Calculus 1

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Calculus 1

Lecture 3

Limits and Continuity

Chapter 2

Limits and Continuity

Limit of a Function and Limit Laws

Limits of Function Values

EXAMPLE 1 How does the function

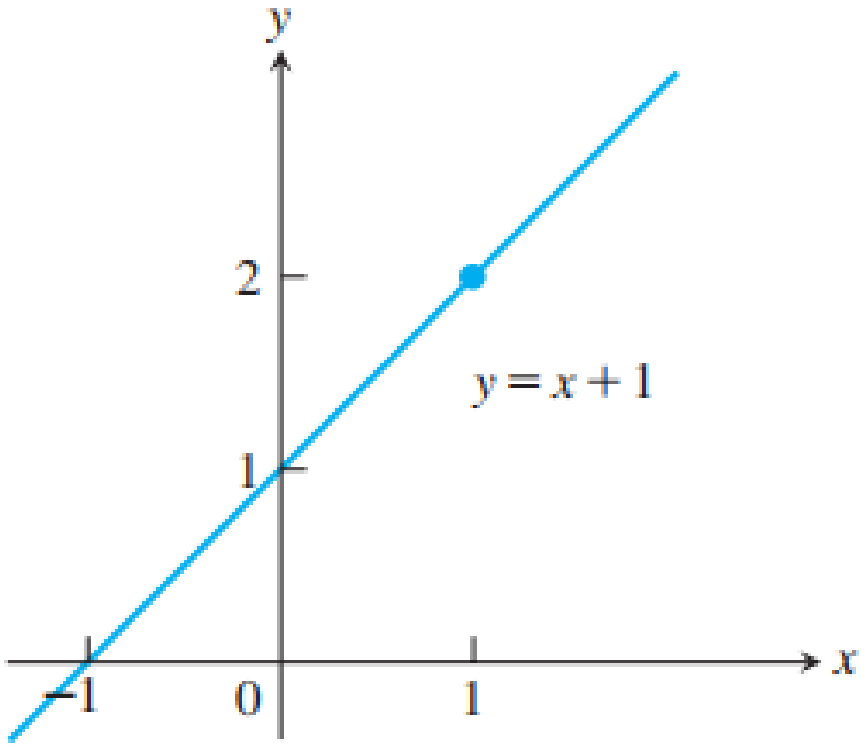
$$f(x) = \frac{x^2 - 1}{x - 1}$$

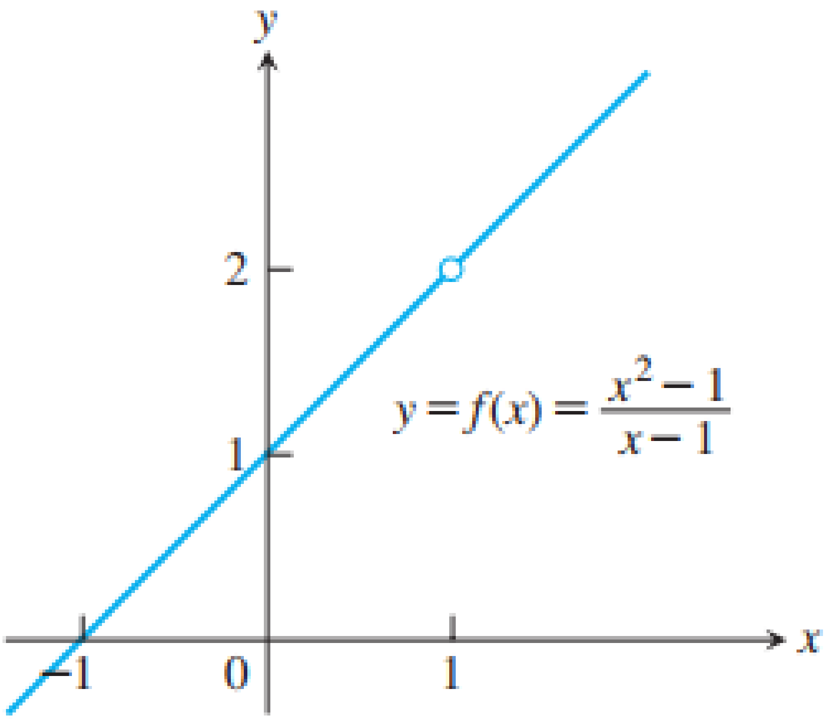
behave near $x = 1$?

Solution

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

Limit of a Function and Limit Laws





Limit of a Function and Limit Laws

itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L." In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

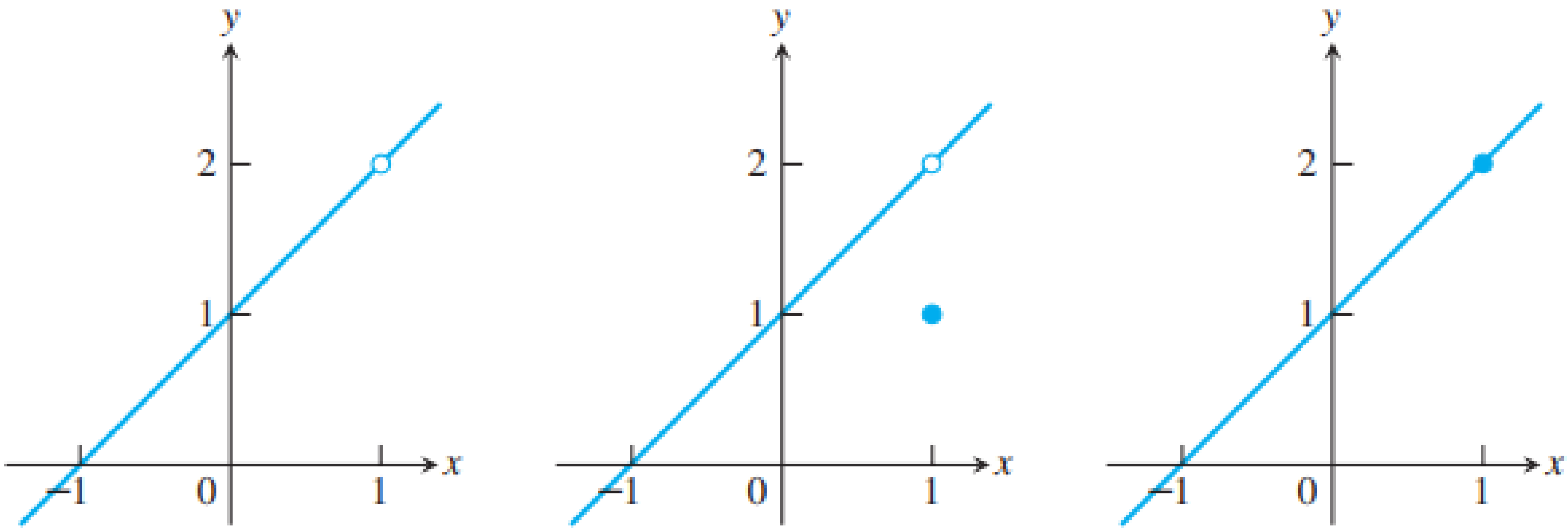
$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Limit of a Function and Limit Laws

