Calculus 1

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Calculus 2

Lecture 4

Infinite Sequences and Series

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Infinite Sequences and Series

- List the terms of a sequence.
- Determine whether a sequence converges or diverges.
- Write a formula for the th term of a sequence.
- Use properties of monotonic sequences and bounded sequences

Sequence

Example:

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & \ldots, & n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots \end{array}$$

Sequence

1 is mapped onto a_1, 2 is mapped onto a_2, and so on. The numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

are the **terms** of the sequence. The number a_n is the ***n*th term** of the sequence, and the entire sequence is denoted by $\{a_n\}$. Occasionally, it is convenient to begin a sequence with a_0, so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ and the domain is the set of nonnegative integers.

Sequence

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \left\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}.$$

Terms of a Sequence

Example: **Listing the Terms of a Sequence**

The terms of the sequence $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$ are

$$\frac{1}{1-2\cdot 1}, \quad \frac{2}{1-2\cdot 2}, \quad \frac{3}{1-2\cdot 3}, \quad \frac{4}{1-2\cdot 4}, \quad \ldots$$

$$-1, \quad -\frac{2}{3}, \quad -\frac{3}{5}, \quad -\frac{4}{7}, \quad \ldots.$$

The terms of the sequence $\{c_n\} = \left\{\frac{n^2}{2^n-1}\right\}$ are

$$\frac{1^2}{2^1-1}, \quad \frac{2^2}{2^2-1}, \quad \frac{3^2}{2^3-1}, \quad \frac{4^2}{2^4-1}, \quad \ldots$$

$$\frac{1}{1}, \quad \frac{4}{3}, \quad \frac{9}{7}, \quad \frac{16}{15}, \quad \ldots.$$

Terms of a Sequence

The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

$$25, \quad 25 - 5 = 20, \quad 20 - 5 = 15, \quad 15 - 5 = 10, \ldots$$

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$$

bounce back and forth between 1 and -1, never converging to a single value.

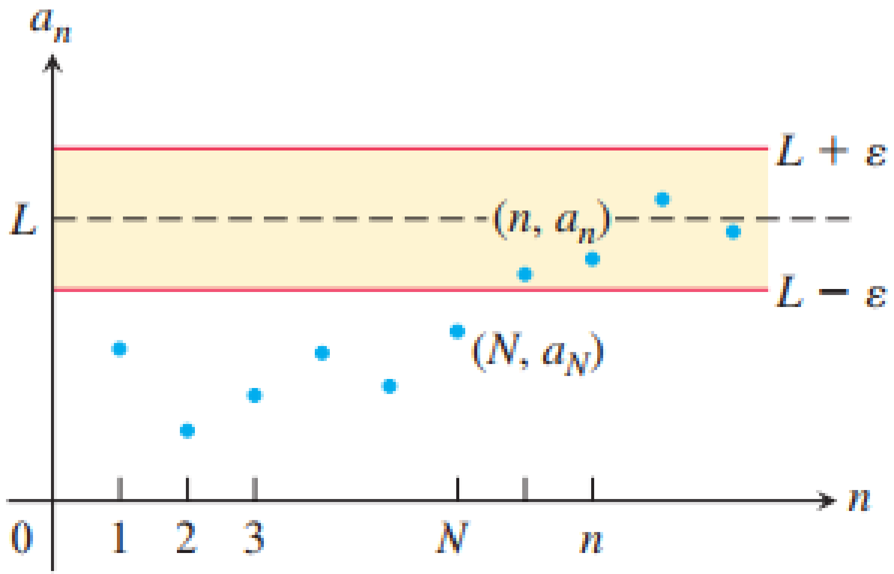
Limit of a Sequence

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).



Calculating Limits of Sequences

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

Limits of Sequences

Example:

(a) $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule and Example 1a

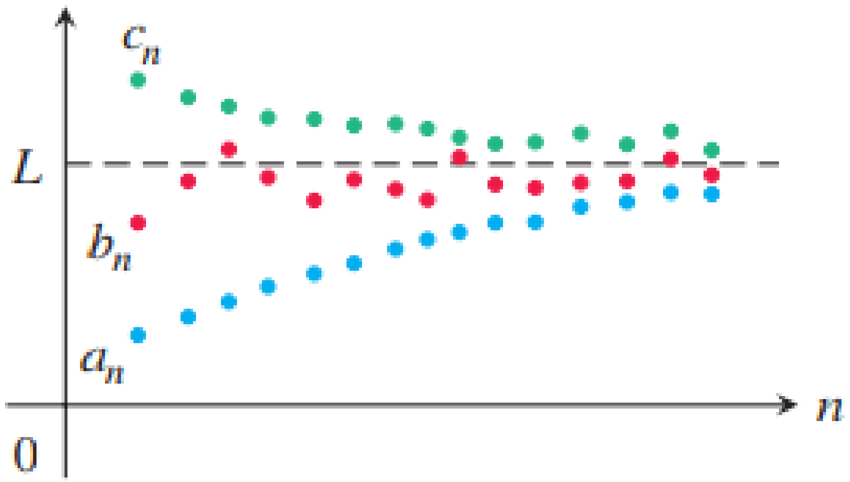
(b) $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ Difference Rule and Example 1a

(c) $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule

(d) $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7.$ Divide numerator and denominator by n^6 and use the Sum and Quotient Rules.

The Sandwich Theorem for Sequences

THEOREM —The Sandwich Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.



