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Calculus 1

Dr. Yamar Hamwi

Al-Manara University

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Calculus 1

Lecture 4

Derivatives

Chapter 3
Derivatives

3.1 Tangents and the Derivative at a Point

3.2 The Derivative as a Function

3.3 Differentiation Rules

3.4 Derivatives of Trigonometric Functions

Tangent Lines and the Derivative at a Point

Finding a Tangent Line to the Graph of a Function

$$y = f(a) + f'(a)(x - a)$$

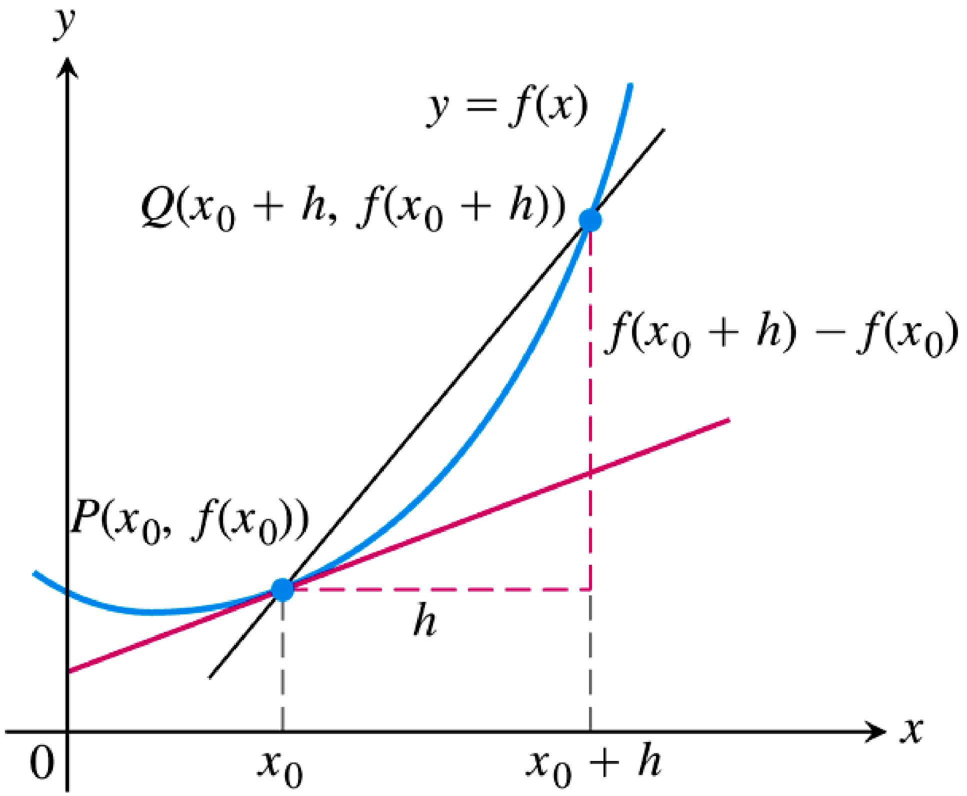


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Rates of Change: Derivative at a Point

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Example

find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together

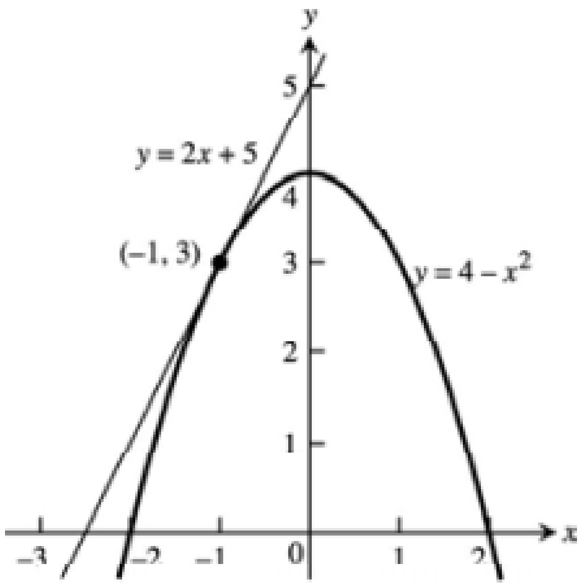
$$y = 4 - x^2, \quad (-1, 3)$$

Solution

$$m = \lim_{h \to 0} \frac{[4-(-1+h)^2]-(4-(-1)^2)}{h} = \lim_{h \to 0} \frac{-(1-2h+h^2)+1}{h}$$

$$= \lim_{h \to 0} \frac{h(2-h)}{h} = 2; \text{ at } (-1,3)\text{: } y = 3 + 2(x-(-1))$$