EXAMPLE 2



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

$\lim\limits_{x \to 1} f(x) = 2$

$f(1)$ غير موجودة

$\lim\limits_{x \to 1} g(x) = 2$

$g(1) = 1$

$\lim\limits_{x \to 1} h(x) = 2$

$h(1) = 2$

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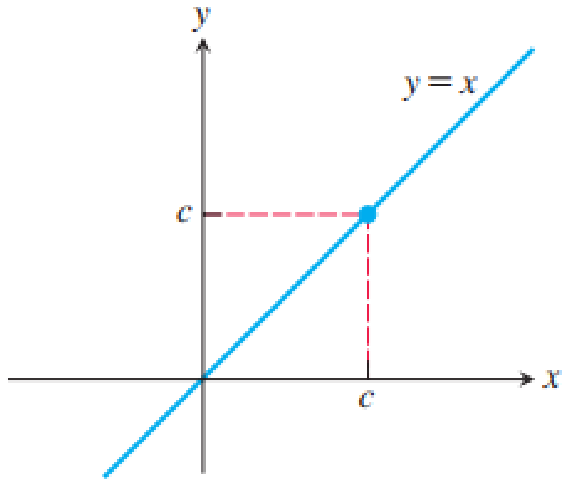
EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$,

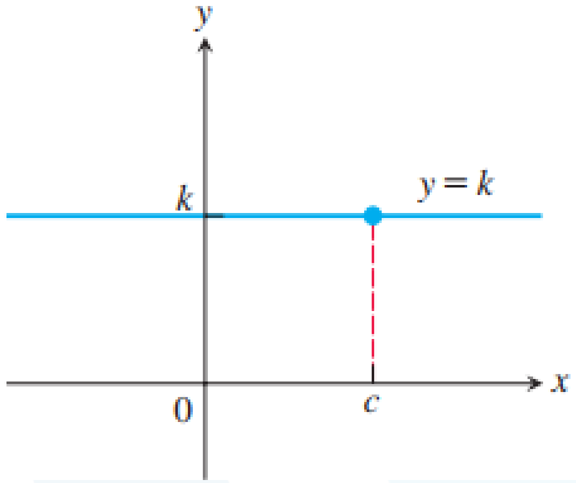
$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$



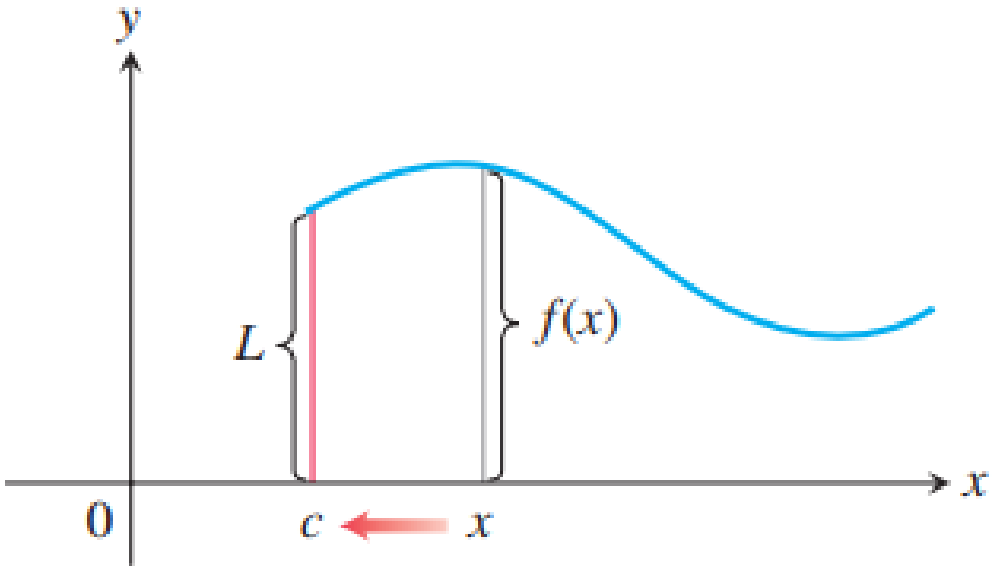
(a) Identity function

(b) If f is the **constant function** $f(x) = k$

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

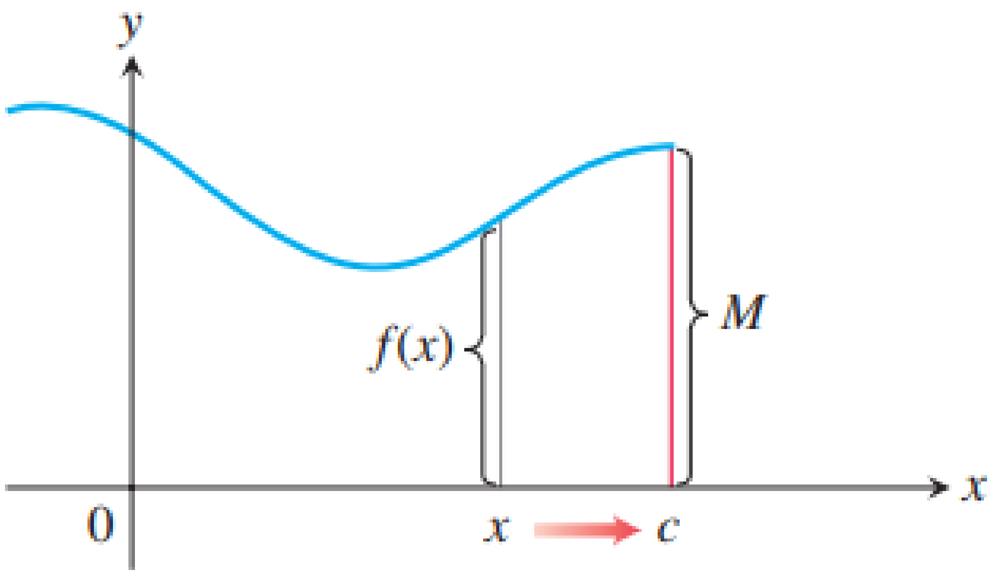


Approaching a Limit from One Side



(a) $\lim_{x \to c^+} f(x) = L$

(a) Right-hand limit as x approaches c.



(b) $\lim_{x \to c^-} f(x) = M$

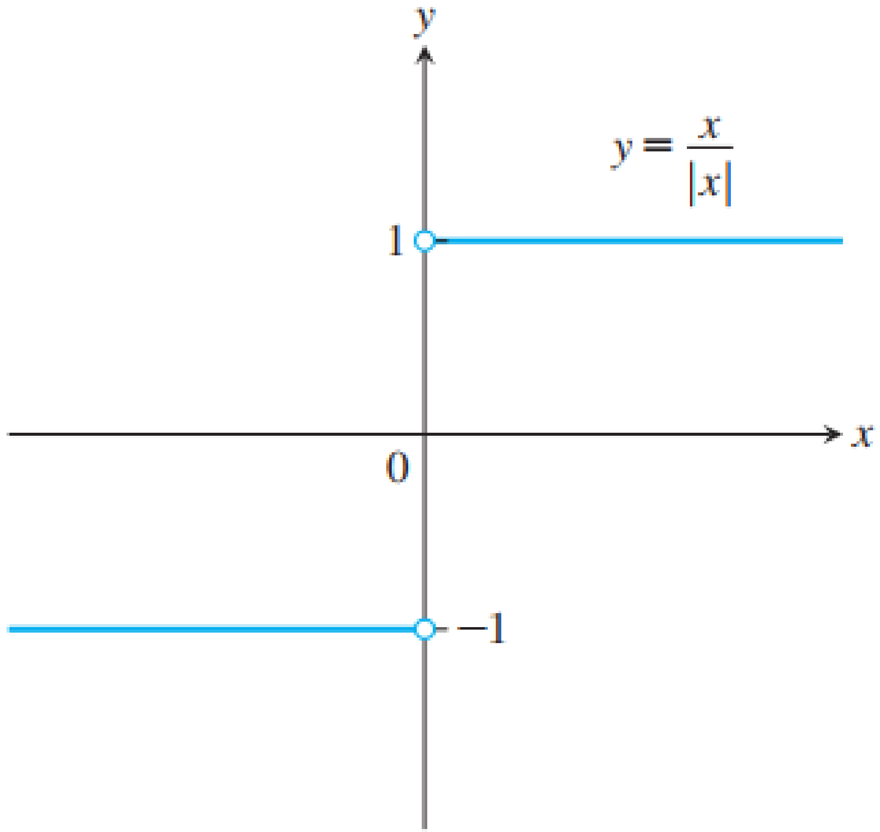
(b) Left-hand limit as x approaches c.