The Sandwich Theorem for Sequences

Example:

(a) $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n};$

(b) $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n};$

(c) $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}.$

Using L'Hôpital's Rule

THEOREM Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n\to\infty} a_n = L \qquad \text{whenever} \qquad \lim_{x\to\infty} f(x) = L.$$

Example: **Show that** $$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Solution:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty} (\ln n)/n = 0$.

Using L'Hôpital's Rule

Example: Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

Solution:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right).$$

Using L'Hôpital's Rule

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0 \text{ form}$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2}$$

L'Hôpital's Rule: differentiate numerator and denominator.

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

Simplify and evaluate.

The sequence $\{a_n\}$ converges to e^2.

Monotonic Sequences and Bounded Sequences

Definition of Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** when its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdot \cdot \cdot \le a_n \le \cdot \cdot \cdot$$

or when its terms are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdot \cdot \cdot \ge a_n \ge \cdot \cdot \cdot .$$

Example:

Determine whether each sequence having the given nth term is monotonic.

a. $a_n = 3 + (-1)^n$

b. $b_n = \dfrac{2n}{1 + n}$

Monotonic Sequences and Bounded Sequences

Solution

a. This sequence alternates between 2 and 4. So, it is not monotonic.

b. This sequence is monotonic because each successive term is greater than its predecessor. To see this, compare the terms b_n and b_{n+1}. [Note that, because n is positive, you can multiply each side of the inequality by $(1 + n)$ and $(2 + n)$ without reversing the inequality sign.]

$$b_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1}$$

$$2n(2+n) \overset{?}{<} (1+n)(2n+2)$$

$$4n + 2n^2 \overset{?}{<} 2 + 4n + 2n^2$$

$$0 < 2$$

Bounded Sequences

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

THEOREM **Bounded Monotonic Sequences**

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Bounded Sequences

Example:

a. $a_n = 3 + (-1)^n$

b. $b_n = \dfrac{2n}{1 + n}$

sequences are bounded. To see this, note that

$$2 \le a_n \le 4, \quad 1 \le b_n \le 2,$$

EXERCISES

Write the first five terms of the sequence

1. $a_n = 3^n$
2. $a_n = \left(-\frac{2}{5}\right)^n$
3. $a_n = \sin \frac{n\pi}{2}$
4. $a_n = \frac{3n}{n+4}$
5. $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$
6. $a_n = 2 + \frac{2}{n} - \frac{1}{n^2}$

EXERCISES

write the next two apparent terms of the sequence. Describe the pattern you used to find these terms

$2, 5, 8, 11, \ldots$

$8, 13, 18, 23, 28, \ldots$

$5, 10, 20, 40, \ldots$

$6, -2, \frac{2}{3}, -\frac{2}{9}, \ldots$

EXERCISES

Proof For any number x, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

EXERCISES

Solution:

$$a_n = \left(1 + \frac{x}{n}\right)^n.$$

Then

$$\ln a_n = \ln\left(1 + \frac{x}{n}\right)^n = n \ln\left(1 + \frac{x}{n}\right) \rightarrow x,$$

as we can see by the following application of L'Hôpital's Rule, in which we differentiate with respect to n:

$$\lim_{n\to\infty} n \ln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{\ln(1 + x/n)}{1/n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{1 + x/n}\right)\cdot\left(-\frac{x}{n^2}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{x}{1 + x/n} = x.$$

$$\left(1 + \frac{x}{n}\right)^n = a_n = e^{\ln a_n} \rightarrow e^x.$$

EXERCISES

Proof $$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Proof

(a) $\dfrac{\ln(n^2)}{n} = \dfrac{2\ln n}{n} \to 2 \cdot 0 = 0$

(b) $\sqrt[n]{n^2} = n^{2/n} = \left(n^{1/n}\right)^2 \to (1)^2 = 1$

(c) $\sqrt[n]{3n} = 3^{1/n}\left(n^{1/n}\right) \to 1 \cdot 1 = 1$

(d) $\left(-\dfrac{1}{2}\right)^n \to 0$

(e) $\left(\dfrac{n-2}{n}\right)^n = \left(1 + \dfrac{-2}{n}\right)^n \to e^{-2}$

EXERCISES

Show that the sequence whose nth term is $a_n = \dfrac{n^2}{2^n - 1}$ converges.

Solution Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0.$$

See Example 1(c), page 584.

So, the sequence converges to 0.

EXERCISES

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the sequence you have chosen converges or diverges.

EXERCISES

Solution First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}, \ldots$$

Consider the function of a real variable $f(x) = 2^x/(2x - 1)$. Applying L'Hôpital's Rule produces

$$\lim_{x\to\infty} \frac{2^x}{2x-1} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2} = \infty.$$

Next, apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{2^n}{2n-1} = \infty.$$

So, the sequence diverges.

EXERCISES

Determine whether the sequence with the given the term is bounded

$$a_n = 4 - \frac{1}{n}$$

$$a_n = \sin \frac{n\pi}{6}$$

$$a_n = \left(\frac{2}{3}\right)^n$$

$$a_n = \frac{\cos n}{n}$$

EXERCISES

determine the convergence or divergence of the sequence with the given th term. If the sequence converges, find its limit

$$a_n = \frac{5}{n+2}$$

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$$

$$a_n = \frac{\ln(n^3)}{2n}$$

$$a_n = 8 + \frac{5}{n}$$

$$a_n = \frac{5^n}{3^n}$$

$$a_n = 2^{1/n}$$

$$a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$$

$$a_n = n \sin \frac{1}{n}$$

$$a_n = \frac{\sin n}{n}$$

$$a_n = \frac{1 + (-1)^n}{n^2}$$

$$a_n = \frac{\cos \pi n}{n^2}$$

$$a_n = -3^{-n}$$

Thank you for your attention