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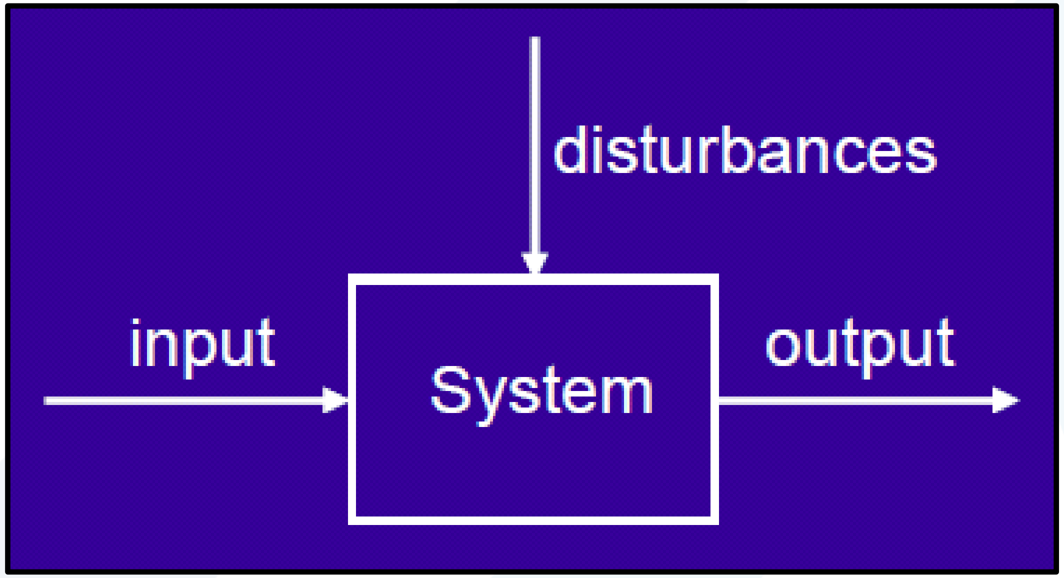
Parameter Estimation for the Second Order Model



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PARAMETER ESTIMATION FOR THE SECOND-ORDER MODEL

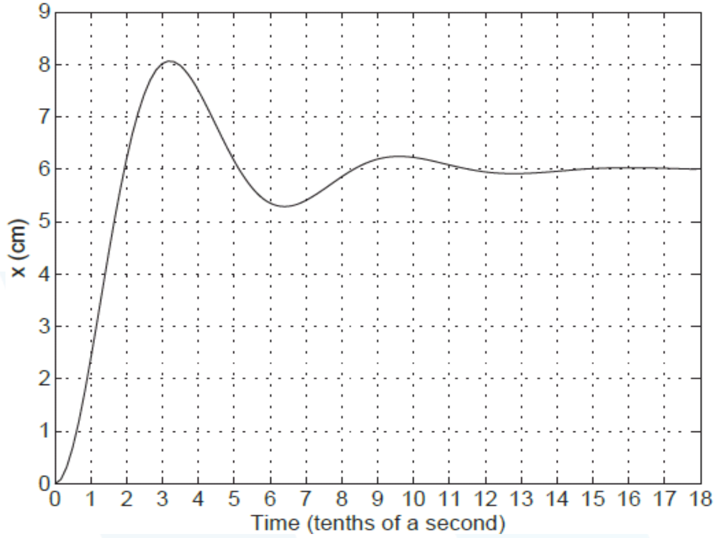
Depending on the available experimental method, we can use either the free or the step response formulas to estimate the parameters of a second-order model.

Example

Figure shows the response of a system to a step input of magnitude 6×10^3 N. The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Estimate the values of m, c, and k.



Solution

From the graph we see that the steady-state response is $x_{ss} = 6$ cm. At steady state, $x_{ss} = f_{ss}/k$, and thus $k = 6 \times 10^3 / 6 \times 10^{-2} = 10^5$ N/m.

