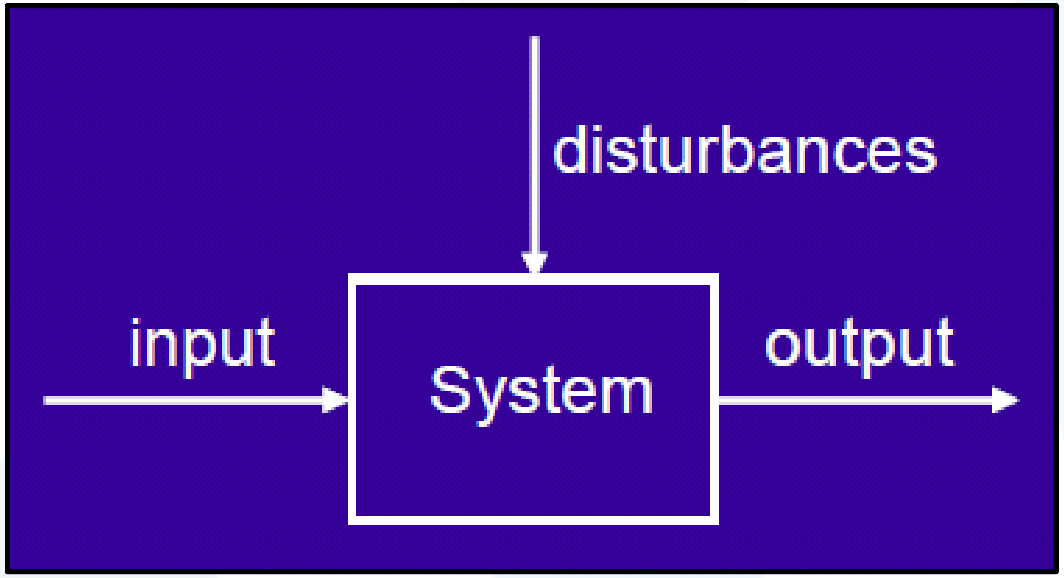
جامعة المنارة
كلية الهندسة
قسم الروبوتيك و الأنظمة الذكية
مقرر النمذجة و المطابقة

Mathematical Techniques to Simulate State Space Model Response



العام الدراسي 2024-2025

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Unit Ramp Response of a System Defined in State Space

Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where u is the unit-ramp function. In what follows, we shall consider a simple example to explain the method. Consider the case where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}(0) = \mathbf{0}$$
$$\mathbf{C} = [1 \quad 0], \quad D = [0]$$

When the initial conditions are zeros, the unit-ramp response is the integral of the unit-step response. Hence the unit-ramp response can be given by

