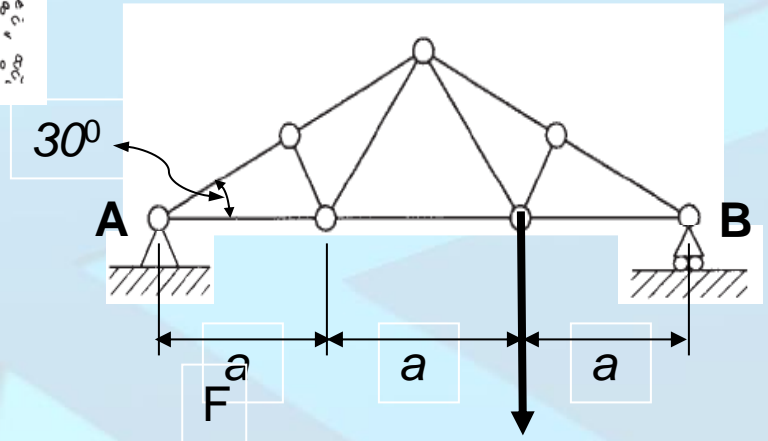
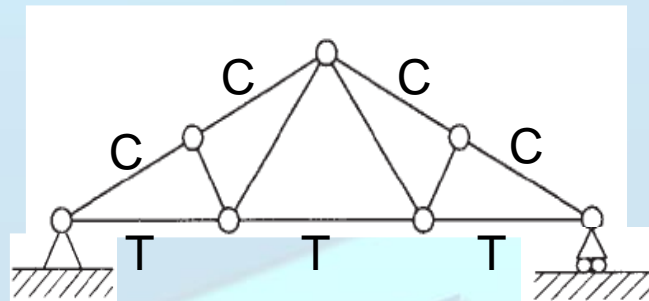
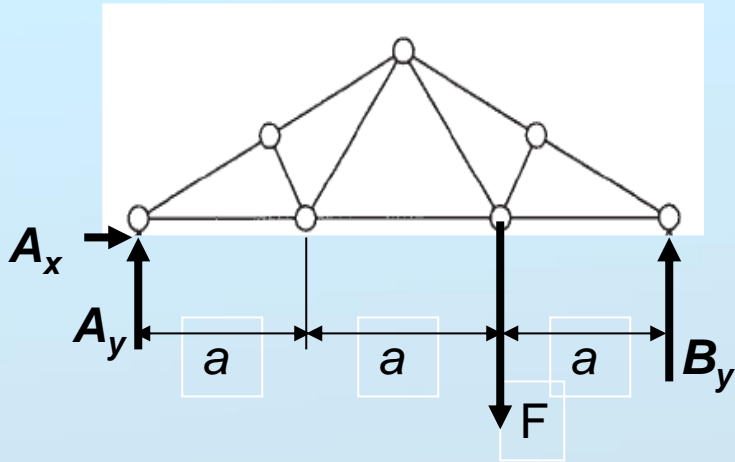
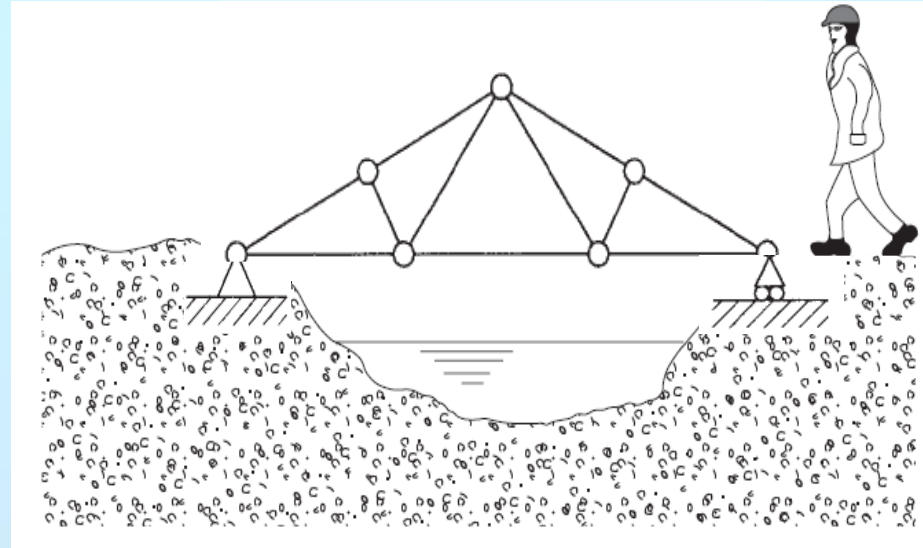


# Beams, Frames,... Arches

Beams are slender structural members that offer resistance to bending. They are among the most important elements in structural engineering.