The peak value from the plot is $x = 8.1$ cm, so the maximum percent overshoot is $M_{\%} = [(8.1 - 6)/6]100 = 35\%$. From Table , we can compute the damping ratio as follows:

$$R = \ln \frac{100}{35} = 1.0498 \qquad \zeta = \frac{R}{\sqrt{\pi^2 + R^2}} = 0.32$$

The peak occurs at $t_p = 0.32$ s. From Table

$$t_p = 0.32 = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{3.316}{\omega_n}$$

Thus, $\omega_n^2 = 107$ and

$$m = \frac{k}{\omega_n^2} = \frac{10^5}{107} = 930 \text{ kg}$$

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Table Step response specifications for the underdamped model $m\ddot{x} + c\dot{x} + kx = f$.

Maximum percent overshoot	$M_{\%} = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
	$\zeta = \dfrac{R}{\sqrt{\pi^2 + R^2}}, \quad R = \ln \dfrac{100}{M_{\%}}$
Peak time	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Delay time	$t_d \approx \dfrac{1 + 0.7\zeta}{\omega_n}$
100% rise time	$t_r = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$
	$\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

From the expression for the damping ratio,

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{c}{2\sqrt{930(10^5)}} = 0.32$$

Thus $c = 6170\ \mathrm{N \cdot s/m}$, and the model is

$$930\ddot{x} + 6170\dot{x} + 10^5 x = f(t)$$

THE LOGARITHMIC DECREMENT

Usually the damping coefficient c is the parameter most difficult to estimate. Mass m and stiffness k can be measured with static tests, but measuring damping requires a dynamic test. If the system exists and dynamic testing can be done with it, then the *logarithmic decrement* provides a good way to estimate the damping ratio ζ, from which we can compute c ($c = 2\zeta\sqrt{mk}$). To see how this is done, use the form $s = -\zeta\omega_n \pm \omega_d j$ for the characteristic roots, and write the free response for the underdamped case as follows:

$$x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

The frequency of the oscillation is ω_d, and thus the period P is $P = 2\pi/\omega_d$. The logarithmic decrement δ is defined as the natural logarithm of the ratio of two successive amplitudes; that is,

$$\delta = \ln\frac{x(t)}{x(t+P)}$$

$$\delta = \ln \frac{Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Be^{-\zeta\omega_n (t+P)}\sin(\omega_d t + \omega_d P + \phi)}$$

Note that $e^{-\zeta\omega_n(t+P)} = e^{-\zeta\omega_n t}e^{-\zeta\omega_n P}$. In addition, because $\omega_d P = 2\pi$ and $\sin(\theta+2\pi) = \sin\theta$, $\sin(\omega_d t + \omega_d P + \phi) = \sin(\omega_d t + \phi)$,