$$z = \int_0^t y \, dt$$

$$\dot{z} = y = x_1$$

Let us define

$$z = x_3$$

$$\dot{x}_3 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

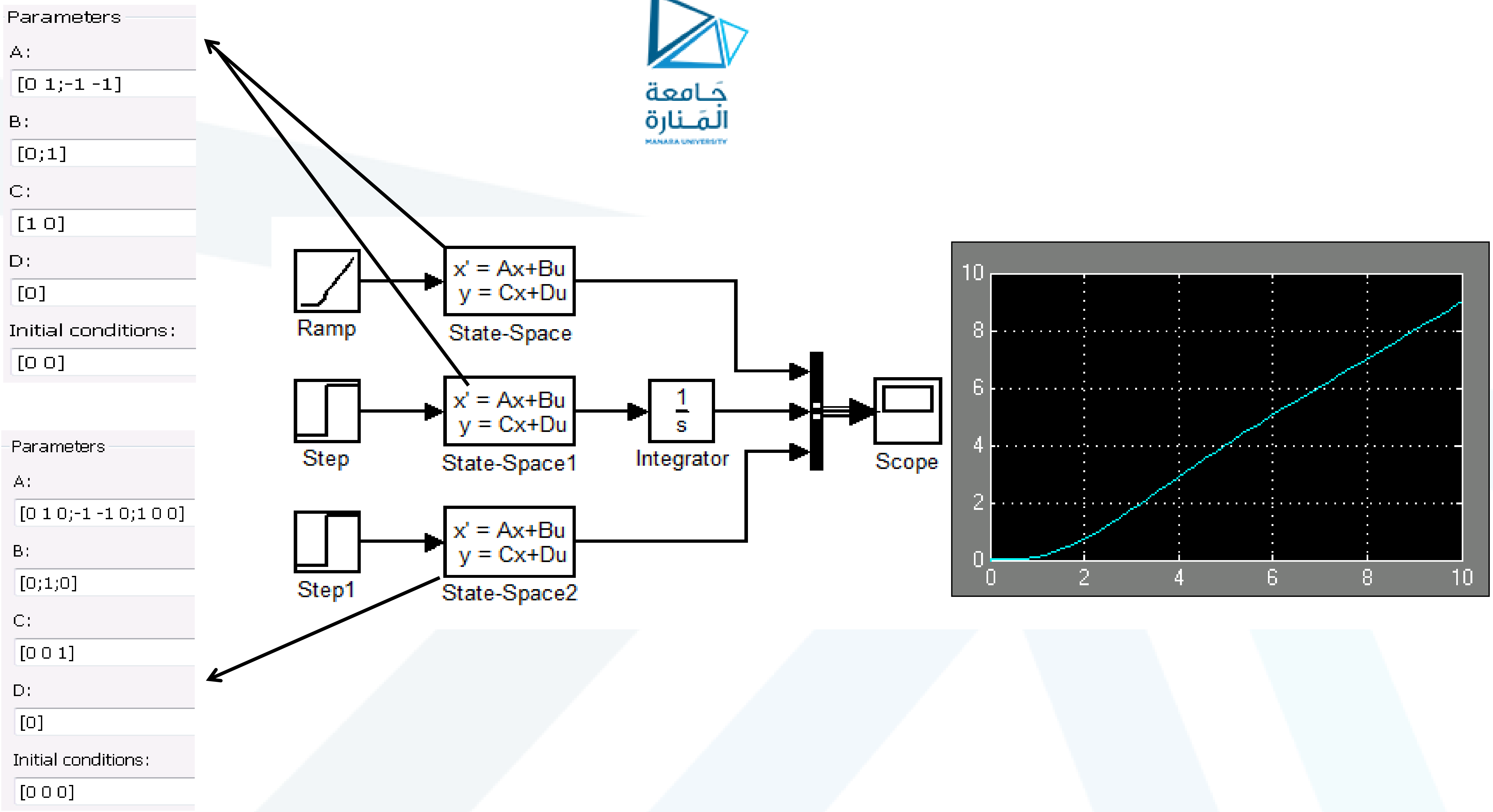
$$z = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where u appearing in Equation is the unit-step function. These equations can be written as

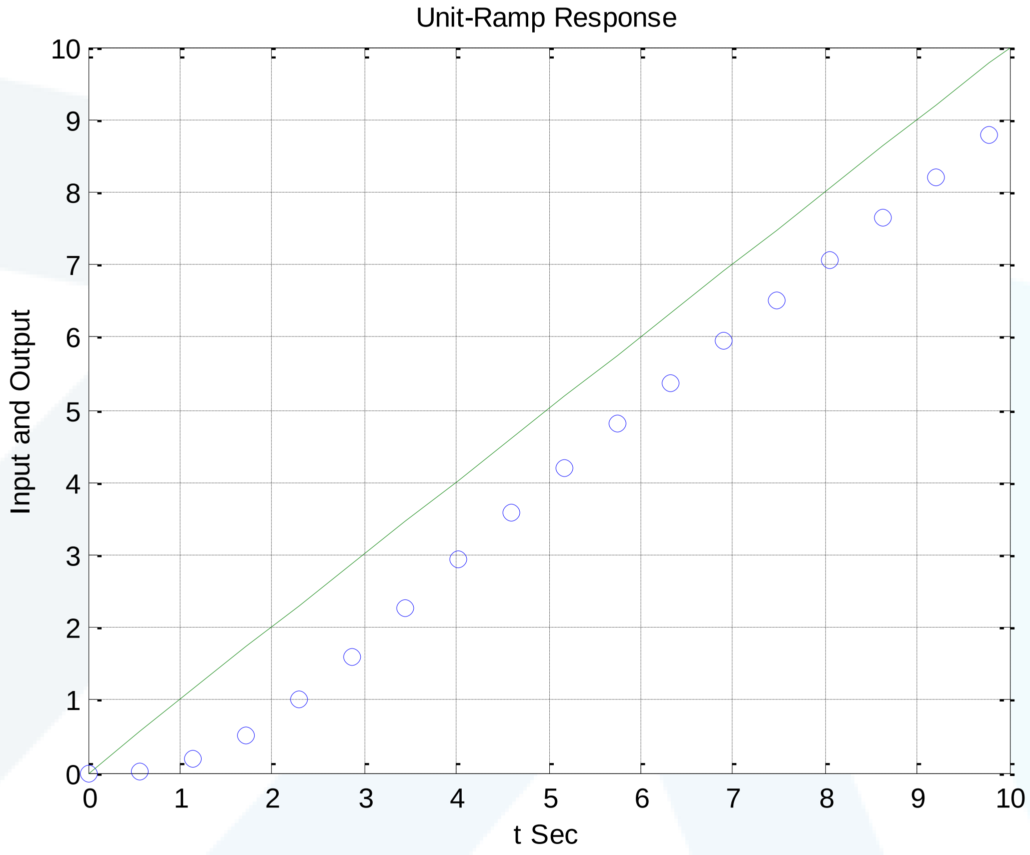
$$\dot{\mathbf{x}} = \mathbf{AAx} + \mathbf{BB}u$$
$$z = \mathbf{CCx} + DDu$$