Approaching a Limit from One Side

EXAMPLE

$$f(x) = x/|x|$$

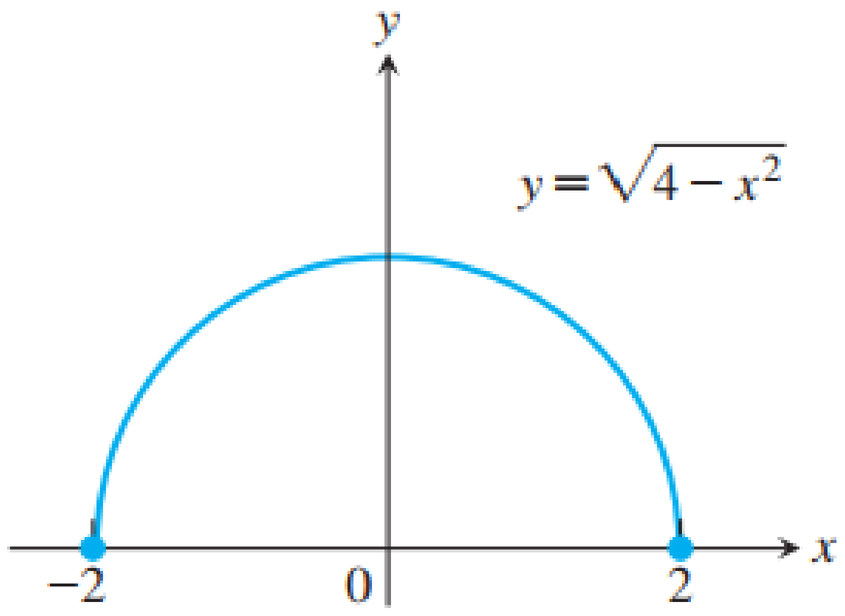
$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$



Approaching a Limit from One Side

EXAMPLE The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$



Approaching a Limit from One Side

THEOREM 6 Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Limit of a Function and Limit Laws

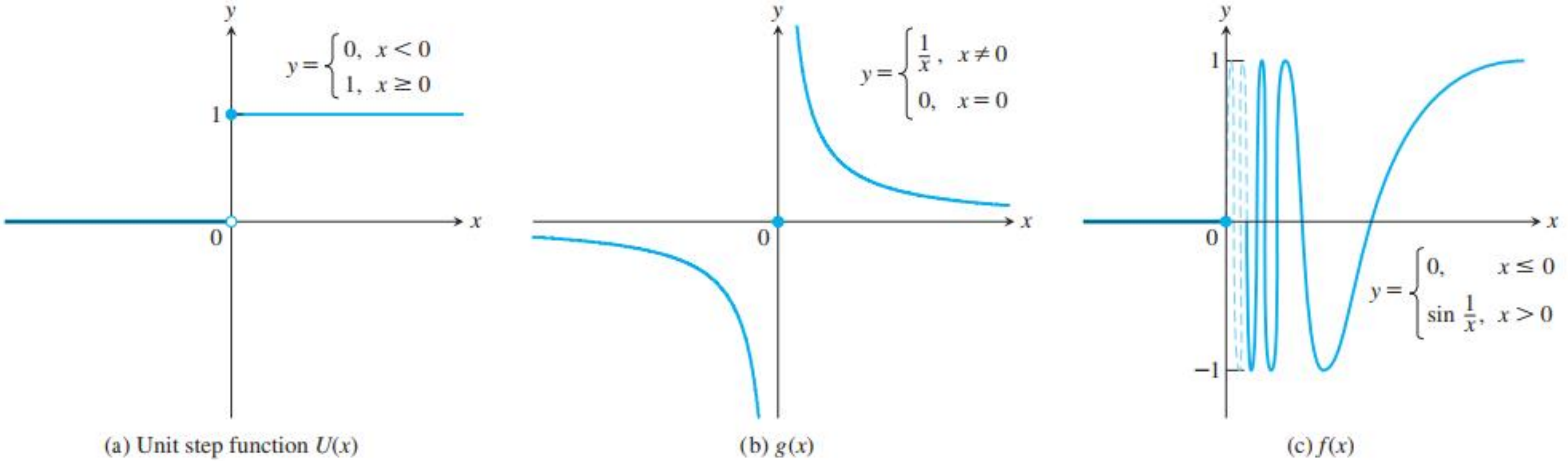
EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Limit of a Function and Limit Laws



$$\lim_{x\to 0^-} U(x)=0$$

$$\lim_{x\to 0^+} U(x)=1$$

$$\lim_{x\to 0^-} U(x)\neq \lim_{x\to 0^+} U(x)$$

النهاية غير موجودة

$$\lim_{x\to 0^-} g(x)=-\infty$$

$$\lim_{x\to 0^+} g(x)=+\infty$$

$$\lim_{x\to 0^-} g(x)\neq \lim_{x\to 0^+} g(x)$$

النهاية غير موجودة

$$\lim_{x\to 0^-} f(x)=0$$

$\lim_{x\to 0^+} f(x)$ النهاية غير موجودة

$-1\leq \sin x\leq +1$ النهاية غير موجودة

the function's values between +1 and -1 in every open interval containing 0

Limit Laws

THEOREM 1—Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

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EXAMPLE 5

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a) $$\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3$$

(b) $$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c} (x^4 + x^2 - 1)}{\lim_{x \to c} (x^2 + 5)} = \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

(c) $$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2} (4x^2 - 3)} = \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} = \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$$

Evaluating Limits of Polynomials and Rational Functions

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Evaluating Limits

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

EXAMPLE 8 Estimate the value of $\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2}$.

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)}$$

$$= \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$$

$$= \frac{1}{\sqrt{x^2+100}+10}.$$

Therefore,

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2+100}+10}$$

$$= \frac{1}{20} = 0.05.$$

Limit Quoti
not 0 at x =

The Sandwich Theorem

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \qquad \text{for all } x \neq 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