$$\Rightarrow y = 2x + 5, \text{ tangent line}$$



Vertical Tangents

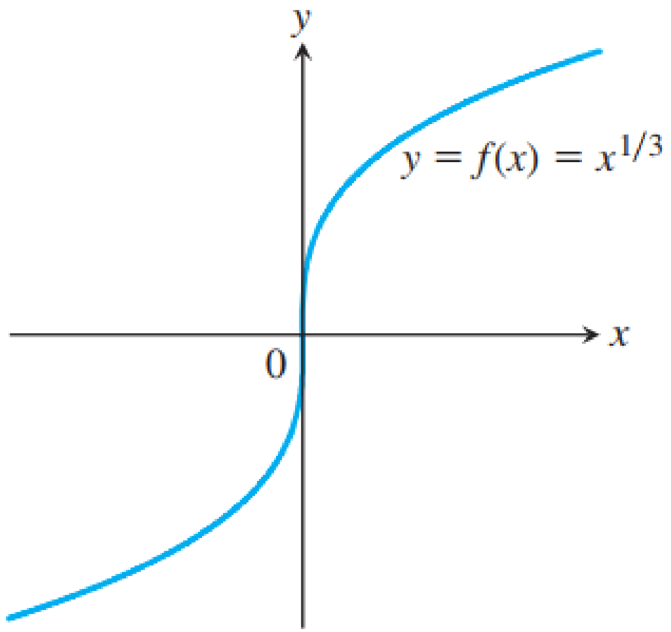
We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$.

Example

For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3} - 0}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty.$$



VERTICAL TANGENT AT ORIGIN

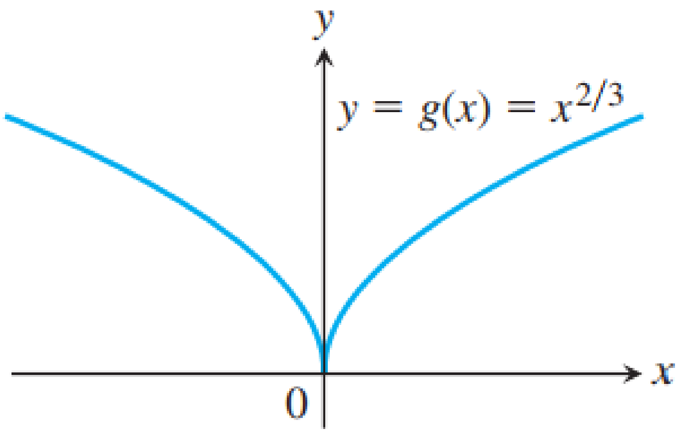
Vertical Tangents

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{2/3} - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT AT ORIGIN

The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The Derivative as a Function

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

Calculating Derivatives from the Definition

EXAMPLE 1 Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \text{Definition} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0.
\end{aligned}$$

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EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the alternative formula to calculate f':

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{\left(\sqrt{z} - \sqrt{x}\right)\left(\sqrt{z} + \sqrt{x}\right)} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

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(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

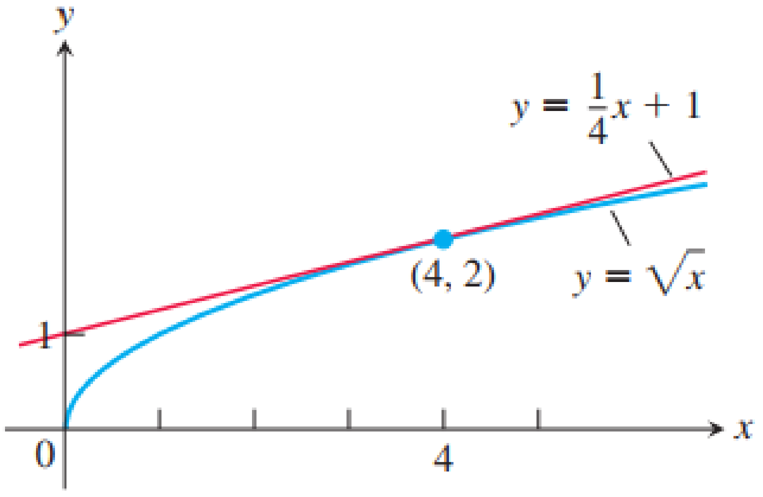

FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Calculating Derivatives from the Definition

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints

Differentiable on an Interval; One-Sided Derivatives

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m, \; |x| = x$$

Differentiable on an Interval; One-Sided Derivatives

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} \\
&= \lim_{h \to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h \to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} \\
&= \lim_{h \to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h \to 0^-} -1 = -1.
\end{aligned}$$

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Differentiable on an Interval; One-Sided Derivatives

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $\left(h, \sqrt{h}\right)$ on a graph of $y = \sqrt{x}$ approach ∞, the graph has a *vertical tangent* at the origin. (See Figure 1.17 on page 9.) ■

Differentiable Functions Are Continuous

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4.

Differentiation Rules

Powers, Multiples, Sums, and Differences

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Differentiation Rules

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x.

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

Differentiation Rules

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

Differentiation Rules

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

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Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

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EXAMPLE 7 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] = (x^2+1)(3x^2) + (x^3+3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ■

Differentiation Rules

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Differentiation Rules

EXAMPLE 8 Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$,

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

Differentiation Rules

EXAMPLE 9 Find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4:

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

Second- and Higher-Order Derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

Example

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

Second- and Higher-Order Derivative

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero. ■

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x$$

(c) $y = \frac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x\cos x - \sin x}{x^2}$$

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 5a and Theorem 7, Section 2.4} \\
&= -\sin x.
\end{aligned}$$

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The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

EXAMPLE 2 We find derivatives

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule}\\ &= \sin x(-\sin x) + \cos x(\cos x)\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

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(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\dfrac{d}{dx}(\cos x) - \cos x\dfrac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

Quotient Rule

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{1 - \sin x}$$

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EXAMPLE Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \quad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

Derivatives of Trigonometric Functions

EXAMPLE 6 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x \qquad \text{Derivative rule for secant function}$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Derivative Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \qquad \text{Derivative rules}$$

$$= \sec^3 x + \sec x \tan^2 x$$

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Derivatives of Trigonometric Functions

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exercises

Testing for Tangents

35. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

36. Does the graph of

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

Solution

35. Slope at origin $= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, $f(x)$ does have a tangent at the origin with slope 0.

