Calculus 2

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Calculus 2

Lecture 5

Infinite Series

Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots \qquad \text{Infinite series}$$

The sum of the first *n* terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$a_1, a_2, \ldots$ are <u>terms</u> of the series.

a_n is the <u>nth term</u>.

Infinite Series

To find the sum of an infinite series, consider the sequence of partial sums listed below.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n$$

$$S_n = \sum_{k=1}^{n} a_k$$

n^{th} partial sum

If S_n has a limit as $n \to \infty$, then the series converges, otherwise it diverges.

Infinite Series

The series in Example is a **telescoping series** of the form

- The nth partial sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

is

$$S_n=1-\frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

- The series

$$\sum_{n=1}^{\infty}1=1+1+1+1+\cdots$$

diverges because $S_n=n$ and the sequence of partial sums diverges.

Telescoping Series

Telescoping Series: $\sum_{n=1}^{\infty}(b_n - b_{n+1})$

$$s_n = b_1 - b_{n+1}$$

$$s = \sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$$

$\sum_{n=1}^{\infty}(b_n - b_{n+1})$ is convergent $\Leftrightarrow$ $\{b_n\}$ is convergent

Example: $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

Telescoping Series

Example

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

From this telescoping form, you can see that the nth partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdot\cdot\cdot + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right) = 1.$$

Geometric Series

In a geometric series, each term is found by multiplying the preceding term by the same number, r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

This converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

$-1 < r < 1$ is the interval of convergence.

Geometric Series

EXAMPLE 1 The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

EXAMPLE 2 The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

Properties of Infinite Series

THEOREM **Properties of Infinite Series**

Let $\Sigma\, a_n$ and $\Sigma\, b_n$ be convergent series, and let A, B, and c be real numbers. If $\Sigma\, a_n = A$ and $\Sigma\, b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

n th-Term Test for Divergence

THEOREM **Limit of the *n*th Term of a Convergent Series**

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

n th-Term Test for Divergence

THEOREM:

$$\sum_{i=1}^{\infty} a_i$$ Convergent $\Rightarrow$ $$\lim_{n\to\infty} a_n = 0$$

THEOREM: THE TEST FOR DIVERGENCE

$$\sum_{i=1}^{\infty} a_i$$ Divergent $\Leftarrow$ $$\lim_{n\to\infty} a_n \neq 0$$ *or* $$\lim_{n\to\infty} a_n \text{ DNE}$$

n th-Term Test for Divergence

Example

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$. $\lim_{n\to\infty} a_n \neq 0$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

n th-Term Test for Divergence

For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the nth term is 0, the nth-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence.

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \cdots + \frac{1}{16}$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}.$$

$$S_{2^n} \geq 1 + \frac{n}{2}.$$

$\{S_{2^n}\}$ diverges to $+\infty$. Therefore, since $\{S_n\}$ has a diverging

p-Series and Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

p-series

is a ***p*-series,** where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Harmonic series

is the **harmonic series.**

Convergence of *p*-Series

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges for $p > 1$ and diverges for $0 < p \leq 1$.

Comparisons of Series

THEOREM **Direct Comparison Test**

Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Comparisons of Series

Example Determine the convergence or divergence

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}.$$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}.$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}.$$ Convergent geometric series

Term-by-term comparison yields

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.

Limit Comparison Test

THEOREM Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Limit Comparison Test

Example

Given Series	Comparison Series	Conclusion
$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

Example

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}.$$

Solution

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

Divergent series

Note that this series diverges by the nth-Term Test. From the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n2^n}{4n^3 + 1}\right)\left(\frac{n^2}{2^n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{4 + (1/n^3)}$$

$$= \frac{1}{4}$$

you can conclude that the series diverges.

summary

	Geometric	Telescoping	General
When convg	$\|r\|<1$	$\{b_{n+1}\}$ convg	$\{s_n\}$ convg
sum	$\frac{a}{1-r}$	$b_1-\lim_{n\to\infty} b_{n+1}$	$\lim_{n\to\infty} s_n$
nth partial sum	$s_n = a\frac{1-r^n}{1-r}$	$s_n = b_1 - b_{n+1}$	$s_n = a_1+\cdots+a_n$ $a_n = s_n - s_{n-1}$

THEOREM: THE TEST FOR DIVERGENCE

$\sum_{i=1}^{\infty} a_i$ Divergent ⟸ $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE

$\{a_n\}_{n=1}^{\infty}$ convg $\to 0$ ⟸ $\sum_{i=1}^{\infty} a_i$ convg ⟺ $\{s_n\}_{n=1}^{\infty}$ convg

Thank you for your attention