$$\mathbf{AA} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \mathbf{C} & 0 \end{array}\right]$$

$$\mathbf{BB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \qquad \mathbf{CC} = [0 \quad 0 \quad 1], \qquad DD = [0]$$


Parameters
A:
[0 1;-1 -1]
B:
[0;1]
C:
[1 0]
D:
[0]
Initial conditions:
[0 0]
Parameters
A:
[0 1 0;-1 -1 0;1 0 0]
B:
[0;1;0]
C:
[0 0 1]
D:
[0]
Initial conditions:
[0 0 0]
Ramp
Step
Step1
x' = Ax+Bu
y = Cx+Du
State-Space
State-Space1
State-Space2
1
s
Integrator
Scope
10
8
6
4
2
0
0
2
4
6
8
10

```
A = [0 1;-1 -1];
B = [0; 1];
C = [1 0];
D = [0];
AA = [A zeros(2,1);C 0];
BB = [B;0];
CC = [0 0 1];
DD = [0];
[z,x,t] = step(AA,BB,CC,DD);
x3 = [0 0 1]*x';
plot(t,x3,'o',t,t,'-')
grid
axis([0 10 0 10])
title('Unit-Ramp Response')
xlabel('t Sec')
ylabel('Input and Output')
```



Response to Initial Condition of a System Defined in State Space

Case 1

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Let us obtain the response $\mathbf{x}(t)$ when the initial condition $\mathbf{x}(0)$ is specified. Assume that there is no external input function acting on this system. Assume also that $\mathbf{x}$ is an n-vector.

First, take Laplace transforms of both sides of Equation

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

This equation can be rewritten as

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{x}(0)$$

Taking the inverse Laplace transform of Equation , we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{x}(0)\,\delta(t)$$

$$\dot{\mathbf{z}} = \mathbf{x}$$

Then Equation can be written as

$$\ddot{\mathbf{z}} = \mathbf{A}\dot{\mathbf{z}} + \mathbf{x}(0)\,\delta(t)$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where $\mathbf{B} = \mathbf{x}(0), \quad u = 1(t)$

Referring to Equation , the state $\mathbf{x}(t)$ is given by $\dot{\mathbf{z}}(t)$. Thus,

$$\mathbf{x} = \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

The solution of Equations gives the response to the initial condition.

Summarizing, the response to the initial condition $\mathbf{x}(0)$ is obtained by solving the following state-space equations:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where $\mathbf{B} = \mathbf{x}(0), \quad u = 1(t)$

Example

Obtain the response of the system subjected to the given initial condition.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

Solution

Obtaining the response of the system to the given initial condition resolves to solving the unit-step response of the following system:

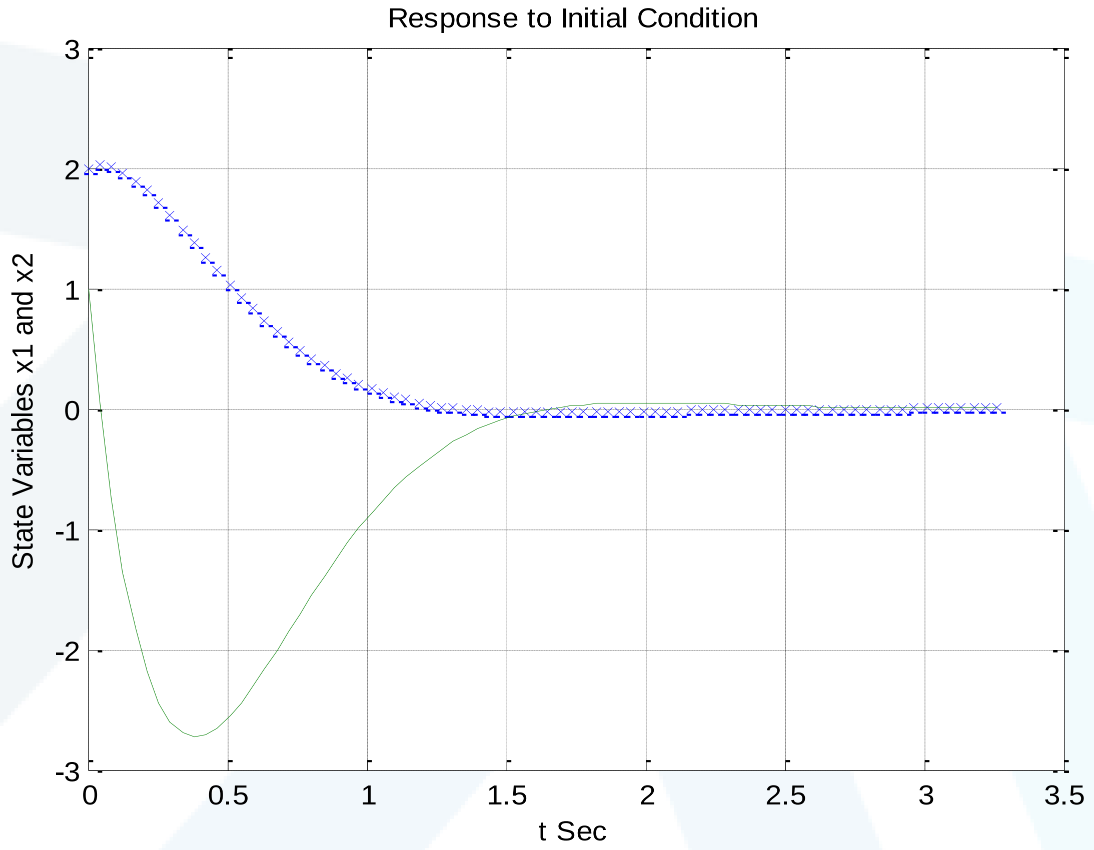
$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

```
A = [0 1;-10 -5];
B = [2;1];
[x,z,t] = step(A,B,A,B);
x1 = [1 0]*x';
x2 = [0 1]*x';
plot(t,x1,'x',t,x2,'-')
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
```



Parameters

A:

[0 1;-10 -5]

B:

[0;0]

C:

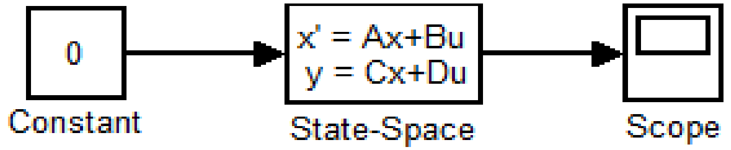
[1 0;0 1]

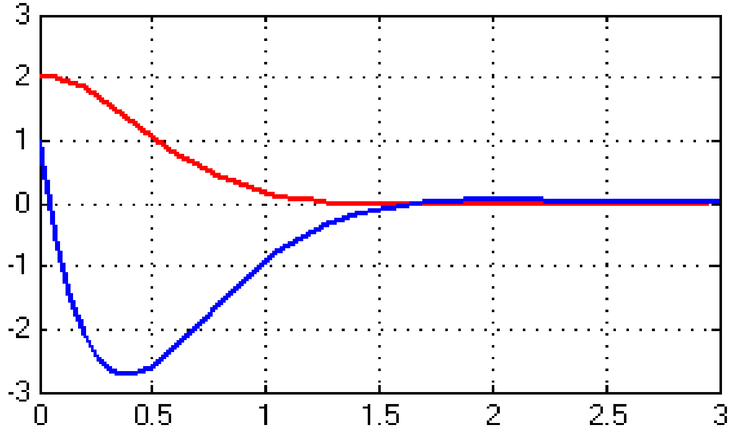
D:

