



# Calculus 2

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Calculus 2

Lecture 5

# Infinite Series



# Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad \text{Infinite series}$$

The sum of the first  $n$  terms

$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

$a_1, a_2, \dots$  are terms of the series.

$a_n$  is the  $n^{\text{th}}$  term.



# Infinite Series

To find the sum of an infinite series, consider the sequence of partial sums listed below.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

⋮

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n$$

$$S_n = \sum_{k=1}^n a_k$$

$n^{\text{th}}$  partial sum

If  $S_n$  has a limit as  $n \rightarrow \infty$ , then the series converges, otherwise it diverges.



# Infinite Series

The series in Example is a **telescoping series** of the form

The  $n$ th partial sum of the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots$$

is

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of  $S_n$  is 1, the series converges and its sum is 1.

The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$$

diverges because  $S_n = n$  and the sequence of partial sums diverges.



# Telescoping Series

**Telescoping Series:**

$$\sum_{n=1}^{\infty} (b_n - b_{n+1})$$

$$s_n = b_1 - b_{n+1}$$

$$s = \sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{n \rightarrow \infty} b_{n+1}$$

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) \text{ is convergent} \iff \{b_n\} \text{ is convergent}$$

**Example:**

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$



# Telescoping Series

## Example

Find the sum of the series  $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$ .

## Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}.$$

From this telescoping form, you can see that the  $n$ th partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) = 1 - \frac{1}{2n + 1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n + 1}\right) = 1.$$



# Geometric Series

In a geometric series, each term is found by multiplying the preceding term by the same number,  $r$ .

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

This converges to  $\frac{a}{1-r}$  if  $|r| < 1$ , and diverges if  $|r| \geq 1$ .

$-1 < r < 1$  is the interval of convergence.





# Geometric Series

**EXAMPLE 1** The geometric series with  $a = 1/9$  and  $r = 1/3$  is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

**EXAMPLE 2** The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with  $a = 5$  and  $r = -1/4$ . It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$



# Properties of Infinite Series

## **THEOREM**      **Properties of Infinite Series**

Let  $\sum a_n$  and  $\sum b_n$  be convergent series, and let  $A$ ,  $B$ , and  $c$  be real numbers. If  $\sum a_n = A$  and  $\sum b_n = B$ , then the following series converge to the indicated sums.

1. 
$$\sum_{n=1}^{\infty} ca_n = cA$$

2. 
$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

3. 
$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$



# $n$ th-Term Test for Divergence

## **THEOREM**      **Limit of the $n$ th Term of a Convergent Series**

If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

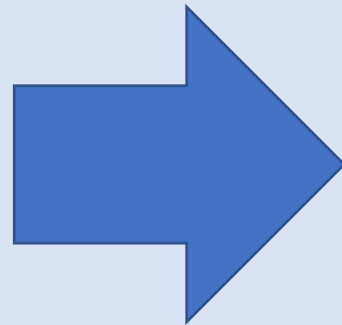


# n th-Term Test for Divergence

## THEOREM:

$$\sum_{i=1}^{\infty} a_i$$

Convergent

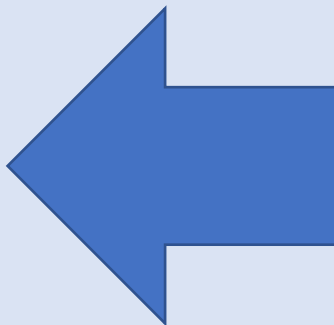


$$\lim_{n \rightarrow \infty} a_n = 0$$

## THEOREM: THE TEST FOR DIVERGENCE

$$\sum_{i=1}^{\infty} a_i$$

Divergent



$$\lim_{n \rightarrow \infty} a_n \neq 0$$

or

$$\lim_{n \rightarrow \infty} a_n \text{ DNE}$$



# n th-Term Test for Divergence

## Example

(a)  $\sum_{n=1}^{\infty} n^2$  diverges because  $n^2 \rightarrow \infty$ .

(b)  $\sum_{n=1}^{\infty} \frac{n+1}{n}$  diverges because  $\frac{n+1}{n} \rightarrow 1$ .  $\lim_{n \rightarrow \infty} a_n \neq 0$

(c)  $\sum_{n=1}^{\infty} (-1)^{n+1}$  diverges because  $\lim_{n \rightarrow \infty} (-1)^{n+1}$  does not exist.