The Sandwich Theorem

Solution Since

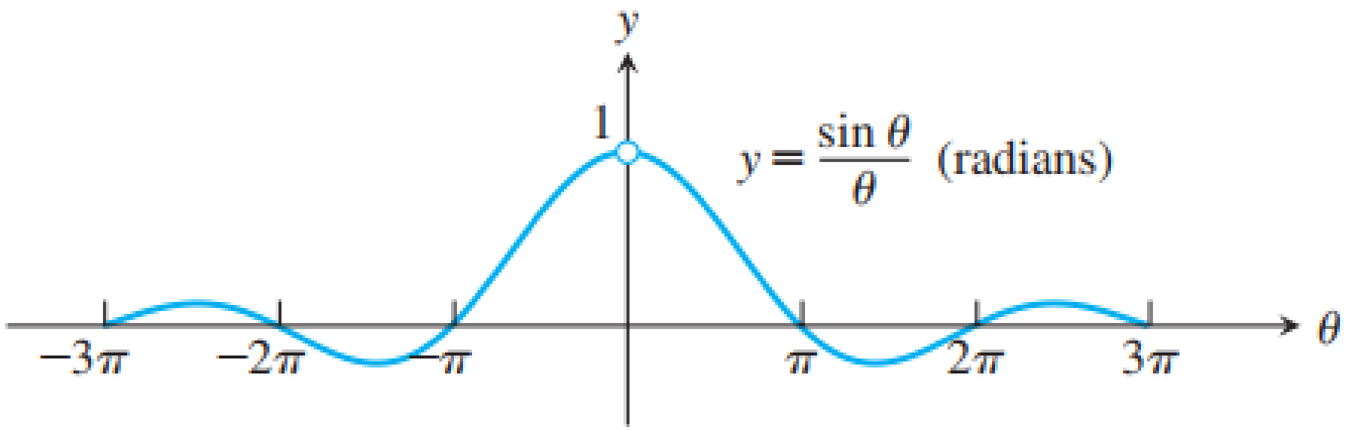
$$\lim_{x\to 0}(1-(x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x\to 0}(1+(x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$

Limits Involving (sin x)/x

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$



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EXAMPLE Show that **(a)** $\lim_{y\to 0} \frac{\cos y - 1}{y} = 0$ and **(b)** $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$$\lim_{h\to 0} \frac{\cos y - 1}{y} = \lim_{h\to 0} -\frac{2\sin^2(y/2)}{y}$$

$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = y/2.$$

$$= -(1)(0) = 0. \qquad \text{Eq. (1) and Example 11a in Section 2.2}$$

Limits Involving (sin x)/x

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x}$$

$$= \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x}$$

Eq. (1) applies with $\theta = 2x$.

$$= \frac{2}{5}(1) = \frac{2}{5}.$$

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EXAMPLE Find $\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of tan t and sec $2t$, we have

$$\begin{aligned}\lim_{t \to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3} \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$

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EXAMPLE Show that for nonzero constants A and B.

Solution

$$\lim_{\theta\to 0}\frac{\sin A\theta}{\sin B\theta}=\lim_{\theta\to 0}\frac{\sin A\theta}{A\theta}A\theta\,\frac{B\theta}{\sin B\theta}\,\frac{1}{B\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin A\theta}{A\theta}\,\frac{B\theta}{\sin B\theta}\,\frac{A}{B}$$

$$=\lim_{\theta\to 0}(1)(1)\,\frac{A}{B}$$

$$=\frac{A}{B}.$$

Continuity at a Point

$\lim_{x\to 1^-} f(x) = 0 \quad \lim_{x\to 1^+} f(x) = 1 \quad f(1) = 1$

so the function *is* continuous from the right at *x* = 1.

$\lim_{x\to 2} f(x) = 1 \qquad f(2) = 2.$

f is not continuous at *x* = 2.

$\lim_{x\to 4^-} f(x) = 1 \qquad f(4) = \frac{1}{2}$

the function is not continuous from the left.

$\lim_{x\to 3} f(x) = 2. \qquad f(3) = 2.$

The function is continuous at x = 3

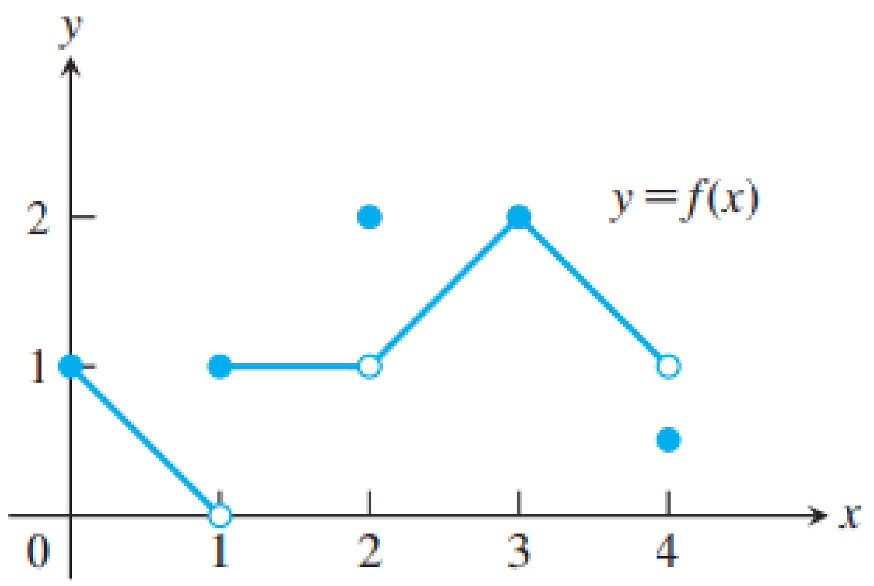

FIGURE : The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

$\lim_{x\to 0^+} f(x) = 1 \qquad f(0) = 1$

The function is continuous from the right at *x* = 0.

Continuity

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

Continuity

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Continuous Functions

THEOREM —Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

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EXAMPLE

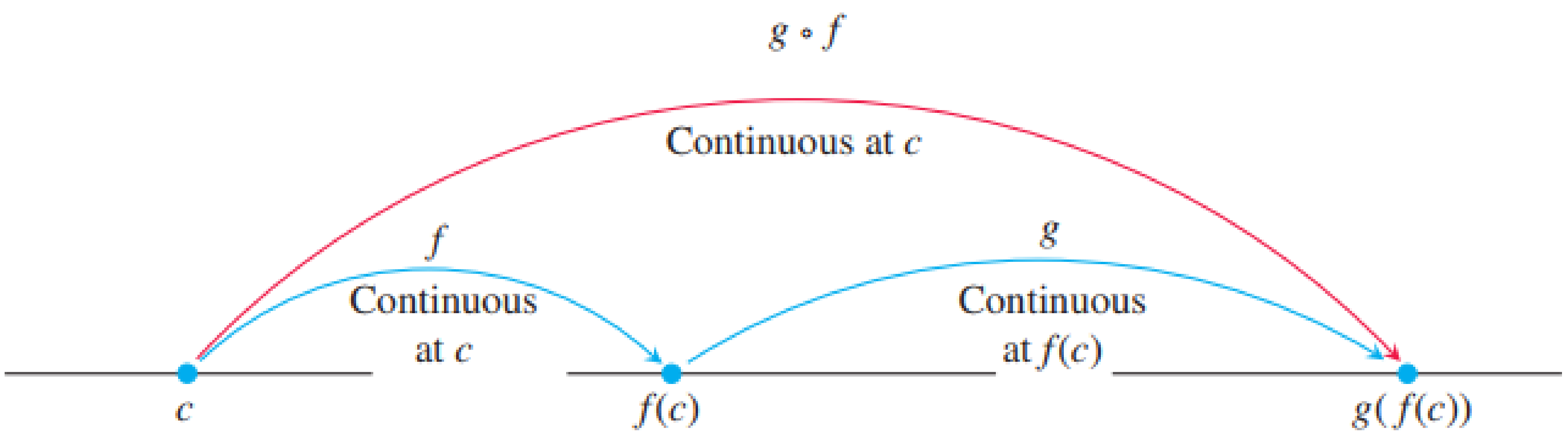
(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2.

EXAMPLE The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$.

Continuity of Compositions of Functions

THEOREM — Compositions of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.