[0;0]

Initial conditions:

[2 1]





Parameters

A:

[0 1;-10 -5]

B:

[2;1]

C:

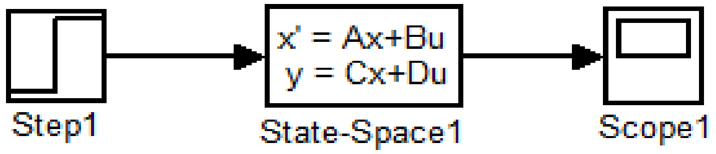
[0 1;-10 -5]

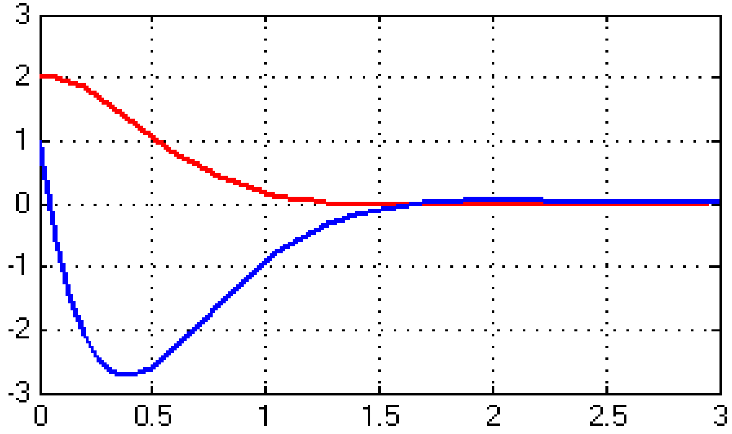
D:

[2;1]

Initial conditions:

[0 0]





Case 2

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

(Assume that $\mathbf{x}$ is an n-vector and $\mathbf{y}$ is an m-vector.)

Similar to case 1, by defining

$$\dot{\mathbf{z}} = \mathbf{x}$$

we can obtain the following equation:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{x}(0)1(t) = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

where

$$\mathbf{B} = \mathbf{x}(0), \qquad u = 1(t)$$

Noting that $\mathbf{x} = \dot{\mathbf{z}}$,

$$\mathbf{y} = \mathbf{C}\dot{\mathbf{z}}$$

$$\mathbf{y} = \mathbf{C}(\mathbf{A}\mathbf{z} + \mathbf{B}u) = \mathbf{C}\mathbf{A}\mathbf{z} + \mathbf{C}\mathbf{B}u$$

The solution of Equations

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$\mathbf{y} = \mathbf{C}\mathbf{A}\mathbf{z} + \mathbf{C}\mathbf{B}u$$

where $\mathbf{B} = \mathbf{x}(0)$ and $u = 1(t)$, gives the response of the system to a given initial condition.

Example

Consider the system subjected to the initial condition as given below.

