



الرياضيات

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التابع الوغاريتمي

مبرهنة 1 :

يوجد تابع وحيد، يرمز له $\ln:]0, +\infty[\rightarrow \mathbb{R}$ ، بحيث :

$$\ln'(x) = \frac{1}{x} ; (\forall x > 0) \text{ and } \ln(1) = 0$$

كذلك يحقق هذا التابع من أجل كلٍّ من $a, b > 0$:

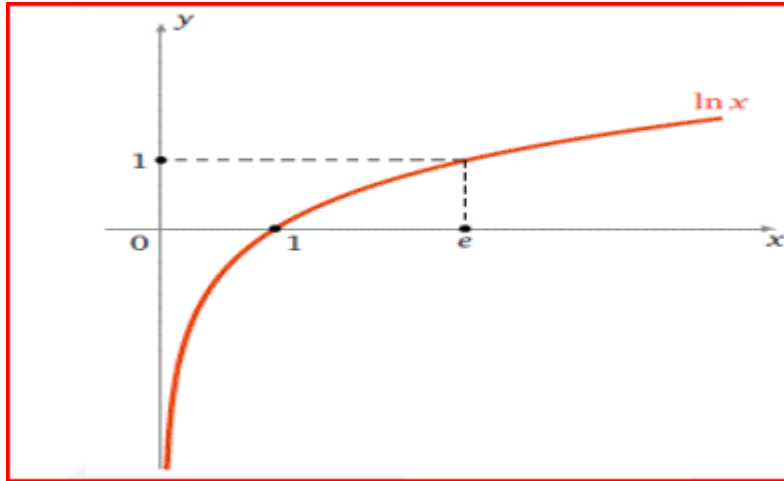
$$.1 \ln(a \times b) = \ln a + \ln b$$

$$.2 \ln\left(\frac{1}{a}\right) = -\ln(a)$$

$$.3 \ln(a^n) = n \ln(a) \quad (\forall n \in \mathbb{N})$$

.4 \ln تابع مستمرّ،

$$.5 \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$





ملاحظة :

\ln يدعى التابع اللوغاريتمي الطبيعيّ أو النيبيري، حيث $\ln e = 1$ ، نعرّف اللوغاريتم بالأساسي a ، ورمزه

\log_a كالآتي :

$$\log_a(x) = \frac{\ln x}{\ln a}$$

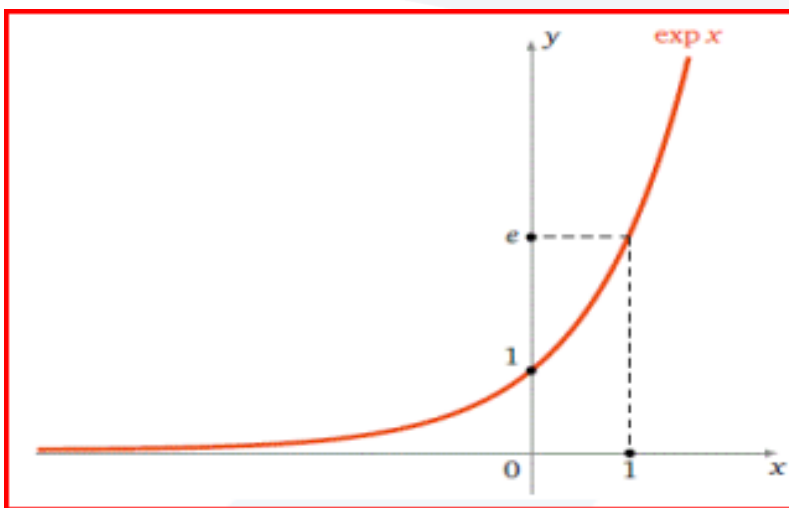
$$\log_a(a) = 1 \text{ و}$$

وعندما يكون $a = 10$ ، نحصل على اللوغاريتم العشري \log_{10} الذي يحقّق $\log_{10}(10) = 1$ ، ويكون

$$\log_{10}(10^n) = n$$

نستخدم التكافؤ الآتي في التطبيقات كثيراً :

$$x = 10^y \Leftrightarrow y = \log_{10} x$$



التّقابل العكسي للتابع $\ln :]0, +\infty[\rightarrow \mathbb{R}$. يدعى التّابع الأسّي، ونرمز له :

$$\exp: \mathbb{R} \rightarrow]0, +\infty[$$

من أجل كلّ $x \in \mathbb{R}$ ، نرمز كذلك للتابع $\exp x$ بالرمز e^x .