36. $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} \sin\frac{1}{h}$. Since $\lim_{h\to 0} \sin\frac{1}{h}$ does not exist, $f(x)$ has no tangent at the origin.

Exercises

37. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

38. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution

37. $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-1-0}{h} = \infty$, and $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty$ $\Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

38. $\lim_{h\to 0^-} \frac{U(0+h)-U(0)}{h} = \lim_{h\to 0^-} \frac{0-1}{h} = \infty$, and $\lim_{h\to 0^+} \frac{U(0+h)-U(0)}{h} = \lim_{h\to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at $(0, 1)$ because the limit does not exist.

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Exercises

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1; \quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$
4. $k(z) = \dfrac{1 - z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$
5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$

1. Step 1: $f(x)=4-x^2$ and $f(x+h)=4-(x+h)^2$

Step 2: $\frac{f(x+h)-f(x)}{h}=\frac{[4-(x+h)^2]-(4-x^2)}{h}=\frac{(4-x^2-2xh-h^2)-4+x^2}{h}=\frac{-2xh-h^2}{h}=\frac{h(-2x-h)}{h}=-2x-h$

Step 3: $f'(x)=\lim\limits_{h\to 0}(-2x-h)=-2x;\ f'(-3)=6,\ f'(0)=0,\ f'(1)=-2$

2. $F(x)=(x-1)^2+1$ and $F(x+h)=(x+h-1)^2+1\Rightarrow F'(x)=\lim\limits_{h\to 0}\frac{[(x+h-1)^2+1]-[(x-1)^2+1]}{h}$

$=\lim\limits_{h\to 0}\frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h}=\lim\limits_{h\to 0}\frac{2xh+h^2-2h}{h}=\lim\limits_{h\to 0}(2x+h-2)=2(x-1);$

$F'(-1)=-4,\ F'(0)=-2,\ F'(2)=2$

3. Step 1: $g(t)=\frac{1}{t^2}$ and $g(t+h)=\frac{1}{(t+h)^2}$

Step 2: $\frac{g(t+h)-g(t)}{h}=\frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h}=\frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2\cdot t^2}\right)}{h}=\frac{t^2-(t^2+2th+h^2)}{(t+h)^2\cdot t^2\cdot h}=\frac{-2th-h^2}{(t+h)^2t^2h}=\frac{h(-2t-h)}{(t+h)^2t^2h}=\frac{-2t-h}{(t+h)^2t^2}$

Step 3: $g'(t)=\lim\limits_{h\to 0}\frac{-2t-h}{(t+h)^2t^2}=\frac{-2t}{t^2\cdot t^2}=\frac{-2}{t^3};\ g'(-1)=2,\ g'(2)=-\frac{1}{4},\ g'\left(\sqrt{3}\right)=-\frac{2}{3\sqrt{3}}$

4. $k(z)=\frac{1-z}{2z}$ and $k(z+h)=\frac{1-(z+h)}{2(z+h)}\Rightarrow k'(z)=\lim\limits_{h\to 0}\frac{\left(\frac{1-(z+h)}{2(z+h)}-\frac{1-z}{2z}\right)}{h}=\lim\limits_{h\to 0}\frac{(1-z-h)z-(1-z)(z+h)}{2(z+h)zh}=\lim\limits_{h\to 0}\frac{z-z^2-zh-z-h+z^2+zh}{2(z+h)zh}$

$=\lim\limits_{h\to 0}\frac{-h}{2(z+h)zh}=\lim\limits_{h\to 0}\frac{-1}{2(z+h)z}=\frac{-1}{2z^2};\ k'(-1)=-\frac{1}{2},\ k'(1)=-\frac{1}{2},\ k'\left(\sqrt{2}\right)=-\frac{1}{4}$

5. Step 1: $p(\theta)=\sqrt{3\theta}$ and $p(\theta+h)=\sqrt{3(\theta+h)}$

Step 2: $\frac{p(\theta+h)-p(\theta)}{h}=\frac{\sqrt{3(\theta+h)}-\sqrt{3\theta}}{h}=\frac{\left(\sqrt{3\theta+3h}-\sqrt{3\theta}\right)}{h}\cdot\frac{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}=\frac{(3\theta+3h)-3\theta}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}$

$=\frac{3h}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}=\frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}$

Step 3: $p'(\theta)=\lim_{h\to 0}\frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}=\frac{3}{\sqrt{3\theta}+\sqrt{3\theta}}=\frac{3}{2\sqrt{3\theta}}$; $p'(1)=\frac{3}{2\sqrt{3}}$, $p'(3)=\frac{1}{2}$, $p'\left(\frac{2}{3}\right)=\frac{3}{2\sqrt{2}}$