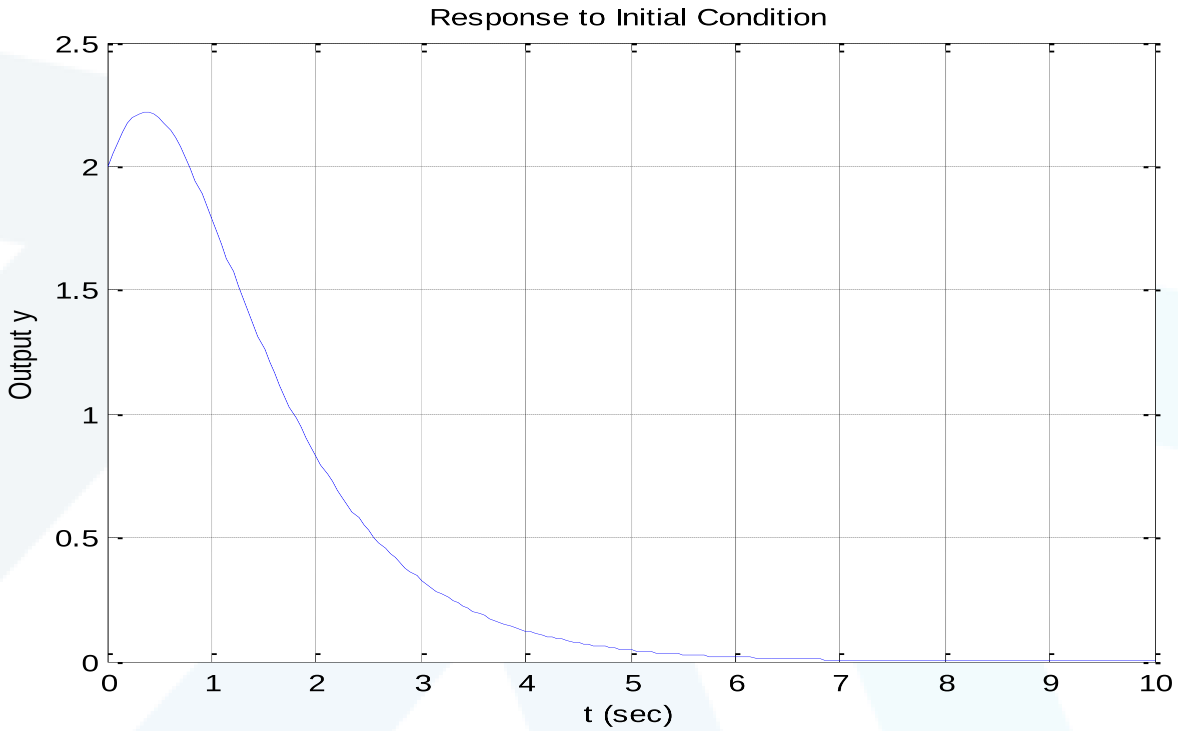
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

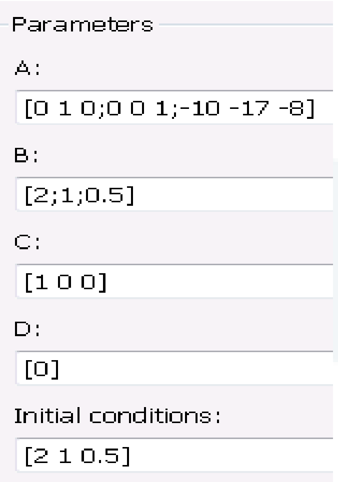
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

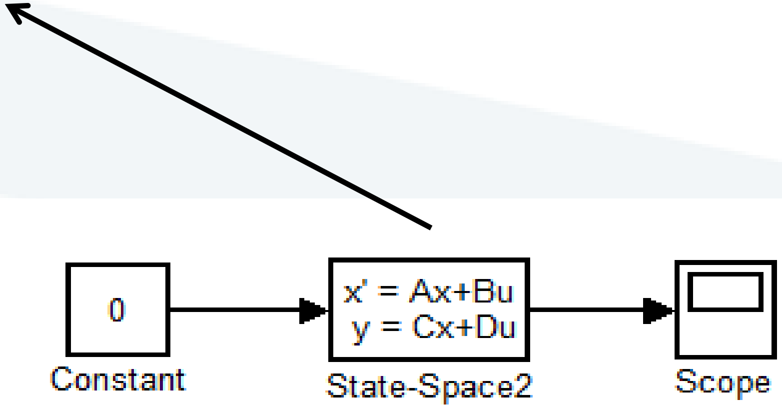
(There is no input or forcing function in this system.) Obtain the response y(t) versus t to the given initial condition