Continuity of Compositions of Functions

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$

(d) $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

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Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

(d) Because the sine function is everywhere-continuous (Exercise 64), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function

Continuity of Compositions of Functions

THEOREM 10—Limits of Continuous Functions
If $\lim_{x\to c} f(x) = b$ and g is continuous at the point b, then

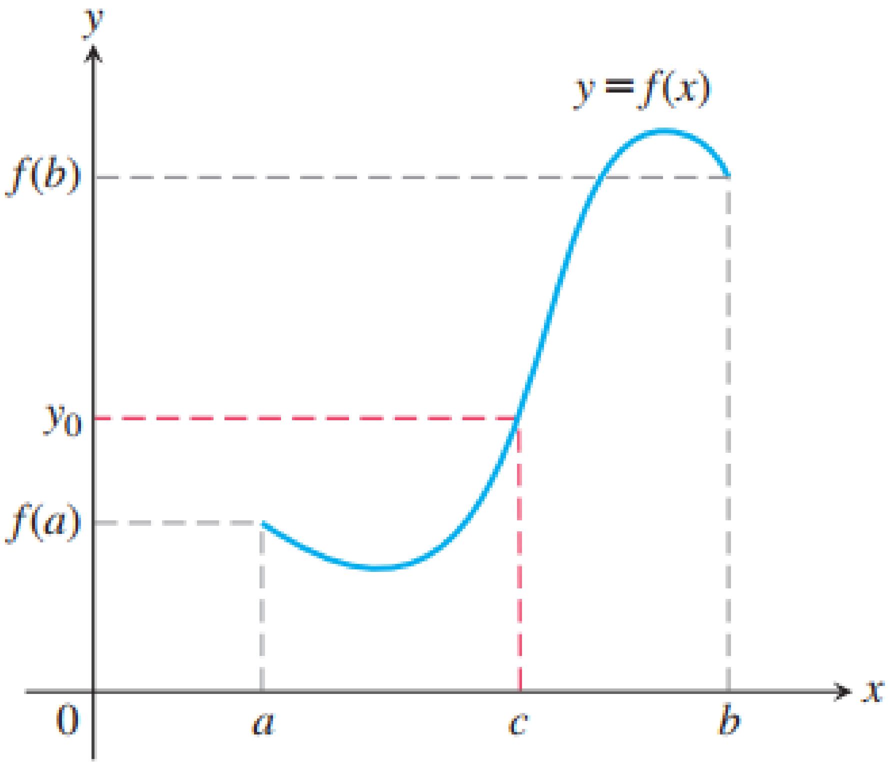
$$\lim_{x\to c} g(f(x)) = g(b).$$

EXAMPLE 9 Applying Theorem 10, we have

$$\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right)$$
$$= \cos(\pi + \sin 2\pi) = \cos\pi = -1.$$

Intermediate Value Theorem for Continuous Functions

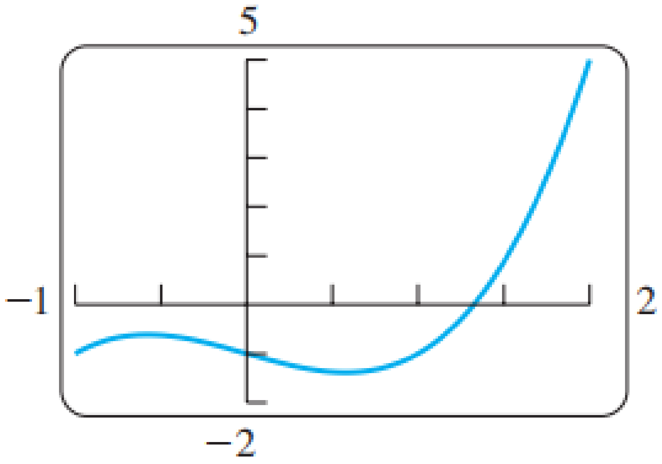
THEOREM 11—The Intermediate Value Theorem for Continuous Functions
If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Intermediate Value Theorem for Continuous Functions

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$.



Intermediate Value Theorem for Continuous Functions

EXAMPLE 11 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 - 4 = 0,$$

and set $f(x) = \sqrt{2x + 5} + x^2 - 4$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x. It follows that $f(x) = \sqrt{2x + 5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 0$. The number c solves the original equation. ■

Finite Limits as $x \to \pm\infty$

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c.

(a)
$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{Sum Rule}$$
$$= 5 + 0 = 5 \qquad \text{Known limits}$$

(b)
$$\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$$
$$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x} \qquad \text{Product Rule}$$
$$= \pi\sqrt{3}\cdot 0\cdot 0 = 0 \qquad \text{Known limits}$$

■

Limits Involving Infinity; Asymptotes of Graphs

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a) $$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$$ Divide numerator and denominator by x^3.

$$=\frac{5+0-0}{3+0}=\frac{5}{3}$$ See Fig. 2.51.

(b) $$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$ Divide numerator and denominator by x^3.

$$=\frac{0+0}{2-0}=0$$ See Fig. 2.52. ■

Horizontal Asymptotes

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Horizontal Asymptotes

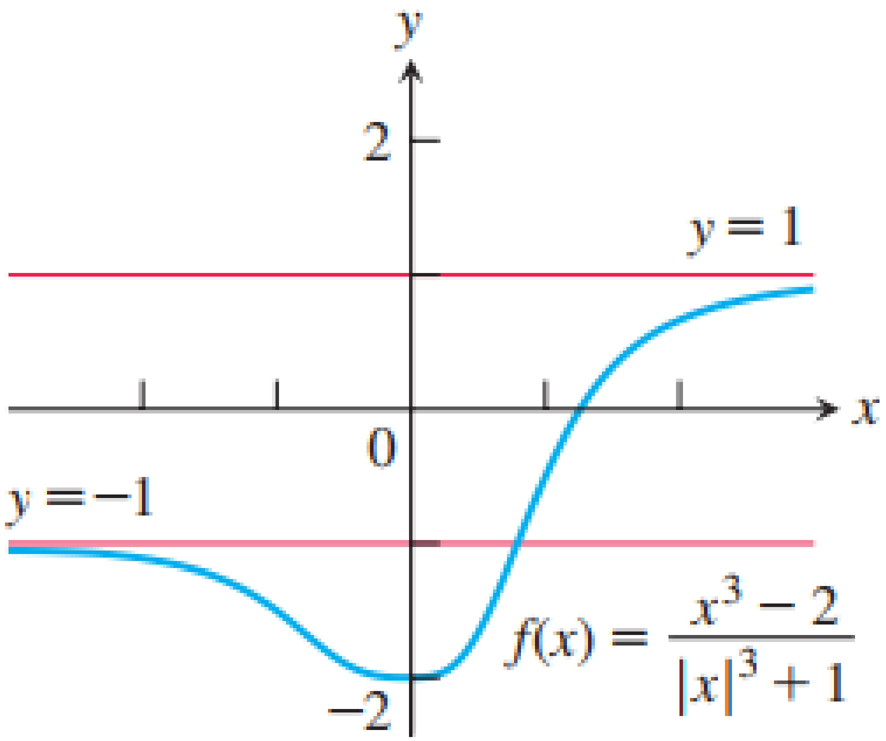
Solution We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

For $x < 0$: $$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x. ■

Horizontal Asymptotes



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EXAMPLE 5 Find **(a)** $\lim_{x \to \infty} \sin(1/x)$ and **(b)** $\lim_{x \to \pm\infty} x \sin(1/x)$.

$$\lim_{x \to \infty} \sin\frac{1}{x} = \lim_{t \to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x \to \infty} x \sin\frac{1}{x} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} x \sin\frac{1}{x} = \lim_{t \to 0^-} \frac{\sin t}{t} = 1.$$

Horizontal Asymptotes

EXAMPLE 7 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.56).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

Oblique Asymptotes

EXAMPLE 9 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 + 0x - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

$$g(x) = \frac{x}{2} + 1$$

Infinite Limits

EXAMPLE 10 Find $\lim_{x \to 1^+} \frac{1}{x-1}$ and $\lim_{x \to 1^-} \frac{1}{x-1}$.