من الجدير بالذّكر أنّ التّابع الأسّي، يأخذ قيماً موجبة تماماً دوماً.

مبرهنة 3 :

يحقّق التّابع الأسّيّ العلاقات الآتية :

- $\ln(\exp x) = x ; \forall x \in \mathbb{R}$ و $\exp(\ln x) = x ; \forall x > 0$
- $(\forall a, b \in \mathbb{R}) \exp(a + b) = \exp a \times \exp b$
- $\exp(nx) = (\exp x)^n$
- $\exp: \mathbb{R} \rightarrow]0, +\infty[$ تابع مستمرّ، ومتزايد تماماً، ويحقّق : $\lim_{x \rightarrow +\infty} \exp x = +\infty$ و $\lim_{x \rightarrow -\infty} \exp x = 0$
- التّابع الأسّيّ قابل للاشتقاق، و $\forall x \in \mathbb{R}$ $\exp' x = \exp x$ وكذلك $\exp x \geq 1 + x$

ملاحظة : التّابع الأسّيّ هو التّابع الوحيد الذي يحقّق أن

$$\exp' x = \exp x \quad \forall x \in \mathbb{R}, \text{ ويحقّق } \exp(1) = e, \text{ حيث } e \approx 2.718 \text{ ويحقّق } \ln e = 1.$$

تعريف:

بالتعريف، من أجل $a > 0$ و $b \in \mathbb{R}$ يكون:

$$a^b = \exp(b \ln a)$$

ملاحظة:

$$\sqrt{a} = a^{\frac{1}{2}} = \exp\left(\frac{1}{2} \ln a\right) \bullet$$

$$\sqrt[n]{a} = a^{\frac{1}{n}} = \exp\left(\frac{1}{n} \ln a\right) \bullet$$

$$a^x = \exp(x \ln a) \bullet$$

التابع a^x $x \mapsto a^x$ ، يمكن كتابته بالشكل $a^x = \exp(x \ln a)$

مبرهنة 4:

من أجل أيّ عددين $a, b \in \mathbb{R}$ و $x, y > 0$ ، يكون:

$$x^{a+b} = x^a \cdot x^b \bullet$$

$$x^{-a} = \frac{1}{x^a} \bullet$$

$$(x \cdot y)^a = x^a \cdot y^a \bullet$$

$$(x^a)^b = x^{ab} \bullet$$

$$\ln(x^a) = a \cdot \ln x \bullet$$

EXAMPLE 8 | Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \frac{1}{3}(5 - \ln 10)$$

What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?

$$e^{\ln 300} = 300 \text{ and } \ln(e^{300}) = 300.$$

Find the domain of $f(x) = \ln(e^x - 3)$.

(a) We must have $e^x - 3 > 0 \Rightarrow e^x > 3 \Rightarrow x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$

Find the exact value of each expression.

35. (a) $\log_5 125$

(b) $\log_3 \left(\frac{1}{27}\right)$

36. (a) $\ln(1/e)$

(b) $\log_{10} \sqrt{10}$

38. (a) $e^{-2 \ln 5}$

(b) $\ln(\ln e^{e^{10}})$

39–41 Express the given quantity as a single logarithm.

39. $\ln 5 + 5 \ln 3$

40. $\ln(a + b) + \ln(a - b) - 2 \ln c$

41. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2} [\ln x - \ln(x^2 + 3x + 2)^2]$

47–50 Solve each equation for x .

47. (a) $e^{7-4x} = 6$

(b) $\ln(3x - 10) = 2$

48. (a) $\ln(x^2 - 1) = 3$

(b) $e^{2x} - 3e^x + 2 = 0$

49. (a) $2^{x-5} = 3$

(b) $\ln x + \ln(x - 1) = 1$

50. (a) $\ln(\ln x) = 1$

51–52 Solve each inequality for x .

51. (a) $\ln x < 0$

(b) $e^x > 5$

52. (a) $1 < e^{3x-1} < 2$

(b) $1 - 2 \ln x < 3$



35. (a) $\log_5 125 = 3$ since $5^3 = 125$.

(b) $\log_3 \frac{1}{27} = -3$ since $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$.

36. (a) $\ln(1/e) = \ln 1 - \ln e = 0 - 1 = -1$

(b) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by (2).