6. $r(s)=\sqrt{2s+1}$ and $r(s+h)=\sqrt{2(s+h)+1}\Rightarrow r'(s)=\lim_{h\to 0}\frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}$

$=\lim_{h\to 0}\frac{\left(\sqrt{2s+h+1}-\sqrt{2s+1}\right)}{h}\cdot\frac{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}=\lim_{h\to 0}\frac{(2s+2h+1)-(2s+1)}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}$

$=\lim_{h\to 0}\frac{2h}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}=\lim_{h\to 0}\frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}}=\frac{2}{\sqrt{2s+1}+\sqrt{2s+1}}=\frac{2}{2\sqrt{2s+1}}=\frac{1}{\sqrt{2s+1}}$;

$r'(0)=1$, $r'(1)=\frac{1}{\sqrt{3}}$, $r'\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}}$

Exercises

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

23. $f(x) = \dfrac{1}{x + 2}$

24. $f(x) = x^2 - 3x + 4$

25. $g(x) = \dfrac{x}{x - 1}$

26. $g(x) = 1 + \sqrt{x}$

Solution

23. $f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{\frac{1}{z+2}-\frac{1}{x+2}}{z-x}=\lim_{z\to x}\frac{(x+2)-(z+2)}{(z-x)(z+2)(x+2)}=\lim_{z\to x}\frac{x-z}{(z-x)(z+2)(x+2)}=\lim_{z\to x}\frac{-1}{(z+2)(x+2)}=\frac{-1}{(x+2)^2}$

24. $f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{(z^2-3z+4)-(x^2-3x+4)}{z-x}=\lim_{z\to x}\frac{z^2-3z-x^2+3x}{z-x}=\lim_{z\to x}\frac{z^2-x^2-3z+3x}{z-x}=\lim_{z\to x}\frac{(z-x)(z+x)-3(z-x)}{z-x}$
$=\lim_{z\to x}\frac{(z-x)[(z+x)-3]}{z-x}=\lim_{z\to x}[(z+x)-3]=2x-3$

25. $g'(x)=\lim_{z\to x}\frac{g(z)-g(x)}{z-x}=\lim_{z\to x}\frac{\frac{z}{z-1}-\frac{x}{x-1}}{z-x}=\lim_{z\to x}\frac{z(x-1)-x(z-1)}{(z-x)(z-1)(x-1)}=\lim_{z\to x}\frac{-z+x}{(z-x)(z-1)(x-1)}=\lim_{z\to x}\frac{-1}{(z-1)(x-1)}=\frac{-1}{(x-1)^2}$

26. $g'(x)=\lim_{z\to x}\frac{g(z)-g(x)}{z-x}=\lim_{z\to x}\frac{(1+\sqrt{z})-(1+\sqrt{x})}{z-x}=\lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x}\cdot\frac{\sqrt{z}+\sqrt{x}}{\sqrt{z}+\sqrt{x}}=\lim_{z\to x}\frac{z-x}{(z-x)(\sqrt{z}+\sqrt{x})}=\lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$

Exercises

In Exercises 41 and 42, determine if the piecewise-defined function is differentiable at the origin.

41. $f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$

42. $g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$

Solution

41. f is not continuous at $x = 0$ since $\lim_{x\to 0} f(x) =$ does not exist and $f(0) = -1$

42. Left-hand derivative: $\lim_{h\to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-} \frac{h^{1/3}-0}{h} = \lim_{h\to 0^-} \frac{1}{h^{2/3}} = +\infty$;

Right-hand derivative: $\lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \frac{h^{2/3}-0}{h} = \lim_{h\to 0^+} \frac{1}{h^{1/3}} = +\infty$;

Then $\lim_{h\to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = +\infty \Rightarrow$ the derivative $g'(0)$ does not exist.

Exercises

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \frac{1}{3s^2} - \frac{5}{2s}$
12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

1. $y=-x^2+3 \Rightarrow \frac{dy}{dx}=\frac{d}{dx}(-x^2)+\frac{d}{dx}(3)=-2x+0=-2x \Rightarrow \frac{d^2y}{dx^2}=-2$

2. $y=x^2+x+8 \Rightarrow \frac{dy}{dx}=2x+1+0=2x+1 \Rightarrow \frac{d^2y}{dx^2}=2$

3. $s=5t^3-3t^5 \Rightarrow \frac{ds}{dt}=\frac{d}{dt}(5t^3)-\frac{d}{dt}(3t^5)=15t^2-15t^4 \Rightarrow \frac{d^2s}{dt^2}=\frac{d}{dt}(15t^2)-\frac{d}{dt}(15t^4)=30t-60t^3$

4. $w=3z^7-7z^3+21z^2 \Rightarrow \frac{dw}{dz}=21z^6-21z^2+42z \Rightarrow \frac{d^2w}{dz^2}=126z^5-42z+42$

7. $w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$

8. $s = -2t^{-1} + 4t^{-2} \Rightarrow \frac{ds}{dt} = 2t^{-2} - 8t^{-3} = \frac{2}{t^2} - \frac{8}{t^3} \Rightarrow \frac{d^2s}{dt^2} = -4t^{-3} + 24t^{-4} = \frac{-4}{t^3} + \frac{24}{t^4}$

9. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} = 12x - 10 + \frac{10}{x^3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - \frac{30}{x^4}$

10. $y = 4 - 2x - x^{-3} \Rightarrow \frac{dy}{dx} = -2 + 3x^{-4} = -2 + \frac{3}{x^4} \Rightarrow \frac{d^2y}{dx^2} = 0 - 12x^{-5} = \frac{-12}{x^5}$

11. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

12. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \dfrac{2x + 5}{3x - 2}$

18. $z = \dfrac{4 - 3x}{3x^2 + x}$

19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$

20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \dfrac{5x + 1}{2\sqrt{x}}$

25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

17. $y=\frac{2x+5}{3x-2}$; use the quotient rule: $u=2x+5$ and $v=3x-2 \Rightarrow u'=2$ and $v'=3 \Rightarrow y'=\frac{vu'-uv'}{v^2}=\frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2}$
$=\frac{6x-4-6x-15}{(3x-2)^2}=\frac{-19}{(3x-2)^2}$

18. $y=\frac{4-3x}{3x^2+x}$; use the quotient rule: $u=4-3x$ and $v=3x^2+x \Rightarrow u'=-3$ and $v'=6x+1 \Rightarrow y'=\frac{vu'-uv'}{v^2}$
$=\frac{(3x^2+x)(-3)-(4-3x)(6x+1)}{(3x^2+x)^2}=\frac{-9x^2-3x+18x^2-21x-4}{(3x^2+x)^2}=\frac{9x^2-24x-4}{(3x^2+x)^2}$

19. $g(x)=\frac{x^2-4}{x+0.5}$; use the quotient rule: $u=x^2-4$ and $v=x+0.5 \Rightarrow u'=2x$ and $v'=1 \Rightarrow g'(x)=\frac{vu'-uv'}{v^2}$
$=\frac{(x+0.5)(2x)-(x^2-4)(1)}{(x+0.5)^2}=\frac{2x^2+x-x^2+4}{(x+0.5)^2}=\frac{x^2+x+4}{(x+0.5)^2}$

20. $f(t)=\frac{t^2-1}{t^2+t-2}=\frac{(t-1)(t+1)}{(t+2)(t-1)}=\frac{t+1}{t+2}, t\neq 1 \Rightarrow f'(t)=\frac{(t+2)(1)-(t+1)(1)}{(t+2)^2}=\frac{t+2-t-1}{(t+2)^2}=\frac{1}{(t+2)^2}$

21. $v=(1-t)\,(1+t^2)^{-1}=\frac{1-t}{1+t^2} \Rightarrow \frac{dv}{dt}=\frac{(1+t^2)(-1)-(1-t)(2t)}{(1+t^2)^2}=\frac{-1-t^2-2t+2t^2}{(1+t^2)^2}=\frac{t^2-2t-1}{(1+t^2)^2}$

22. $w=\frac{x+5}{2x-7} \Rightarrow w'=\frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2}=\frac{2x-7-2x-10}{(2x-7)^2}=\frac{-17}{(2x-7)^2}$

23. $f(s)=\frac{\sqrt{s}-1}{\sqrt{s}+1} \Rightarrow f'(s)=\frac{(\sqrt{s}+1)\left(\frac{1}{2\sqrt{s}}\right)-(\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2}=\frac{(\sqrt{s}+1)-(\sqrt{s}-1)}{2\sqrt{s}(\sqrt{s}+1)^2}=\frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$

NOTE: $\frac{d}{ds}(\sqrt{s})=\frac{1}{2\sqrt{s}}$ from Example 2 in Section 3.2

24. $u=\frac{5x+1}{2\sqrt{x}} \Rightarrow \frac{du}{dx}=\frac{(2\sqrt{x})(5)-(5x+1)\left(\frac{1}{\sqrt{x}}\right)}{4x}=\frac{5x-1}{4x^{3/2}}$

25. $v=\frac{1+x-4\sqrt{x}}{x}\Rightarrow v'=\frac{x\left(1-\frac{2}{\sqrt{x}}\right)-(1+x-4\sqrt{x})}{x^2}=\frac{2\sqrt{x}-1}{x^2}$

26. $r=2\left(\frac{1}{\sqrt{\theta}}+\sqrt{\theta}\right)\Rightarrow r'=2\left(\frac{\sqrt{\theta}(0)-1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta}+\frac{1}{2\sqrt{\theta}}\right)=-\frac{1}{\theta^{3/2}}+\frac{1}{\theta^{1/2}}$

27. $y=\frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u=1$ and $v=(x^2-1)(x^2+x+1)\Rightarrow u'=0$ and

$v'=(x^2-1)(2x+1)+(x^2+x+1)(2x)=2x^3+x^2-2x-1+2x^3+2x^2+2x=4x^3+3x^2-1\Rightarrow\frac{dy}{dx}=\frac{vu'-uv'}{v^2}$

$=\frac{0-1(4x^3+3x^2-1)}{(x^2-1)^2(x^2+x+1)^2}=\frac{-4x^3-3x^2+1}{(x^2-1)^2(x^2+x+1)^2}$

28. $y=\frac{(x+1)(x+2)}{(x-1)(x-2)}=\frac{x^2+3x+2}{x^2-3x+2}\Rightarrow y'=\frac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}=\frac{-6x^2+12}{(x-1)^2(x-2)^2}=\frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

Find the first and second derivatives of the functions in Exercises 45–52.