$$\delta = \ln e^{\zeta\omega_n P} = \zeta\omega_n P$$

Because $P = 2\pi/\omega_d = 2\pi/\omega_n\sqrt{1-\zeta^2}$, we have

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

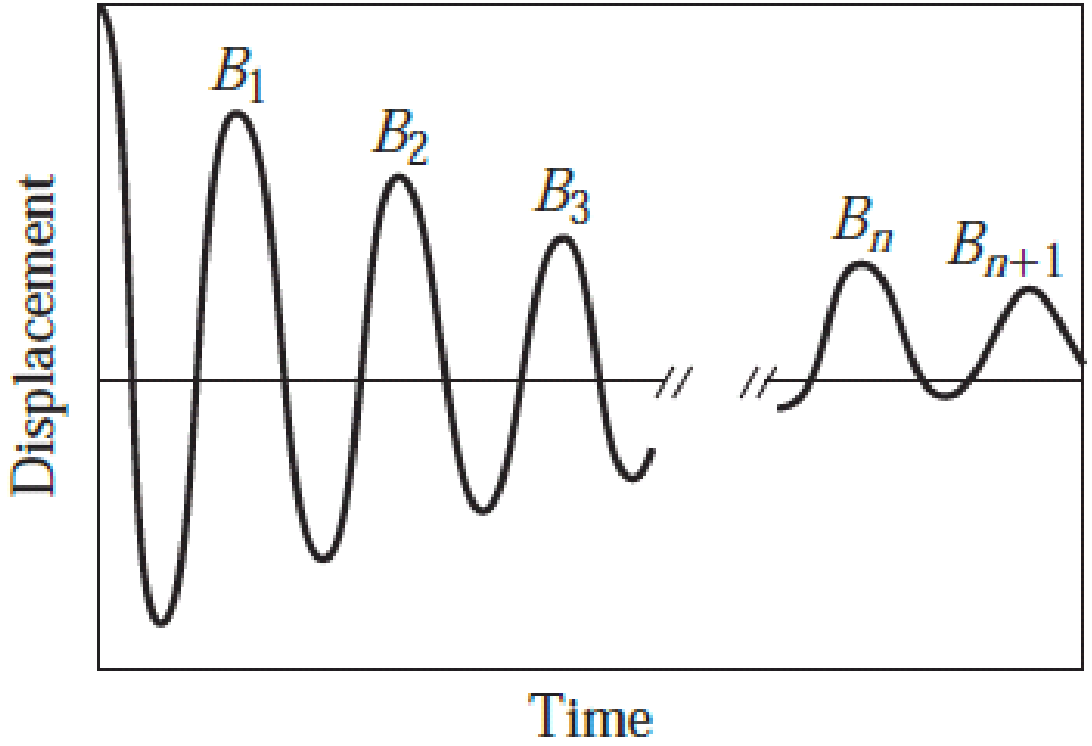
We can solve this for ζ to obtain

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

If we have a plot of $x(t)$ from a test, we can measure two values x at two times t and $t + P$. These values can be measured at two successive peaks in x. The x values are then substituted to compute δ. Equation gives the value of ζ, from which we compute $c = 2\zeta\sqrt{mk}$.

The plot of $x(t)$ will contain some measurement error, and for this reason, the preceding method is usually modified to use measurements of two peaks n cycles apart Let the peak values be denoted B_1, B_2, etc. and note that

$$\ln\left(\frac{B_1}{B_2}\frac{B_2}{B_3}\frac{B_3}{B_4}\cdots\frac{B_n}{B_{n+1}}\right) = \ln\left(\frac{B_1}{B_{n+1}}\right)$$



B_1
B_2
B_3
B_n
B_{n+1}
Displacement
Time

or

$$\ln \frac{B_1}{B_2} + \ln \frac{B_2}{B_3} + \ln \frac{B_3}{B_4} \cdots \ln \frac{B_n}{B_{n+1}} = \ln \frac{B_1}{B_{n+1}}$$

Thus

$$\delta + \delta + \delta + \cdots + \delta = n\delta = \ln \frac{B_1}{B_{n+1}}$$

or

$$\delta = \frac{1}{n} \ln \frac{B_1}{B_{n+1}}$$

We normally take the first peak to be B_1, because this is the highest peak and least subject to measurement error, but this is not required. The above formula applies to any two points n cycles apart.

Example

Measurement of the free response of a certain system whose mass is 500 kg shows that after six cycles the amplitude of the displacement is 10% of the first amplitude. Also, the time for these six cycles to occur was measured to be 30 s. Estimate the system's damping c and stiffness k.

Solution

From the given data, $n = 6$ and $B_7 / B_1 = 0.1$. Thus,

$$\delta = \frac{1}{6} \ln\left(\frac{B_1}{B_7}\right) = \frac{1}{6} \ln 10 = \frac{2.302}{6} = 0.384$$

$$\zeta = \frac{0.384}{\sqrt{4\pi^2 + (0.384)^2}} = 0.066$$

Because the measured time for six cycles was 30 s, the period P is $P = 30/6 = 5$ s. Thus $\omega_d = 2\pi/P = 2\pi/5$. The damped frequency is related to the undamped frequency as

$$\omega_d = \frac{2\pi}{5} = \omega_n\sqrt{1-\zeta^2} = \omega_n\sqrt{1-(0.066)^2}$$

Thus, $\omega_n = 1.26$ and

$$k = m\omega_n^2 = 500(1.26)^2 = 794 \text{ N/m}$$

The damping constant is calculated as follows:

$$c = 2\zeta\sqrt{mk} = 2(0.066)\sqrt{500(794)} = 83.2 \text{ N}\cdot\text{s/m}$$

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