38. (a) $e^{-2 \ln 5} = (e^{\ln 5})^{-2} \stackrel{(6)}{=} 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

(b) $\ln(\ln e^{e^{10}}) \stackrel{(6)}{=} \ln(e^{10}) \stackrel{(6)}{=} 10$

39. $\ln 5 + 5 \ln 3 = \ln 5 + \ln 3^5$
 $= \ln(5 \cdot 3^5)$
 $= \ln 1215$

40. $\ln(a + b) + \ln(a - b) - 2 \ln c = \ln[(a + b)(a - b)] - \ln c^2$
 $= \ln \frac{(a + b)(a - b)}{c^2}$
or $\ln \frac{a^2 - b^2}{c^2}$

$$\begin{aligned} 41. \quad \frac{1}{3} \ln(x+2)^3 + \frac{1}{2} [\ln x - \ln(x^2 + 3x + 2)^2] &= \ln[(x+2)^3]^{1/3} + \frac{1}{2} \ln \frac{x}{(x^2 + 3x + 2)^2} \\ &= \ln(x+2) + \ln \frac{\sqrt{x}}{x^2 + 3x + 2} \\ &= \ln \frac{(x+2)\sqrt{x}}{(x+1)(x+2)} \\ &= \ln \frac{\sqrt{x}}{x+1} \end{aligned}$$

Note that since $\ln x$ is defined for $x > 0$, we have $x + 1$, $x + 2$, and $x^2 + 3x + 2$ all positive, and hence their logarithms are defined.

47. (a) $e^{7-4x} = 6 \Leftrightarrow 7 - 4x = \ln 6 \Leftrightarrow 7 - \ln 6 = 4x \Leftrightarrow x = \frac{1}{4}(7 - \ln 6)$

(b) $\ln(3x - 10) = 2 \Leftrightarrow 3x - 10 = e^2 \Leftrightarrow 3x = e^2 + 10 \Leftrightarrow x = \frac{1}{3}(e^2 + 10)$

48. (a) $\ln(x^2 - 1) = 3 \Leftrightarrow x^2 - 1 = e^3 \Leftrightarrow x^2 = 1 + e^3 \Leftrightarrow x = \pm\sqrt{1 + e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \Leftrightarrow (e^x - 1)(e^x - 2) = 0 \Leftrightarrow e^x = 1 \text{ or } e^x = 2 \Leftrightarrow x = \ln 1 \text{ or } x = \ln 2, \text{ so } x = 0 \text{ or } \ln 2$.

49. (a) $2^{x-5} = 3 \Leftrightarrow \log_2 3 = x - 5 \Leftrightarrow x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \Leftrightarrow \ln(2^{x-5}) = \ln 3 \Leftrightarrow (x - 5) \ln 2 = \ln 3 \Leftrightarrow x - 5 = \frac{\ln 3}{\ln 2} \Leftrightarrow x = 5 + \frac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x - 1) = \ln(x(x - 1)) = 1 \Leftrightarrow x(x - 1) = e^1 \Leftrightarrow x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}(1 \pm \sqrt{1 + 4e})$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}(1 + \sqrt{1 + 4e})$.

50. (a) $\ln(\ln x) = 1 \Leftrightarrow e^{\ln(\ln x)} = e^1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow e^{\ln x} = e^e \Leftrightarrow x = e^e$

51. (a) $\ln x < 0 \Rightarrow x < e^0 \Rightarrow x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

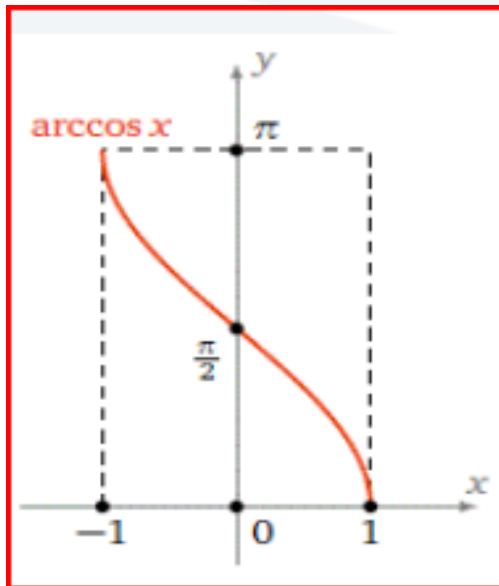
(b) $e^x > 5 \Rightarrow \ln e^x > \ln 5 \Rightarrow x > \ln 5$