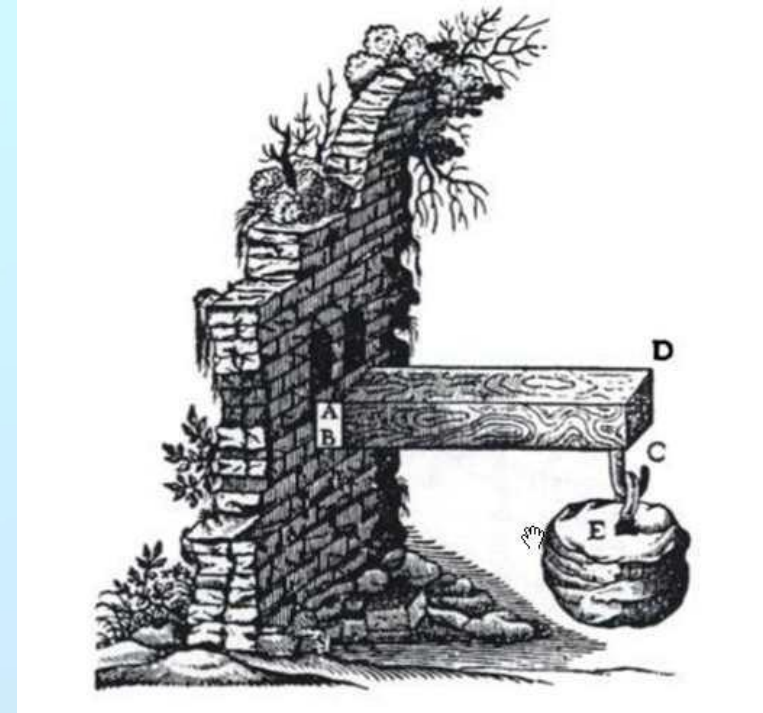
الجزان (ومفردها جائز) عناصر إنشائية هامة نحيلة تتلقى حمولات (أي قوى وعزوم) عرضية وتقاومها بالانحناء





الجيزان (ومفردها جائز) عناصر إنشائية هامة نحيلة تتلقى حمولات (أي قوى وعزوم) عرضية وتقاومها بالانعطاف

The basic function of a structure is to carry loads for which it has been designed & transmit forces from their points of applications to the supports. For example



Observing... the behavior of structures under the action of external loads, led to 2Qs (two questions) then to 2Ss+2Ss:

How strong is a str.? Strength notion & Stress concept.

How stiff is a str.? Stiffness notion & Strain concept. (*deformation concept*)

يهتم ميكانيك المواد بدراسة سلوك العناصر الإنشائية من خلال: **المقاومة** ومعيار قياسها **الإجهادات**، و**الصلابة** ومعيار قياسها **التشوهات**.

## Knowledge of the internal forces is important in order :

1. to determine the load-bearing capacity of a beam,
2. to compute the properties (area,...) of the cross-section required to sustain a given load,
3. or to compute the deformation as we will see later.