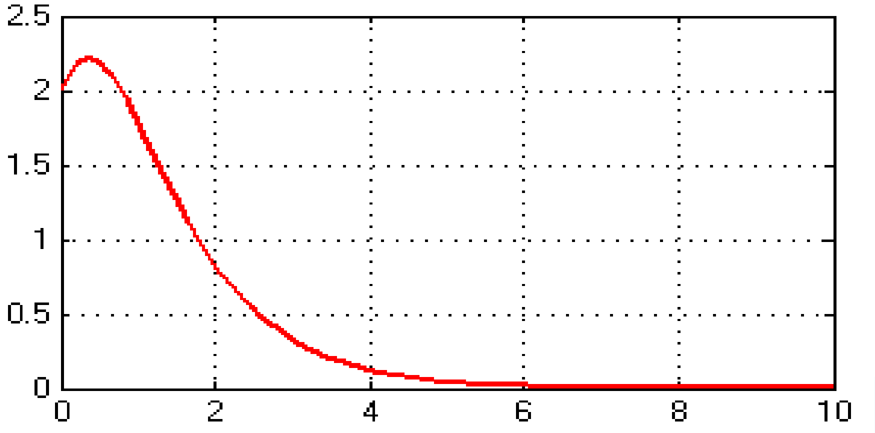
Solution

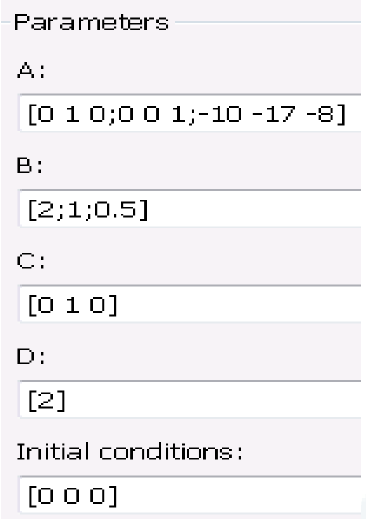
```
A = [0 1 0;0 0 1;-10 -17 -8];
B = [2;1;0.5];
C=[1 0 0];
[y,x,t] = step(A,B,C*A,C*B);
plot(t,y)
grid;
title('Response to Initial Condition')
xlabel('t (sec)')
ylabel('Output y')
```

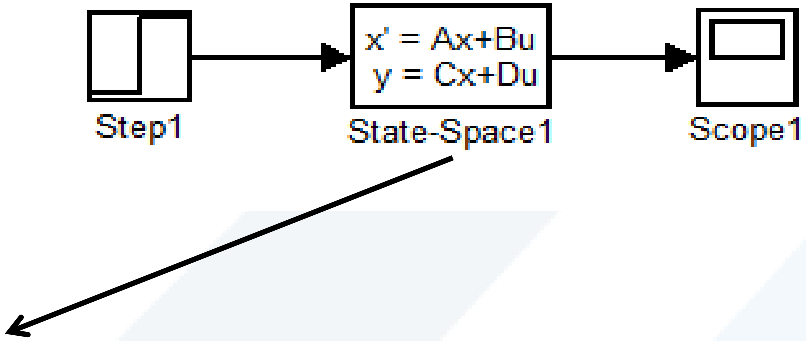


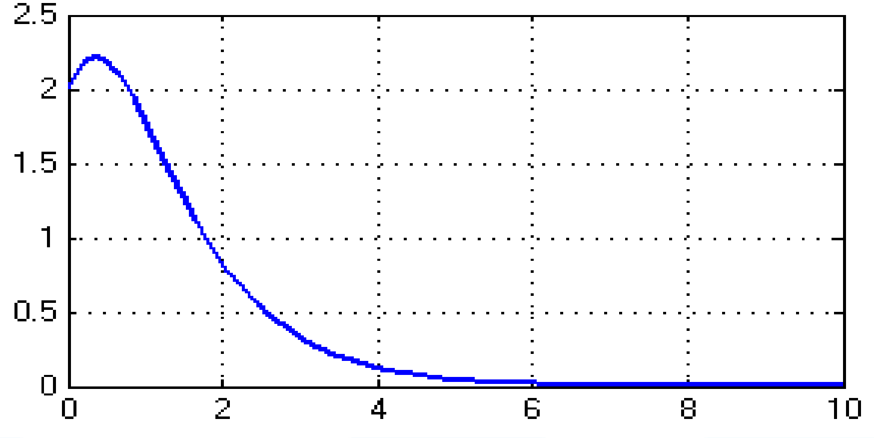

Parameters
A:
[0 1 0;0 0 1;-10 -17 -8]
B:
[2;1;0.5]
C:
[1 0 0]
D:
[0]
Initial conditions:
[2 1 0.5]