45. $y = \dfrac{x^3 + 7}{x}$

46. $s = \dfrac{t^2 + 5t - 1}{t^2}$

47. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

48. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

49. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$

50. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

45. $y=\frac{x^3+7}{x}=x^2+7x^{-1}\Rightarrow\frac{dy}{dx}=2x-7x^{-2}=2x-\frac{7}{x^2}\Rightarrow\frac{d^2y}{dx^2}=2+14x^{-3}=2+\frac{14}{x^3}$

46. $s=\frac{t^2+5t-1}{t^2}=1+\frac{5}{t}-\frac{1}{t^2}=1+5t^{-1}-t^{-2}\Rightarrow\frac{ds}{dt}=0-5t^{-2}+2t^{-3}=-5t^{-2}+2t^{-3}=\frac{-5}{t^2}+\frac{2}{t^3}$
$\Rightarrow\frac{d^2s}{dt^2}=10t^{-3}-6t^{-4}=\frac{10}{t^3}-\frac{6}{t^4}$

47. $r=\frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}=\frac{\theta^3-1}{\theta^3}=1-\frac{1}{\theta^3}=1-\theta^3\Rightarrow\frac{dr}{d\theta}=0+3\theta^{-4}=3\theta^{-4}=\frac{3}{\theta^4}\Rightarrow\frac{d^2r}{d\theta^2}=-12\theta^{-5}=\frac{-12}{\theta^5}$

48. $u=\frac{(x^2+x)(x^2-x+1)}{x^4}=\frac{x(x+1)(x^2-x+1)}{x^4}=\frac{x(x^3+1)}{x^4}=\frac{x^4+x}{x^4}=1+\frac{x}{x^4}=1+x^{-3}$
$\Rightarrow\frac{du}{dx}=0-3x^{-4}=-3x^{-4}=\frac{-3}{x^4}\Rightarrow\frac{d^2u}{dx^2}=12x^{-5}=\frac{12}{x^5}$

49. $w=\left(\frac{1+3z}{3z}\right)(3-z)=\left(\frac{1}{3}z^{-1}+1\right)(3-z)=z^{-1}-\frac{1}{3}+3-z=z^{-1}+\frac{8}{3}-z\Rightarrow\frac{dw}{dz}=-z^{-2}+0-1=-z^{-2}-1=\frac{-1}{z^2}-1$
$\Rightarrow\frac{d^2w}{dz^2}=2z^{-3}-0=2z^{-3}=\frac{2}{z^3}$

50. $p=\frac{q^2+3}{(q-1)^3+(q+1)^3}=\frac{q^2+3}{(q^3-3q^2+3q-1)+(q^3+3q^2+3q+1)}=\frac{q^2+3}{2q^3+6q}=\frac{q^2+3}{2q(q^2+3)}=\frac{1}{2q}=\frac{1}{2}q^{-1}\Rightarrow\frac{dp}{dq}=-\frac{1}{2}q^{-2}=-\frac{1}{2q^2}$
$\Rightarrow\frac{d^2p}{dq^2}=q^{-3}=\frac{1}{q^3}$

Exercises

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

54. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

53. $u(0)=5, u'(0)=-3, v(0)=-1, v'(0)=2$

(a) $\frac{d}{dx}(uv)=uv'+vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0}=u(0)v'(0)+v(0)u'(0)=5\cdot 2+(-1)(-3)=13$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0}=\frac{v(0)u'(0)-u(0)v'(0)}{(v(0))^2}=\frac{(-1)(-3)-(5)(2)}{(-1)^2}=-7$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)=\frac{uv'-vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0}=\frac{u(0)v'(0)-v(0)u'(0)}{(u(0))^2}=\frac{(5)(2)-(-1)(-3)}{(5)^2}=\frac{7}{25}$

(d) $\frac{d}{dx}(7v-2u)=7v'-2u' \Rightarrow \frac{d}{dx}(7v-2u)\,|_{x=0}=7v'(0)-2u'(0)=7\cdot 2-2(-3)=20$

54. $u(1)=2, u'(1)=0, v(1)=5, v'(1)=-1$

(a) $\frac{d}{dx}(uv)\,|_{x=1}=u(1)v'(1)+v(1)u'(1)=2\cdot(-1)+5\cdot 0=-2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1}=\frac{v(1)u'(1)-u(1)v'(1)}{(v(1))^2}=\frac{5\cdot 0-2\cdot(-1)}{(5)^2}=\frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1}=\frac{u(1)v'(1)-v(1)u'(1)}{(u(1))^2}=\frac{2\cdot(-1)-5\cdot 0}{(2)^2}=-\frac{1}{2}$

(d) $\frac{d}{dx}(7v-2u)\,|_{x=1}=7v'(1)-2u'(1)=7\cdot(-1)-2\cdot 0=-7$

53. $u(0)=5, u'(0)=-3, v(0)=-1, v'(0)=2$

(a) $\frac{d}{dx}(uv)=uv'+vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0)+v(0)u'(0) = 5\cdot 2+(-1)(-3)=13$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0)-u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)=\frac{uv'-vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0)-v(0)u'(0)}{(u(0))^2} = \frac{(5)(2)-(-1)(-3)}{(5)^2} = \frac{7}{25}$