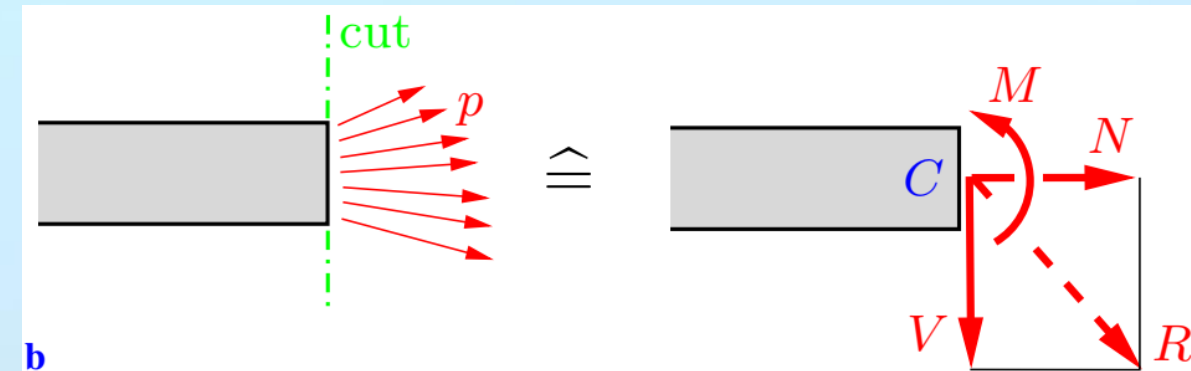
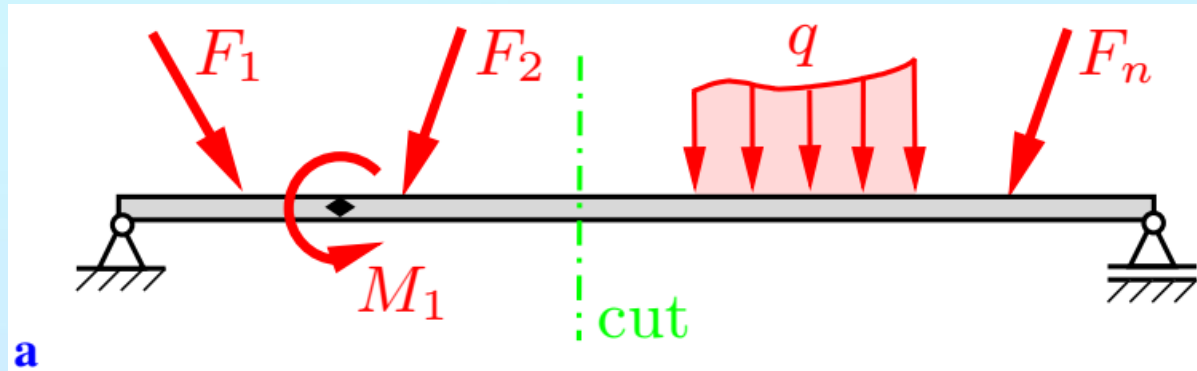
1. قدرة التحمل

2. خصائص المقطع القادر على تلقي الحمولة ونقلها بأمان

3. التشوهات

For simplicity, the discussion is limited to statically determinate plane problems,

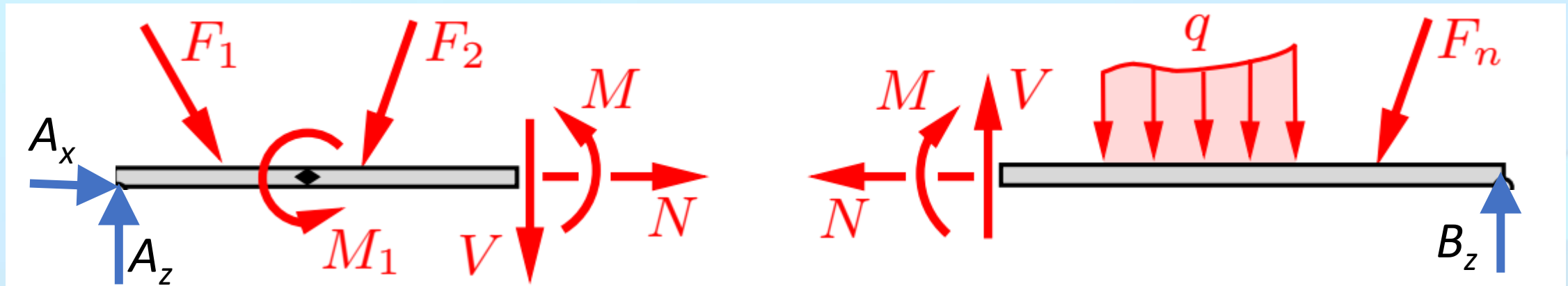
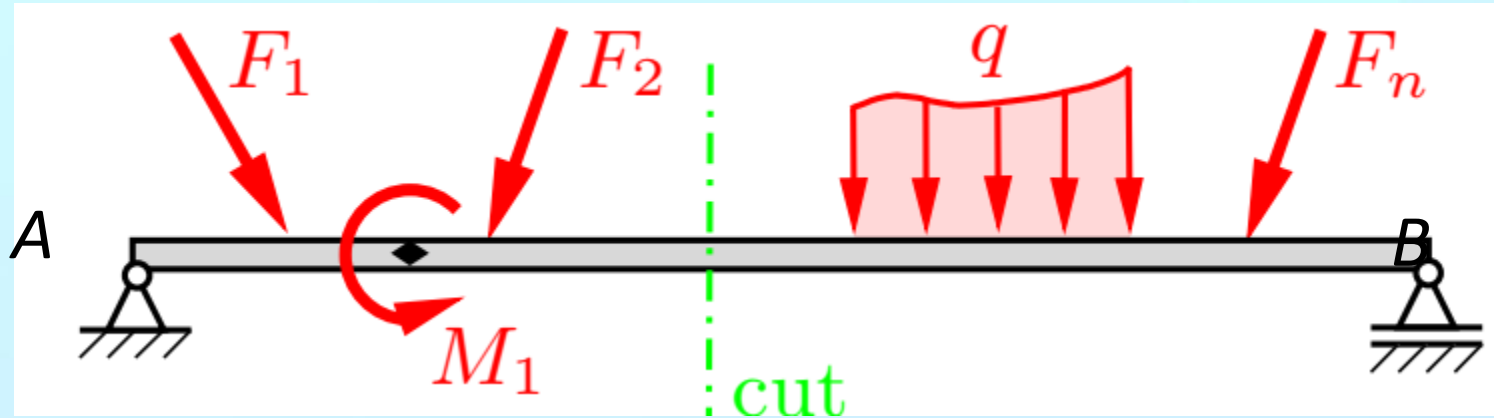
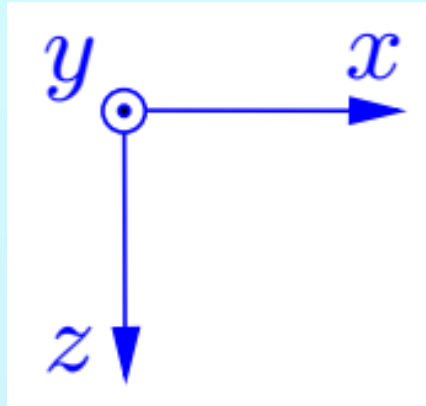
المسائل المقررة والمستوية