0
Constant
x' = Ax+Bu
y = Cx+Du
State-Space2
Scope


2.5
2
1.5
1
0.5
0
0
2
4
6
8
10

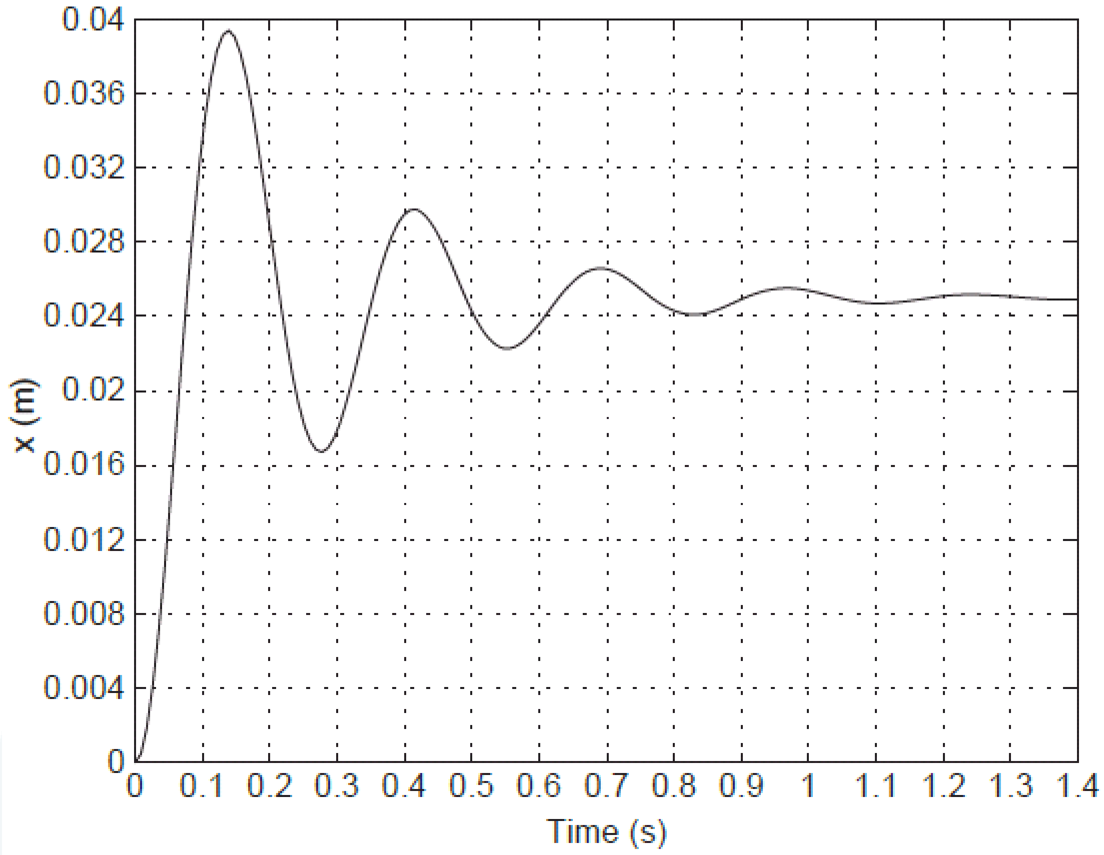

Parameters
A:
[0 1 0;0 0 1;-10 -17 -8]
B:
[2;1;0.5]
C:
[0 1 0]
D:
[2]
Initial conditions:
[0 0 0]


Step1
x' = Ax+Bu
y = Cx+Du
State-Space1
Scope1


2.5
2
1.5
1
0.5
0
0
2
4
6
8
10

Example

Figure shows the response of a system to a step input of magnitude 1000 N. The equation of motion is $m\ddot{x} + c\dot{x} + kx = f(t)$. Estimate the values of m, c, and k.



Example

A mass-spring-damper system has a mass of 100 kg. Its free response amplitude decays such that the amplitude of the 30th cycle is 20% of the amplitude of the 1st cycle. It takes 60 s to complete 30 cycles. Estimate the damping constant c and the spring constant k.

Example

(a) Use the least-squares method to fit the linear function $y = mx + b$ to the function $y = ax^2 + bx$ over the range $0 \leq x \leq L$. (b) Apply the results to the case where $a = 3$, $b = 5$, and $L = 2$.

انتهت المحاضرة