$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{x-1} = -\infty.$$

EXAMPLE 13 Find $\lim_{x \to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

$$\begin{aligned} \lim_{x \to -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \to -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x \to -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty, \end{aligned}$$

Vertical Asymptotes

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Vertical Asymptotes

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.65). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

Exercices

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x \to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

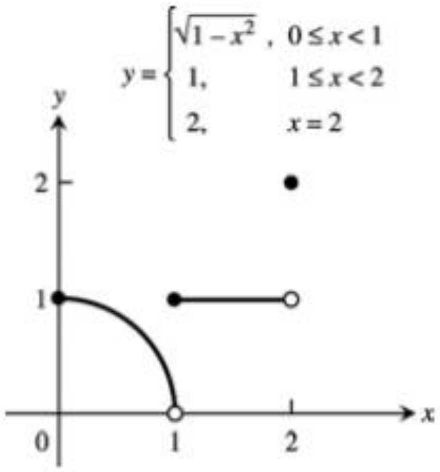
d. At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

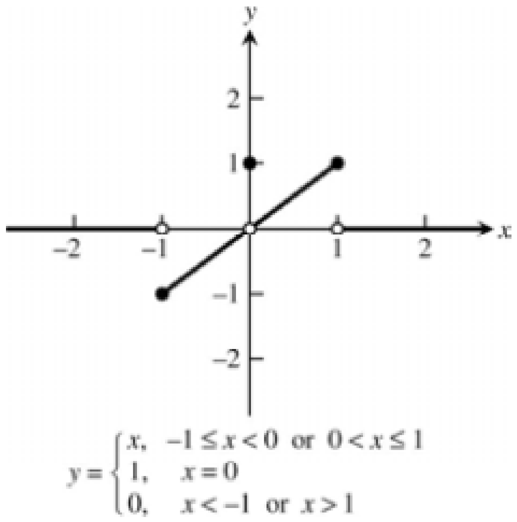
10. $f(x) = \begin{cases} x, & -1 \le x < 0, \quad \text{or} \quad 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \quad \text{or} \quad x > 1 \end{cases}$

Solution

9. (a) domain: $0 \le x \le 2$
   range: $0 < y \le 1$ and $y = 2$
   (b) $\lim_{x \to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
   (c) $x = 2$
   (d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
    range: $-1 \le y \le 1$
    (b) $\lim_{x \to c} f(x)$ exists for c belonging to
    $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
    (c) none
    (d) none



Exercices **Find the limits in Exercises**

14. $\lim_{x \to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

15. $\lim_{h \to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

16. $\lim_{h \to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

17. **a.** $\lim_{x \to -2^+} (x+3)\frac{|x+2|}{x+2}$ **b.** $\lim_{x \to -2^-} (x+3)\frac{|x+2|}{x+2}$

Solution

14. $\lim_{x\to 1^-}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)=\left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right)=\left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right)=1$

15. $\lim_{h\to 0^+}\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}=\lim_{h\to 0^+}\left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)=\lim_{h\to 0^+}\frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}$

$=\lim_{h\to 0^+}\frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}=\frac{0+4}{\sqrt{5}+\sqrt{5}}=\frac{2}{\sqrt{5}}$

16. $\lim_{h\to 0^-}\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}=\lim_{h\to 0^-}\left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$=\lim_{h\to 0^-}\frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)}=\lim_{h\to 0^-}\frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)}=\frac{-(0+11)}{\sqrt{6}+\sqrt{6}}=-\frac{11}{2\sqrt{6}}$

17. (a) $\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}=\lim_{x\to -2^+}(x+3)\frac{(x+2)}{(x+2)}$ $\quad(|x+2|=(x+2)\text{ for }x>-2)$

$=\lim_{x\to -2^+}(x+3)=((-2)+3)=1$

(b) $\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2}=\lim_{x\to -2^-}(x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad(|x+2|=-(x+2)\text{ for }x<-2)$

$=\lim_{x\to -2^-}(x+3)(-1)=-(-2+3)=-1$

Exercices

23. $\lim_{y\to 0} \frac{\sin 3y}{4y}$

24. $\lim_{h\to 0^-} \frac{h}{\sin 3h}$

25. $\lim_{x\to 0} \frac{\tan 2x}{x}$

26. $\lim_{t\to 0} \frac{2t}{\tan t}$

27. $\lim_{x\to 0} \frac{x \csc 2x}{\cos 5x}$

28. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x$

29. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$

30. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

31. $\lim_{\theta\to 0} \frac{1 - \cos\theta}{\sin 2\theta}$

32. $\lim_{x\to 0} \frac{x - x\cos x}{\sin^2 3x}$

33. $\lim_{t\to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

34. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

35. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$

36. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

37. $\lim_{\theta\to 0} \theta \cos\theta$

38. $\lim_{\theta\to 0} \sin\theta \cot 2\theta$

39. $\lim_{x\to 0} \frac{\tan 3x}{\sin 8x}$

40. $\lim_{y\to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

Solution

23. $\lim_{y\to 0}\frac{\sin 3y}{4y}=\frac{1}{4}\lim_{y\to 0}\frac{3\sin 3y}{3y}=\frac{3}{4}\lim_{y\to 0}\frac{\sin 3y}{3y}=\frac{3}{4}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{3}{4}$ (where $\theta=3y$)

24. $\lim_{h\to 0^-}\frac{h}{\sin 3h}=\lim_{h\to 0^-}\left(\frac{1}{3}\cdot\frac{3h}{\sin 3h}\right)=\frac{1}{3}\lim_{h\to 0^-}\frac{1}{\left(\frac{\sin 3h}{3h}\right)}=\frac{1}{3}\left(\frac{1}{\lim\limits_{\theta\to 0^-}\frac{\sin\theta}{\theta}}\right)=\frac{1}{3}\cdot 1=\frac{1}{3}$ (where $\theta=3h$)

25. $\lim_{x\to 0}\frac{\tan 2x}{x}=\lim_{x\to 0}\frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x}=\lim_{x\to 0}\frac{\sin 2x}{x\cos 2x}=\left(\lim_{x\to 0}\frac{1}{\cos 2x}\right)\left(\lim_{x\to 0}\frac{2\sin 2x}{2x}\right)=1\cdot 2=2$

26. $\lim_{t\to 0}\frac{2t}{\tan t}=2\lim_{t\to 0}\frac{t}{\left(\frac{\sin t}{\cos t}\right)}=2\lim_{t\to 0}\frac{t\cos t}{\sin t}=2\left(\lim_{t\to 0}\cos t\right)\left(\frac{1}{\lim\limits_{t\to 0}\frac{\sin t}{t}}\right)=2\cdot 1\cdot 1=2$

27. $\lim_{x\to 0}\frac{x\csc 2x}{\cos 5x}=\lim_{x\to 0}\left(\frac{x}{\sin 2x}\cdot\frac{1}{\cos 5x}\right)=\left(\frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin 2x}\right)\left(\lim_{x\to 0}\frac{1}{\cos 5x}\right)=\left(\frac{1}{2}\cdot 1\right)(1)=\frac{1}{2}$