52. (a) $1 < e^{3x-1} < 2 \Rightarrow \ln 1 < 3x - 1 < \ln 2 \Rightarrow 0 < 3x - 1 < \ln 2 \Rightarrow 1 < 3x < 1 + \ln 2 \Rightarrow \frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

(b) $1 - 2 \ln x < 3 \Rightarrow -2 \ln x < 2 \Rightarrow \ln x > -1 \Rightarrow x > e^{-1}$

التوابع المثلثية العكسية

التابع \arccos



ليكن لدينا التابع $\cos: \mathbb{R} \rightarrow [-1, +1]$ ، $x \mapsto \cos x$ ، حتى يكون هذا التابع تقابلاً، ينبغي أن نقصر التابع على المجال $[0, \pi]$ ، حيث يكون التابع مستمراً و متزايداً تماماً، وبهذا التابع $\cos: [0, \pi] \rightarrow [-1, +1]$ ، ويملك تقابلاً عكسياً \arccos المعرف بالشكل :

$$\arccos x : [-1, +1] \rightarrow [0, \pi]$$

التوابع المثلثية العكسية

وبحسب تعريف التّقابل العكسيّ :

$$\cos(\arccos x) = x \quad , \quad \forall x \in [-1, +1]$$

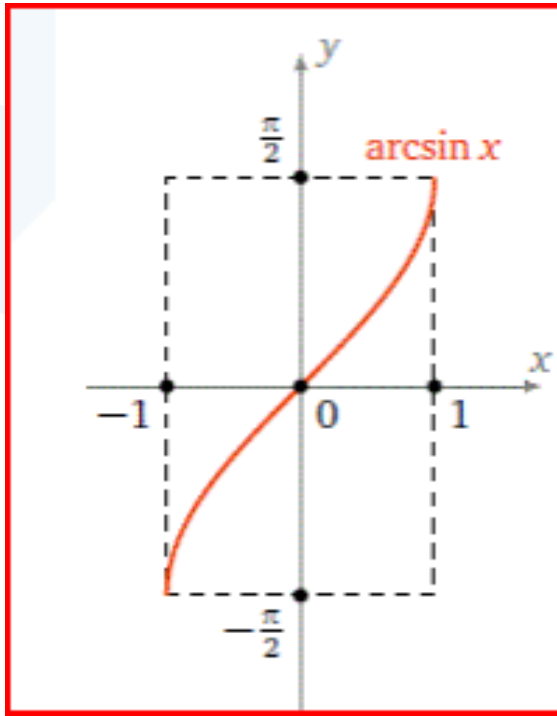
$$\arccos(\cos x) = x \quad , \quad \forall x \in [0, \pi]$$

بكلام آخر:

$$\arccos(y) = x \Leftrightarrow \cos x = y \quad , \quad \forall x \in [0, \pi]$$

وأخيراً : لنرى مشتقّ \arccos

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad \forall x \in]-1, +1[$$



مقصود التابع \sin على المجال $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ المعرف بالشكل :

$$\sin: [-\frac{\pi}{2}, +\frac{\pi}{2}] \rightarrow [-1, +1]$$

يمثل تقابلاً، وتقابله العكسي هو التابع \arcsin

$$\arcsin: [-1, +1] \rightarrow [-\frac{\pi}{2}, +\frac{\pi}{2}]$$

$$\sin(\arcsin x) = x, \quad \forall x \in [-1, +1]$$

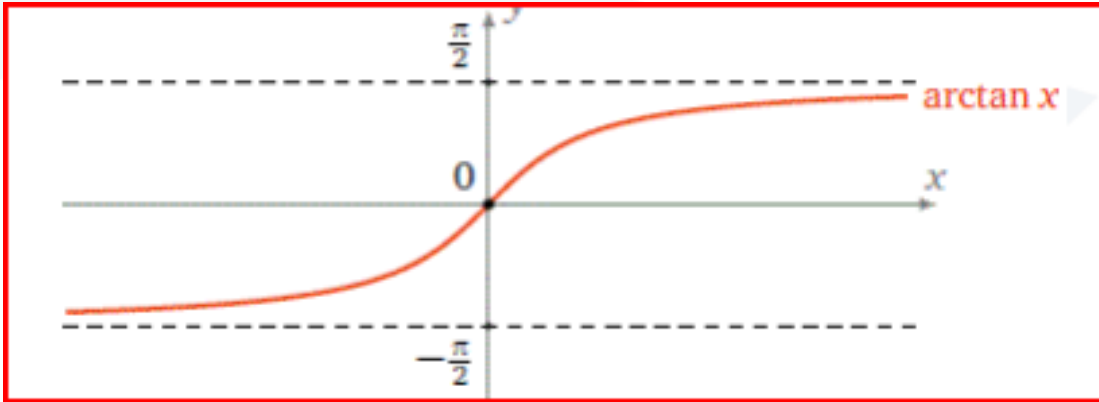
$$\arcsin(\sin x) = x, \quad \forall x \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$$