The quantities  $N$ ,  $V$  &  $M$  are called the stress resultants (قوى المقطع أو القوى الداخلية أو محصلات الاجهادات)

In particular  $N$  is called the normal force (القوة الناعمية),  $V$  is called the shear force (قوة القص) and  $M$  is called the bending moment (عزم الانعطاف).

In order to determine the stress resultants (القوى الداخلية), the beam is divided by a cut into two segments (method of sections) (طريقة المقطع)

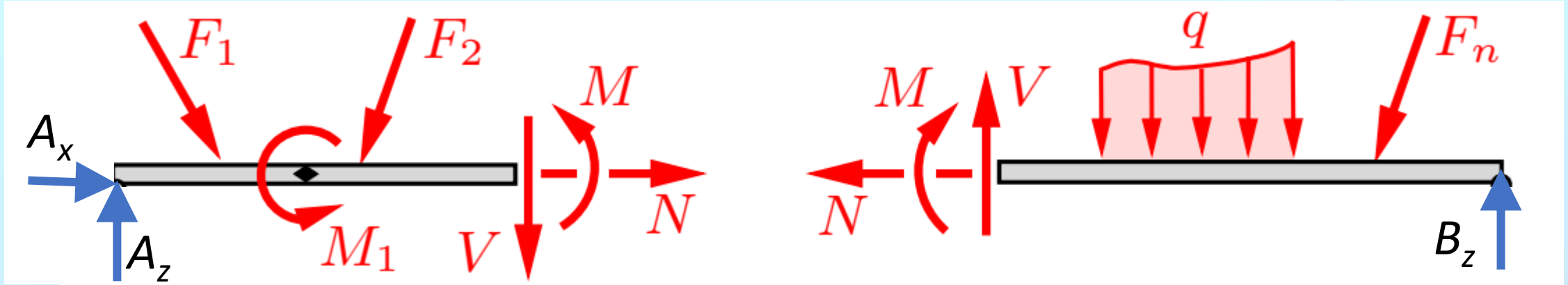


A F. B. D. of each segment includes all of the forces acting on it, i.e., the applied loads (forces and couples), the support reactions & the stress resultants acting at the cut sections. (مخطط جسم حر)

لكل جزء مع حملاته وردود أفعال مسانده وقوى المقطع

According to Newton's third law (action = reaction) stress resultant act in opposite directions at the two faces of the segments of the beam.

وفق قانون نيوتن الثالث (رد الفعل يساوي ويعاكس الفعل) تكون القوى الداخلية على وجهي القطع متساوية ومتعاكسة.