28. $\lim_{x\to 0}6x^2(\cot x)(\csc 2x)=\lim_{x\to 0}\frac{6x^2\cos x}{\sin x\sin 2x}=\lim_{x\to 0}\left(3\cos x\cdot\frac{x}{\sin x}\cdot\frac{2x}{\sin 2x}\right)=3\cdot 1\cdot 1=3$

29. $\lim_{x\to 0}\frac{x+x\cos x}{\sin x\cos x}=\lim_{x\to 0}\left(\frac{x}{\sin x\cos x}+\frac{x\cos x}{\sin x\cos x}\right)=\lim_{x\to 0}\left(\frac{x}{\sin x}\cdot\frac{1}{\cos x}\right)+\lim_{x\to 0}\frac{x}{\sin x}$

$$=\lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)\cdot\lim_{x\to 0}\left(\frac{1}{\cos x}\right)+\lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)=(1)(1)+1=2$$

30. $\lim_{x\to 0} \frac{x^2-x+\sin x}{2x} = \lim_{x\to 0}\left(\frac{x}{2}-\frac{1}{2}+\frac{1}{2}\left(\frac{\sin x}{x}\right)\right) = 0-\frac{1}{2}+\frac{1}{2}(1) = 0$

31. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$

$= \lim_{\theta\to 0} \frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)} = \frac{0}{(2)(2)} = 0$

32. $\lim_{x\to 0} \frac{x-x\cos x}{\sin^2 3x} = \lim_{x\to 0} \frac{x(1-\cos x)}{\sin^2 3x} = \lim_{x\to 0} \frac{\frac{x(1-\cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x\to 0} \frac{\frac{1-\cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)}{\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$

33. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ since $\theta = 1-\cos t \to 0$ as $t \to 0$

34. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ since $\theta = \sin h \to 0$ as $h \to 0$

35. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\sin 2\theta}\cdot\frac{2\theta}{2\theta}\right) = \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

36. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0}\left(\frac{\sin 5x}{\sin 4x}\cdot\frac{4x}{5x}\cdot\frac{5}{4}\right) = \frac{5}{4}\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{4x}{\sin 4x}\right) = \frac{5}{4}\cdot 1\cdot 1 = \frac{5}{4}$

37. $\lim_{\theta\to 0} \theta\cos\theta = 0\cdot 1 = 0$

38. $\lim_{\theta\to 0} \sin\theta\cot 2\theta = \lim_{\theta\to 0} \sin\theta\frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta\to 0} \sin\theta\frac{\cos 2\theta}{2\sin\theta\cos\theta} = \lim_{\theta\to 0} \frac{\cos 2\theta}{2\cos\theta} = \frac{1}{2}$

39. $\lim_{x\to 0} \frac{\tan 3x}{\sin 8x} = \lim_{x\to 0} \left(\frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\sin 8x}\right) = \lim_{x\to 0} \left(\frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\sin 8x}\cdot\frac{8x}{3x}\cdot\frac{3}{8}\right)$

$$= \frac{3}{8}\lim_{x\to 0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8}\cdot 1\cdot 1\cdot 1 = \frac{3}{8}$$

40. $\lim_{y\to 0} \frac{\sin 3y\cot 5y}{y\cot 4y} = \lim_{y\to 0} \frac{\sin 3y\sin 4y\cos 5y}{y\cos 4y\sin 5y} = \lim_{y\to 0} \left(\frac{\sin 3y}{y}\right)\left(\frac{\sin 4y}{\cos 4y}\right)\left(\frac{\cos 5y}{\sin 5y}\right)\left(\frac{3\cdot 4\cdot 5y}{3\cdot 4\cdot 5y}\right)$

$$= \lim_{y\to 0} \left(\frac{\sin 3y}{3y}\right)\left(\frac{\sin 4y}{4y}\right)\left(\frac{5y}{\sin 5y}\right)\left(\frac{\cos 5y}{\cos 4y}\right)\left(\frac{3\cdot 4}{5}\right) = 1\cdot 1\cdot 1\cdot 1\cdot\frac{12}{5} = \frac{12}{5}$$

Exercices

33. $\lim_{y \to 1} \sec (y \sec^2 y - \tan^2 y - 1)$

34. $\lim_{x \to 0} \tan\left(\frac{\pi}{4} \cos (\sin x^{1/3})\right)$

35. $\lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$ 36. $\lim_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

Solution

33. $\lim_{y \to 1} \sec(y\sec^2 y - \tan^2 y - 1) = \lim_{y \to 1} \sec(y\sec^2 y - \sec^2 y) = \lim_{y \to 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1) =$ $\sec 0 = 1$, and function continuous at $y = 1$.

34. $\lim_{x \to 0} \tan\left[\frac{\pi}{4}\cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4}\cos(\sin(0))\right] = \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

35. $\lim_{t \to 0} \cos\left[\frac{\pi}{\sqrt{19-3\sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19-3\sec 0}}\right] = \cos\frac{\pi}{\sqrt{16}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

Exercices

45. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

Solution

45. As defined, $\lim\limits_{x \to 2^-} f(x) = 12$ and $\lim\limits_{x \to 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For $f(x)$ to be continuous we must have $12 = 2a^2 - 2a \Rightarrow a = 3$ or $a = -2$.

Exercices

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \le -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \ge 1 \end{cases}$$

Solution

47. As defined, $\lim_{x\to -1^-} f(x) = -2$ and $\lim_{x\to -1^+} f(x) = a(-1) + b = -a + b$, and $\lim_{x\to 1^-} f(x) = a(1) + b = a + b$ and $\lim_{x\to 1^+} f(x) = 3$. For $f(x)$ to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow a = \frac{5}{2}$ and $b = \frac{1}{2}$.

Exercices

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

Solution

48. As defined, $\lim_{x\to 0^-} g(x) = a(0) + 2b = 2b$ and $\lim_{x\to 0^+} g(x) = (0)^2 + 3a - b = 3a - b$, and $\lim_{x\to 2^-} g(x) = (2)^2 + 3a - b = 4 + 3a - b$ and $\lim_{x\to 0^+} g(x) = 3(2) - 5 = 1$. For $g(x)$ to be continuous we must have $2b = 3a - b$ and $4 + 3a - b = 1 \Rightarrow a = -\frac{3}{2}$ and $b = -\frac{3}{2}$.

Exercices

25. $\lim_{x\to-\infty}\left(\frac{1-x^3}{x^2+7x}\right)^5$

26. $\lim_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}}$

27. $\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}$

28. $\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}$

29. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

30. $\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

31. $\lim_{x\to\infty}\frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

32. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

33. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}$

34. $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}$

35. $\lim_{x\to\infty}\frac{x-3}{\sqrt{4x^2+25}}$

36. $\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

25. $$\lim_{x\to-\infty}\left(\frac{1-x^3}{x^2-7x}\right)^5=\lim_{x\to-\infty}\left(\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5=\left(\lim_{x\to-\infty}\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}}\right)^5=\left(\frac{0+\infty}{1-0}\right)^5=\infty$$

26. $$\lim_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}}=\lim_{x\to\infty}\sqrt{\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}}=\sqrt{\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}}=\sqrt{\frac{0-0}{1+0-0}}=\sqrt{0}=0$$

27. $$\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}=\lim_{x\to\infty}\frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}}=0$$

28. $$\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}=\lim_{x\to\infty}\frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1}=-1$$

29. $$\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}=\lim_{x\to-\infty}\frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}}=\lim_{x\to-\infty}\frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)}=1$$

30. $$\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}=\lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1-\frac{1}{x}}=\infty$$