وبكلام آخر:

$$\arcsin(y) = x \Leftrightarrow \sin x = y, \quad \forall x \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$$

وأخيراً: لنرى مشتق \arccos

$$\arcsin' x = \frac{+1}{\sqrt{1-x^2}}; \quad \forall x \in]-1, +1[$$



مقصود التابع tan على المجال $]-\frac{\pi}{2}, +\frac{\pi}{2}[$ المعرف بالشكل :

$$\tan:]-\frac{\pi}{2}, +\frac{\pi}{2}[\rightarrow \mathbb{R}$$

يمثل تقابلاً، وتقابله العكسي هو التابع arctan

$$\arctan: \mathbb{R} \rightarrow]-\frac{\pi}{2}, +\frac{\pi}{2}[$$

لدينا حسب تعريف التّقابل العكسي :

$$\tan(\arctan x) = x, \quad \forall x \in \mathbb{R}$$

$$\arctan(\tan x) = x, \quad \forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$$

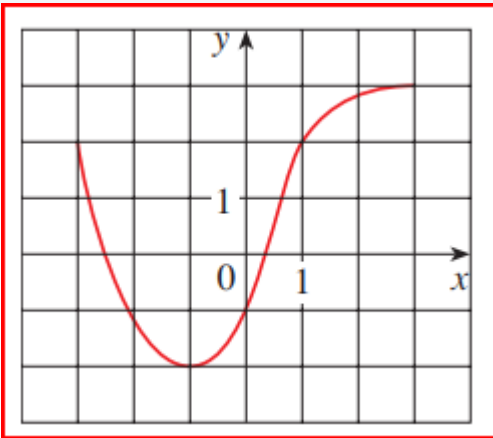
وبكلام آخر:

$$\arctan(y) = x \Leftrightarrow \tan x = y, \quad \forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[$$

وأخيراً: لنرى مشتقّ arccos

$$\arctan' x = \frac{1}{1+x^2} \quad \forall x \in \mathbb{R}$$

Thank you for your attention



1. The graph of a function f is given at the left.
 - (a) State the value of $f(-1)$.
 - (b) Estimate the value of $f(2)$.
 - (c) For what values of x is $f(x) = 2$?
 - (d) Estimate the values of x such that $f(x) = 0$.
 - (e) State the domain and range of f .

3. Find the domain of the function.

(a) $f(x) = \frac{2x + 1}{x^2 + x - 2}$

(b) $g(x) = \frac{\sqrt[3]{x}}{x^2 + 1}$

(c) $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

1. (a) Locate -1 on the x -axis and then go down to the point on the graph with an x -coordinate of -1 . The corresponding y -coordinate is the value of the function at $x = -1$, which is -2 . So, $f(-1) = -2$.
- (b) Using the same technique as in part (a), we get $f(2) \approx 2.8$.
- (c) Locate 2 on the y -axis and then go left and right to find all points on the graph with a y -coordinate of 2 . The corresponding x -coordinates are the x -values we are searching for. So $x = -3$ and $x = 1$.
- (d) Using the same technique as in part (c), we get $x \approx -2.5$ and $x \approx 0.3$.
- (e) The domain is all the x -values for which the graph exists, and the range is all the y -values for which the graph exists. Thus, the domain is $[-3, 3]$, and the range is $[-2, 3]$.

3. (a) Set the denominator equal to 0 and solve to find restrictions on the domain: $x^2 + x - 2 = 0 \Rightarrow$

$(x - 1)(x + 2) = 0 \Rightarrow x = 1$ or $x = -2$. Thus, the domain is all real numbers except 1 or -2 or, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

(b) Note that the denominator is always greater than or equal to 1, and the numerator is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

(c) Note that the function h is the sum of two root functions. So h is defined on the intersection of the domains of these two root functions. The domain of a square root function is found by setting its radicand greater than or equal to 0. Now,

$4 - x \geq 0 \Rightarrow x \leq 4$ and $x^2 - 1 \geq 0 \Rightarrow (x - 1)(x + 1) \geq 0 \Rightarrow x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

تمرين

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f .

