$, then

(1) $AI_n = A$ (2) $I_m A = A$

- Transpose of a matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

$$A=\begin{bmatrix}2\\8\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

- Properties of transposes:

  (1) $(A^T)^T = A$

  (2) $(A+B)^T = A^T + B^T$

  (3) $(cA)^T = c(A)^T$

  (4) $(AB)^T = B^T + A^T$

- A square matrix A is symmetric if $A^T = A$
- A square matrix A is skew-symmetric if $A^T = -A$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

$A = A^T \Rightarrow a = 2,\ \ b = 3,\ \ c = 5$

If $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c?

$A = -A^T \Rightarrow a = -1,\ \ b = -2,\ \ c = -3$

- Notes:

(1) AA^T is symmetric.

(2) Every square matrix $A \in M_n(R)$ can be expressed as the sum of a symmetric matrix B and a skew-symmetric matrix C.

$$B = \tfrac{1}{2}(A + A^T), \quad C = \tfrac{1}{2}(A - A^T)$$

- Noncommutativity of Matrix Multiplication

$\underset{m\times n}{A}\underset{n\times p}{B} \neq BA$ Three situations:

(1) If $m \neq p$, then AB is defined, BA is undefined

(2) If $m = p$, $m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

(3) If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$ (Sizes are the same, $AB \neq BA$)

- Example 16: (AB and BA are not equal)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix} \quad AB \neq BA$$

- Cancelation Law

$AC = BC,\ C \neq O$

(1) If C is invertible, then $A = B$ (Cancellation is valid)

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Example 17: (An example in which cancellation is not valid)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}, \quad BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So $AC = BC$ but $A \neq B$

4. The Inverse of a Matrix

Consider $A \in M_n$. If there exists a matrix $B \in M_n$. such that $AB = BA = I_n$, then

(1) A is invertible (or nonsingular) (2) B is the inverse of A

- Note: A matrix that does not have an inverse is called noninvertible (or singular).
- Notes:

  (1) The inverse of a matrix is unique.

  (2) The inverse of A is denoted by A^{-1}, (3) $AA^{-1} = A^{-1}A = I$.

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$\begin{bmatrix} A & | & I \end{bmatrix} \longrightarrow \begin{bmatrix} I & | & A^{-1} \end{bmatrix}$$ Gauss-Jordan Elimination

- Note: If A can't be row reduced to I, then A is singular.

- Example 18: (Find the inverse of the following matrix)

$$A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ -6 & 2 & 3\end{bmatrix}$$

$$\left[A \;\vdots\; I\right]=\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ -6 & 2 & 3 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ -6 & 2 & 3 & 0 & 0 & 1\end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & -4 & 3 & 6 & 0 & 1\end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & -1 & 2 & 4 & 1\end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & -4 & -1\end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

So the matrix A is invertible, and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

- Power of a square matrix:

(1) $A^0 = I$

(2) $A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$

(3) $A^r \cdot A^s = A^{r+s} \quad r, s\text{: integers}$

(4) $(A^r)^s = A^{rs}$

(5) $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

- Theorem 2: (Properties of inverse matrices)

  If A is an invertible matrix, k is a positive integer, and c is a scalar $\neq 0$, then

  (1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

  (2) A^k is invertible and $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k = A^{-k}$

  (3) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}, \; c \neq 0$

  (4) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

- Theorem 3: (The inverse of a product)

  If $A, B \in M_n$ are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

- Note: $(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$
- Theorem 4: (Cancellation properties)

  If C is an invertible matrix, then the following properties hold:

  (1) If $AC = BC$, then $A = B$ (Right cancellation property)

  (2) If $CA = CB$, then $A = B$ (Left cancellation property)
- Note: If C is not invertible, then cancellation is not valid.
- Theorem 5: (Systems of equations with unique solutions)

  If A is an invertible matrix, then the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

- Example 19: Use an inverse matrix to solve each system

$$(a)\quad \begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$(a)\ \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$(b)\ \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Elementary Matrices

- An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.
- Three elementary matrices:

(1) $R_{ij} = r_{ij}(I)$ — Interchange two rows

(2) $R_i^{(k)} = r_i^{(k)}(I) \qquad (k \neq 0)$ — Multiply a row by a nonzero constant

(3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ — Add a multiple of a row to another row

- Note: Only do a single elementary row operation.
- Example 20: (Elementary matrices and non elementary matrices)

$$(a) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} r_2^{(2)}(I_2) \qquad (b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (c) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} r_{12}^{(2)}(I_2)$$

$$(d)\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} r_{23}(I_3) \qquad (e)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \qquad (f)\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}$$

- **Theorem 6: (Representing elementary row operations)**

  Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by EA. ($r(I) = E$, $r(A) = EA$)

- **Example 21: (Elementary matrices and elementary row operation)**

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix} \quad (r_{12}(A) = R_{12}A)$$

$$(b)\ \begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}\ (r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\ \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}\ (r_{12}^{(2)}(A)=R_{12}^{(2)}A)$$

- Example 22: (Using elementary matrices)

  Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A=\begin{bmatrix}0&1&3&5\\1&-3&0&2\\2&-6&2&0\end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}, \qquad E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2A_1 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}, \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3A_2 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \qquad E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$B = E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- **Definition:** A Matrix B is **row-equivalent** to A if there exists a finite number of elementary matrices such that: $B = E_k E_{k-1} \dots E_1 A$.