31. $$\lim_{x\to\infty}\frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}=\lim_{x\to\infty}\frac{2x^{1/15}-\frac{1}{x^{19/15}}+\frac{7}{x^{8/5}}}{1+\frac{3}{x^{3/5}}+\frac{1}{x^{11/10}}}=\infty$$

32. $$\lim_{x\to-\infty}\frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}=\lim_{x\to-\infty}\frac{\frac{1}{x^{2/3}}-5+\frac{3}{x}}{2+\frac{1}{x^{1/3}}-\frac{4}{x}}=-\frac{5}{2}$$

33. $$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}=\lim_{x\to\infty}\frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}}=\lim_{x\to\infty}\frac{\sqrt{(x^2+1)/x^2}}{(x+1)/x}=\lim_{x\to\infty}\frac{\sqrt{1+1/x^2}}{(1+1/x)}=\frac{\sqrt{1+0}}{(1+0)}=1$$

34. $$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}=\lim_{x\to-\infty}\frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}}=\lim_{x\to-\infty}\frac{\sqrt{(x^2+1)/x^2}}{(x+1)/(-x)}=\lim_{x\to\infty}\frac{\sqrt{1+1/x^2}}{(-1-1/x)}=\frac{\sqrt{1+0}}{(-1-0)}=-1$$

35. $$\lim_{x\to\infty}\frac{x-3}{\sqrt{4x^2+25}}=\lim_{x\to\infty}\frac{(x-3)/\sqrt{x^2}}{\sqrt{4x^2+25}/\sqrt{x^2}}=\lim_{x\to\infty}\frac{(x-3)/x}{\sqrt{(4x^2+25)/x^2}}=\lim_{x\to\infty}\frac{(1-3/x)}{\sqrt{4+25/x^2}}=\frac{(1-0)}{\sqrt{4+0}}=\frac{1}{2}$$

36. $$\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}=\lim_{x\to-\infty}\frac{(4-3x^3)/\sqrt{x^6}}{\sqrt{x^6+9}/\sqrt{x^6}}=\lim_{x\to-\infty}\frac{(4-3x^3)/(-x^3)}{\sqrt{(x^6+9)/x^6}}=\lim_{x\to\infty}\frac{(-4/x^3+3)}{\sqrt{1+9/x^6}}=\frac{(0+3)}{\sqrt{1+0}}=3$$

Find the limits in Exercises 80–86.

80. $\lim_{x \to \infty} \left(\sqrt{x + 9} - \sqrt{x + 4} \right)$

81. $\lim_{x \to \infty} \left(\sqrt{x^2 + 25} - \sqrt{x^2 - 1} \right)$

82. $\lim_{x \to -\infty} \left(\sqrt{x^2 + 3} + x \right)$

83. $\lim_{x \to -\infty} \left(2x + \sqrt{4x^2 + 3x - 2} \right)$

84. $\lim_{x \to \infty} \left(\sqrt{9x^2 - x} - 3x \right)$

85. $\lim_{x \to \infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right)$

86. $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x} \right)$

80. $$\lim_{x\to\infty}\left(\sqrt{x+9}-\sqrt{x+4}\right)=\lim_{x\to\infty}\left[\sqrt{x+9}-\sqrt{x+4}\right]\cdot\left[\frac{\sqrt{x+9}+\sqrt{x+4}}{\sqrt{x+9}+\sqrt{x+4}}\right]=\lim_{x\to\infty}\frac{(x+9)-(x+4)}{\sqrt{x+9}+\sqrt{x+4}}$$

$$=\lim_{x\to\infty}\frac{5}{\sqrt{x+9}+\sqrt{x+4}}=\lim_{x\to\infty}\frac{\frac{5}{\sqrt{x}}}{\sqrt{1+\frac{9}{x}}+\sqrt{1+\frac{4}{x}}}=\frac{0}{1+1}=0$$

81. $$\lim_{x\to\infty}\left(\sqrt{x^2+25}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\left[\sqrt{x^2+25}-\sqrt{x^2-1}\right]\cdot\left[\frac{\sqrt{x^2+25}+\sqrt{x^2-1}}{\sqrt{x^2+25}+\sqrt{x^2-1}}\right]=\lim_{x\to\infty}\frac{(x^2+25)-(x^2-1)}{\sqrt{x^2+25}+\sqrt{x^2-1}}$$

$$=\lim_{x\to\infty}\frac{26}{\sqrt{x^2+25}+\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{\frac{26}{x}}{\sqrt{1+\frac{25}{x^2}}+\sqrt{1-\frac{1}{x^2}}}=\frac{0}{1+1}=0$$

82. $$\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)=\lim_{x\to-\infty}\left[\sqrt{x^2+3}+x\right]\cdot\left[\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}\right]=\lim_{x\to-\infty}\frac{(x^2+3)-(x^2)}{\sqrt{x^2+3}-x}=\lim_{x\to-\infty}\frac{3}{\sqrt{x^2+3}-x}$$

$$=\lim_{x\to-\infty}\frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1+\frac{3}{x^2}}-\frac{x}{\sqrt{x^2}}}=\lim_{x\to-\infty}\frac{-\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}=\frac{0}{1+1}=0$$

83. $\lim_{x\to-\infty}\left(2x+\sqrt{4x^2+3x-2}\right)=\lim_{x\to-\infty}\left[2x+\sqrt{4x^2+3x-2}\right]\cdot\left[\frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}\right]=\lim_{x\to-\infty}\frac{(4x^2)-(4x^2+3x-2)}{2x-\sqrt{4x^2+3x-2}}$

$=\lim_{x\to-\infty}\frac{-3x+2}{2x-\sqrt{4x^2+3x-2}}=\lim_{x\to-\infty}\frac{\frac{-3x+2}{\sqrt{x^2}}}{\frac{2x}{\sqrt{x^2}}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}=\lim_{x\to-\infty}\frac{\frac{-3x+2}{-x}}{\frac{2x}{-x}-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$

$=\lim_{x\to-\infty}\frac{3-\frac{2}{x}}{-2-\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}=\frac{3-0}{-2-2}=-\frac{3}{4}$

84. $\lim_{x\to\infty}\left(\sqrt{9x^2-x}-3x\right)=\lim_{x\to\infty}\left[\sqrt{9x^2-x}-3x\right]\cdot\left[\frac{\sqrt{9x^2-x}+3x}{\sqrt{9x^2-x}+3x}\right]=\lim_{x\to\infty}\frac{(9x^2-x)-(9x^2)}{\sqrt{9x^2-x}+3x}=\lim_{x\to\infty}\frac{-x}{\sqrt{9x^2-x}+3x}$

$=\lim_{x\to\infty}\frac{-\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2}-\frac{x}{x^2}}+\frac{3x}{x}}=\lim_{x\to\infty}\frac{-1}{\sqrt{9-\frac{1}{x}}+3}=\frac{-1}{3+3}=-\frac{1}{6}$

85. $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)=\lim_{x\to\infty}\left[\sqrt{x^2+3x}-\sqrt{x^2-2x}\right]\cdot\left[\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}\right]=\lim_{x\to\infty}\frac{(x^2+3x)-(x^2-2x)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$

$=\lim_{x\to\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}=\lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}=\frac{5}{1+1}=\frac{5}{2}$

86. $\lim_{x\to\infty}\sqrt{x^2+x}-\sqrt{x^2-x}=\lim_{x\to\infty}\left[\sqrt{x^2+x}-\sqrt{x^2-x}\right]\cdot\left[\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right]=\lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$

$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}=\frac{2}{1+1}=1$

Thank you for your attention