(d)  $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$  diverges because  $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$ .



# $n$ th-Term Test for Divergence

For the series  $\sum_{n=1}^{\infty} \frac{1}{n}$ , you have

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Because the limit of the  $n$ th term is 0, the  $n$ th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence.

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \cdots + \frac{1}{16}$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{4}{2}$$

$$S_{2^n} \geq 1 + \frac{n}{2}$$

$\{S_{2^n}\}$  diverges to  $+\infty$ . Therefore, since  $\{S_n\}$  has a diverging



# p-Series and Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

*p*-series

is a *p*-series, where *p* is a positive constant. For *p* = 1, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Harmonic series

is the harmonic series.

## Convergence of *p*-Series

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges for  $p > 1$  and diverges for  $0 < p \leq 1$ .



# Comparisons of Series

## **THEOREM** **Direct Comparison Test**

Let  $0 < a_n \leq b_n$  for all  $n$ .

1. If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.
2. If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  diverges.





# Comparisons of Series

**Example** Determine the convergence or divergence

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

**Solution**

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

**Solution** This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

Convergent geometric series

Term-by-term comparison yields

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.



# Limit Comparison Test

## THEOREM Limit Comparison Test

If  $a_n > 0$ ,  $b_n > 0$ , and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

where  $L$  is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.



# Limit Comparison Test

## Example

### Given Series

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$$

$$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$$

### Comparison Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

### Conclusion

Both series converge.

Both series diverge.

Both series converge.



# Limit Comparison Test

## Example

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}$$

## Solution

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

Divergent series

Note that this series diverges by the  $n$ th-Term Test. From the limit

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \left( \frac{n2^n}{4n^3 + 1} \right) \left( \frac{n^2}{2^n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4 + (1/n^3)} \\ &= \frac{1}{4}\end{aligned}$$

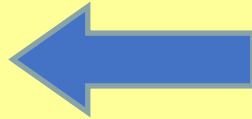
you can conclude that the series diverges.

# summary

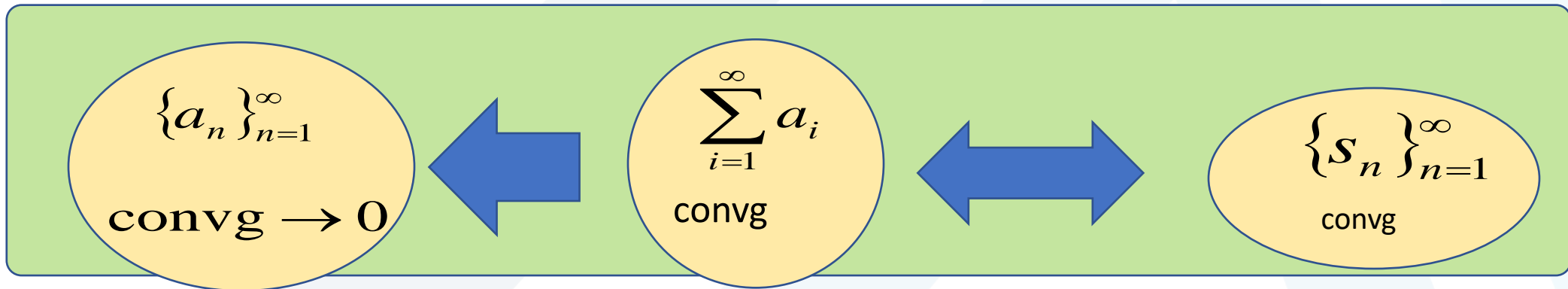
	Geometric	Telescoping	General
When convg	$ r  < 1$	$\{b_{n+1}\} \text{ convg}$	$\{s_n\} \text{ convg}$
sum	$\frac{a}{1-r}$	$b_1 - \lim_{n \rightarrow \infty} b_{n+1}$	$\lim_{n \rightarrow \infty} s_n$
nth partial sum	$s_n = a \frac{1-r^n}{1-r}$	$s_n = b_1 - b_{n+1}$	$s_n = a_1 + \dots + a_n$ $a_n = s_n - s_{n-1}$

**THEOREM:** THE TEST FOR DIVERGENCE

$$\sum_{i=1}^{\infty} a_i \text{ Divergent}$$



$$\lim_{n \rightarrow \infty} a_n \neq 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} a_n \text{ DNE}$$





**Thank you for your attention**