- **Theorem 7: (Elementary matrices are invertible)**
  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- **Notes:** (1) $(R_{ij})^{-1} = R_{ij}$ (2) $(R_i^{(k)})^{-1} = R_i^{(1/k)}$ (3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- # Example 23: (Inverse of elementary matrices)

Elementary Matrix | **Inverse Matrix**

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$

- Theorem 8: (A property of invertible matrices)

  A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

- Example 24: Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow{r_2^{(1/2)}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

Thus $A = (R_1^{(-1)})^{-1} (R_{12}^{(-3)})^{-1} (R_2^{(\frac{1}{2})})^{-1} (R_{21}^{(-2)})^{-1} = R_1^{(-1)} R_{12}^{(3)} R_2^{(2)} R_{21}^{(2)}$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- **Note:** If A is invertible then

  $E_k \cdots E_3 E_2 E_1 A = I$ $\qquad A^{-1} = E_k \cdots E_3 E_2 E_1$ $\qquad A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$

- **Theorem 9: (Equivalent conditions)**

  If A is an $n \times n$ matrix, then the following statements are equivalent.

  (1) A is invertible.

  (2) $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every $n \times 1$ column matrix $\boldsymbol{b}$.

  (3) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

  (4) A is row-equivalent to I_n.

  (5) A can be written as the product of elementary matrices.

- LU-factorization:

  If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU -factorization of A

- Note: If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU–factorization of A.

$$E_k \cdots E_2 E_1 A = U \Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U = LU \qquad (L = E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

- Example 25: (LU–factorization)

$$(a)\ A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$$

(a)

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1&2\\0&-2\end{bmatrix}=U=R_{12}^{(-1)}A\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&-3&0\\0&1&3\\2&-10&2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&-4&2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&0&14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\2&-4&1\end{bmatrix}$$

- Solving $A\boldsymbol{x} = \boldsymbol{b}$ with an LU-factorization of A

  $A\boldsymbol{x} = \boldsymbol{b}$ If $A = LU$, then $LU\boldsymbol{x} = \boldsymbol{b}$ Let $\boldsymbol{y} = U\boldsymbol{x}$, then $L\boldsymbol{y} = \boldsymbol{b}$, two steps:

  (1) Write $\boldsymbol{y} = U\boldsymbol{x}$, and solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ using forward substitution.

  (2) Solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ using backward substitution.

- Example 26: (Solving a linear system using LU–factorization)

$$\begin{aligned} x_1 - 3x_2 \qquad\quad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $\boldsymbol{y} = U\boldsymbol{x}$, and solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 = -14 \end{aligned}$$

(2) Solve the following system $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= -1 \\ x_2 &= -1 - 3x_3 = -1 - (3)(-1) = 2 \\ x_1 &= -5 + 3x_2 = -5 + 3(2) = 1 \end{aligned}$$

Thus, the solution is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

6. Complex Matrices

- **Conjugate of a matrix:** $A \in M_{m\times n}(C) = \left[a_{ij}\right]_{m\times n} \Rightarrow \overline{A} \in M_{m\times n}(C) = \left[\overline{a_{ij}}\right]_{m\times n}$

$$A = \begin{bmatrix} 1+i & 1 \\ i & 1-i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 1-i & 1 \\ -i & 1+i \end{bmatrix}$$

- **Properties of the conjugate of a matrix:**

(1) $\overline{\overline{A}} = A$ (2) $\overline{A \pm B} = \overline{A} \pm \overline{B}$ (3) $\overline{AB} = \overline{A}\,\overline{B}$

(4) $\overline{cA} = \overline{c}\,\overline{A}, \quad c \in C$ (5) $(\overline{A})^T = \overline{A^T}$

(6) If A is invertible, then $(\overline{A})^{-1} = \overline{A^{-1}}$

- **Conjugate transpose of a matrix:** $A \in M_{m\times n}(C) \Rightarrow A^* = \overline{A^T} \in M_{n\times m}(C)$

$$A=\begin{bmatrix}1+i & -i & 0\\ 2 & 3-2i & i\end{bmatrix}\Rightarrow A^*=\overline{A^T}=\begin{bmatrix}1-i & 2\\ i & 3+2i\\ 0 & -i\end{bmatrix}$$

- Properties of the conjugate transpose:

(1) $(A^*)^*=A$ (2) $(A\pm B)^*=A^*\pm B^*$

(3) $(AB)^*=B^*A^*$ (4) $(cA)^*=\bar{c}\,A^*,\quad c\in C$

- A square matrix $A\in M_n(C)$ is Hermitian if $A^*=A$

$$A=\begin{bmatrix}2 & 2+i & 4\\ 2-i & 3 & i\\ 4 & -i & 1\end{bmatrix}=A^*$$

- A square matrix $A \in M_n(C)$ is skew-Hermitian if $A^* = -A$

$$A = \begin{bmatrix} -i & 2+i \\ -2+i & 0 \end{bmatrix} = -A^*$$

- Notes:

(1) Diagonal entries of an Hermitian matrix are real.

(2) Diagonal entries of a Skew-Hermitian matrix are purely imaginary or zero.

(3) Every square matrix $A \in M_n(C)$ can be expressed as the sum of a Hermitian matrix B and a skew-Hermitian matrix C.

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2}(A - A^*)$$

Applications

- Systems of linear equations arise in a wide variety of applications.
  - Fitting a polynomial function to a set of data points in the plane.
  - Networks and Kirchhoff's Laws for electricity.
  - Solving puzzles (Sudoku puzzles).
- Matrices are used in cryptography to encode and decode information.
- Matrices are used in Finding the least squares regression line for a set of data.
- Matrix algebra is used to analyze an economic system.