7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.

(a) $f \circ g$

(b) $g \circ f$

(c) $g \circ g \circ g$

Determine whether each of the following functions is even, odd, or neither even nor odd.

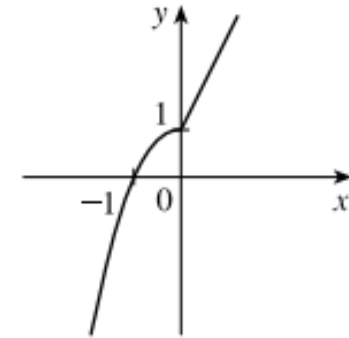
(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

6. (a) $f(-2) = 1 - (-2)^2 = -3$ and $f(1) = 2(1) + 1 = 3$

(b) For $x \leq 0$ plot $f(x) = 1 - x^2$ and, on the same plane, for $x > 0$ plot the graph of $f(x) = 2x + 1$.



7. (a) $(f \circ g)(x) = f(g(x)) = f(2x - 3) = (2x - 3)^2 + 2(2x - 3) - 1 = 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) - 3 = 2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = g(g(g(x))) = g(g(2x - 3)) = g(2(2x - 3) - 3) = g(4x - 9) = 2(4x - 9) - 3$
 $= 8x - 18 - 3 = 8x - 21$

$$\begin{aligned} \text{(a)} \quad f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

$$\text{(b)} \quad g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

$$\text{(c)} \quad h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

19–20 Find the domain of each function.

19. (a) $f(x) = \frac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \frac{1 + x}{e^{\cos x}}$

20. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

23. If $f(x) = 5^x$, show that

$$\frac{f(x + h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right)$$

37. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

19. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \Leftrightarrow e^{1-x^2} = 1 \Leftrightarrow 1 - x^2 = 0 \Leftrightarrow x = \pm 1$. Thus, the function $f(x) = \frac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

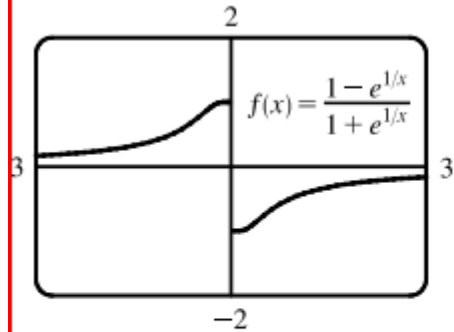
(b) The denominator is never equal to zero, so the function $f(x) = \frac{1+x}{e^{\cos x}}$ has domain \mathbb{R} , or $(-\infty, \infty)$.

20. (a) The sine and exponential functions have domain \mathbb{R} , so $g(t) = \sin(e^{-t})$ also has domain \mathbb{R} .

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

23. If $f(x) = 5^x$, then $\frac{f(x+h) - f(x)}{h} = \frac{5^{x+h} - 5^x}{h} = \frac{5^x 5^h - 5^x}{h} = \frac{5^x(5^h - 1)}{h} = 5^x \left(\frac{5^h - 1}{h} \right)$.

الحل



From the graph, it appears that f is an odd function (f is undefined for $x = 0$).

To prove this, we must show that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1} \\ &= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x) \end{aligned}$$

so f is an odd function.

Find the domain and sketch the graph of the function.

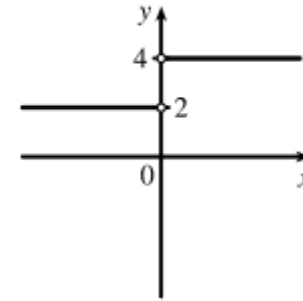
$$47. G(x) = \frac{3x + |x|}{x}$$

$$48. g(x) = |x| - x$$

تمرين

47. $G(x) = \frac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

$$G(x) = \begin{cases} \frac{3x + x}{x} & \text{if } x > 0 \\ \frac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \frac{4x}{x} & \text{if } x > 0 \\ \frac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$



48. $g(x) = |x| - x = \begin{cases} x - x & \text{if } x \geq 0 \\ -x - x & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$.

The domain is \mathbb{R} , or $(-\infty, \infty)$.