(d) $\frac{d}{dx}(7v-2u)=7v'-2u' \Rightarrow \frac{d}{dx}(7v-2u)\big|_{x=0} = 7v'(0)-2u'(0) = 7\cdot 2-2(-3)=20$

54. $u(1)=2, u'(1)=0, v(1)=5, v'(1)=-1$

(a) $\frac{d}{dx}(uv)\big|_{x=1} = u(1)v'(1)+v(1)u'(1) = 2\cdot(-1)+5\cdot 0=-2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1)-u(1)v'(1)}{(v(1))^2} = \frac{5\cdot 0-2\cdot(-1)}{(5)^2} = \frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1)-v(1)u'(1)}{(u(1))^2} = \frac{2\cdot(-1)-5\cdot 0}{(2)^2} = -\frac{1}{2}$

(d) $\frac{d}{dx}(7v-2u)\big|_{x=1} = 7v'(1)-2u'(1) = 7\cdot(-1)-2\cdot 0=-7$

69. Find the value of a that makes the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

70. Find the values of a and b that make the following function differentiable for all x-values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

69. $g'(x)=\begin{cases}2x-3 & x>0\\ a & x<0\end{cases}$, since g is differentiable at $x=0 \Rightarrow \lim_{x\to 0^+}(2x-3)=-3$ and $\lim_{x\to 0^-} a=a \Rightarrow a=-3$

70. $f'(x)=\begin{cases}a & x>-1\\ 2bx & x<-1\end{cases}$, since f is differentiable at $x=-1 \Rightarrow \lim_{x\to -1^+} a=a$ and $\lim_{x\to -1^-}(2bx)=-2b \Rightarrow a=-2b$, and since f is continuous at $x=-1 \Rightarrow \lim_{x\to -1^+}(ax+b)=-a+b$ and $\lim_{x\to -1^-}(bx^2-3)=b-3 \Rightarrow -a+b=b-3$ $\Rightarrow a=3 \Rightarrow 3=-2b \Rightarrow b=-\frac{3}{2}$.

Derivatives

In Exercises 1–18, find dy/dx.

1. $y = -10x + 3\cos x$
2. $y = \frac{3}{x} + 5\sin x$
3. $y = x^2 \cos x$
4. $y = \sqrt{x}\sec x + 3$
5. $y = \csc x - 4\sqrt{x} + \frac{7}{e^x}$
6. $y = x^2 \cot x - \frac{1}{x^2}$
7. $f(x) = \sin x \tan x$
8. $g(x) = \frac{\cos x}{\sin^2 x}$
9. $y = xe^{-x}\sec x$
10. $y = (\sin x + \cos x)\sec x$
11. $y = \frac{\cot x}{1 + \cot x}$
12. $y = \frac{\cos x}{1 + \sin x}$
13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
15. $y = (\sec x + \tan x)(\sec x - \tan x)$
16. $y = x^2 \cos x - 2x\sin x - 2\cos x$
17. $f(x) = x^3 \sin x \cos x$
18. $g(x) = (2 - x)\tan^2 x$

In Exercises 19–22, find ds/dt.

19. $s = \tan t - e^{-t}$
20. $s = t^2 - \sec t + 5e^t$
21. $s = \frac{1 + \csc t}{1 - \csc t}$
22. $s = \frac{\sin t}{1 - \cos t}$

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22. $s = \frac{\sin t}{1 - \cos t}$

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = x^2 \cos x \Rightarrow \frac{dy}{dx} = x^2(-\sin x) + 2x\cos x = -x^2 \sin x + 2x\cos x$

4. $y = \sqrt{x}\sec x + 3 \Rightarrow \frac{dy}{dx} = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}} + 0 = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}}$

5. $y = \csc x - 4\sqrt{x} + \frac{7}{e^x} \quad \Rightarrow \frac{dy}{dx} = -\csc x\cot x - \frac{4}{2\sqrt{x}} - \frac{7}{e^x}$

6. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2\csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2\csc^2 x + 2x\cot x + \frac{2}{x^3}$

7. $f(x) = \sin x\tan x \Rightarrow f'(x) = \sin x\sec^2 x + \cos x\tan x = \sin x\sec^2 x + \cos x\frac{\sin x}{\cos x} = \sin x(\sec^2 x + 1)$

8. $g(x) = \frac{\cos x}{\sin^2 x} = \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = \csc x\cot x \Rightarrow g'(x) = \csc x(-\csc^2 x) + (-\csc x\cot x)\cot x = -\csc^3 x - \csc x\cot^2 x$
$= -\csc x(\csc^2 x + \cot^2 x)$

9. $y = xe^{-x}\sec x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x)e^{-x}\sec x + x\frac{d}{dx}(e^{-x})\sec x + xe^{-x}\frac{d}{dx}(\sec x) = e^{-x}\sec x - xe^{-x}\sec x + xe^{-x}\sec x\tan x =$
$e^{-x}\sec x(1 - x + x\tan x)$

10. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

11. $y = \frac{\cot x}{1+\cot x} \Rightarrow \frac{dy}{dx} = \frac{(1+\cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}$

12. $y = \frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$
$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$

13. $y=\frac{4}{\cos x}+\frac{1}{\tan x}=4\sec x+\cot x\Rightarrow\frac{dy}{dx}=4\sec x\tan x-\csc^2 x$

14. $y=\frac{\cos x}{x}+\frac{x}{\cos x}\Rightarrow\frac{dy}{dx}=\frac{x(-\sin x)-(\cos x)(1)}{x^2}+\frac{(\cos x)(1)-x(-\sin x)}{\cos^2 x}=\frac{-x\sin x-\cos x}{x^2}+\frac{\cos x+x\sin x}{\cos^2 x}$

15. $y=(\sec x+\tan x)(\sec x-\tan x)\Rightarrow\frac{dy}{dx}=(\sec x+\tan x)\frac{d}{dx}(\sec x-\tan x)+(\sec x-\tan x)\frac{d}{dx}(\sec x+\tan x)$

$=(\sec x+\tan x)(\sec x\tan x-\sec^2 x)+(\sec x-\tan x)(\sec x\tan x+\sec^2 x)$

$=(\sec^2 x\tan x+\sec x\tan^2 x-\sec^3 x-\sec^2 x\tan x)+(\sec^2 x\tan x-\sec x\tan^2 x+\sec^3 x-\tan x\sec^2 x)=0.$

$\left(\text{Note also that } y=\sec^2 x-\tan^2 x=(\tan^2 x+1)-\tan^2 x=1\Rightarrow\frac{dy}{dx}=0.\right)$

16. $y=x^2\cos x-2x\sin x-2\cos x\Rightarrow\frac{dy}{dx}=(x^2(-\sin x)+(\cos x)(2x))-(2x\cos x+(\sin x)(2))-2(-\sin x)$

$=-x^2\sin x+2x\cos x-2x\cos x-2\sin x+2\sin x=-x^2\sin x$

17. $f(x)=x^3\sin x\cos x\Rightarrow f'(x)=x^3\sin x(-\sin x)+x^3\cos x(\cos x)+3x^2\sin x\cos x$

$=-x^3\sin^2 x+x^3\cos^2 x+3x^2\sin x\cos x$

18. $g(x) = (2-x)\tan^2 x \Rightarrow g'(x) = (2-x)(2\tan x\sec^2 x) + (-1)\tan^2 x = 2(2-x)\tan x\sec^2 x - \tan^2 x$
$= 2(2-\text{x})\tan x(\sec^2 x - \tan x)$

19. $s = \tan t - e^{-t} \Rightarrow \frac{ds}{dt} = \sec^2 t + e^{-t}$

20. $s = t^2 - \sec t + 5e^t \Rightarrow \frac{ds}{dt} = 2t - \sec t\tan t + 5e^t$

21. $s = \frac{1+\csc t}{1-\csc t} \Rightarrow \frac{ds}{dt} = \frac{(1-\csc t)(-\csc t\cot t)-(1+\csc t)(\csc t\cot t)}{(1-\csc t)^2} = \frac{-\csc t\cot t+\csc^2 t\cot t-\csc t\cot t-\csc^2 t\cot t}{(1-\csc t)^2} = \frac{-2\csc t\cot t}{(1-\csc t)^2}$

22. $s = \frac{\sin t}{1-\cos t} \Rightarrow \frac{ds}{dt} = \frac{(1-\cos t)(\cos t)-(\sin t)(\sin t)}{(1-\cos t)^2} = \frac{\cos t-\cos^2 t-\sin^2 t}{(1-\cos t)^2} = \frac{\cos t-1}{(1-\cos t)^2} = -\frac{1}{1-\cos t} = \frac{1}{\cos t-1}$

33. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

34. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2\sin x$.

b. $y = 9\cos x$.

33. (a) $y=\csc x \Rightarrow y'=-\csc x\cot x \Rightarrow y''=-((\csc x)(-\csc^2 x)+(\cot x)(-\csc x\cot x))=\csc^3 x+\csc x\cot^2 x$
$=(\csc x)(\csc^2 x+\cot^2 x)=(\csc x)(\csc^2 x+\csc^2 x-1)=2\csc^3 x-\csc x$

(b) $y=\sec x \Rightarrow y'=\sec x\ \tan x \Rightarrow y''=(\sec x)(\sec^2 x)+(\tan x)(\sec x\ \tan x)=\sec^3 x+\sec x\ \tan^2 x$
$=(\sec x)(\sec^2 x+\tan^2 x)=(\sec x)(\sec^2 x+\sec^2 x-1)=2\sec^3 x-\sec x$

34. (a) $y=-2\ \sin x \Rightarrow y'=-2\cos x \Rightarrow y''=-2(-\sin x)=2\sin x \Rightarrow y'''=2\cos x \Rightarrow\ y^{(4)}=-2\ \sin x$

(b) $y=9\cos x \Rightarrow y'=-9\sin x \Rightarrow y''=-9\cos x \Rightarrow y'''=-9(-\sin x)=9\sin x \Rightarrow y^{(4)}=9\cos x$

Thank you for your attention