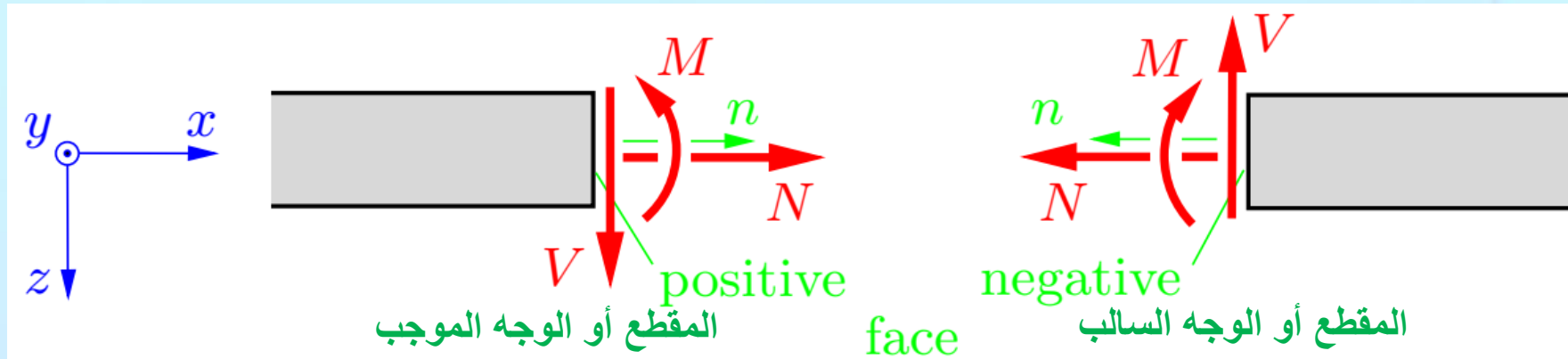


Since each part of the beam is in equilibrium, the 3 conditions of equilibrium for either part can be used to compute the three unknown stress resultants.

إن جزئي الجائز على طرفي القطع في حالة توازن، لذا فمن الممكن استخدام شروط التوازن الثلاثة لكل منهما من أجل حساب القوى الداخلية (قوى المقطع) المؤثرة على كل منهما.

Before we can provide examples for the determination of the stress resultants, a sign convention must be introduced. Consider the two adjoining portions of the same beam shown in the next figure. The coordinate  $x$  coincides with the direction of the axis of the beam and points to the right; the coordinate  $z$  points downward.

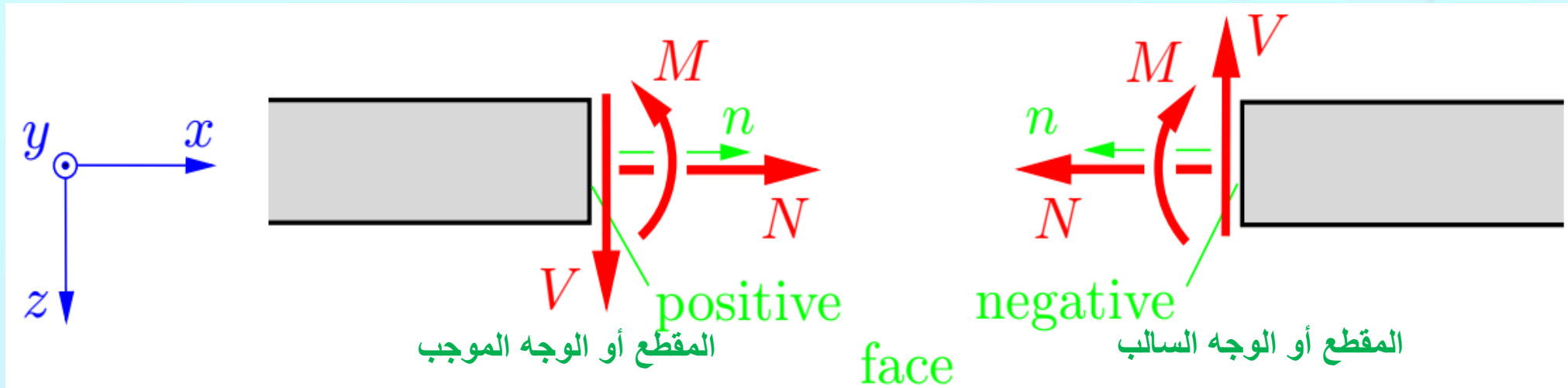
### جملة الاحداثيات والوجهين (المقطعين) الموجب والسالب



By cutting the beam, a left-hand face and a right-hand face are obtained (figure above). They are characterized by a normal vector  $n$  that points outward from the interior of the beam. If the vector  $n$  points in the positive (negative) direction of the  $x$ -axis, the corresponding face is called positive (negative).

The following sign convention is adopted:

اصطلاح إشارة القوى الداخلية:



Positive stress resultants at a **positive (negative) face** point in the **positive (negative) directions of the coordinates.**

Here, the bending moment  $M$  has to be interpreted as a vector pointing in the direction of the  $y$ -axis (*positive direction according to the right-hand rule*)

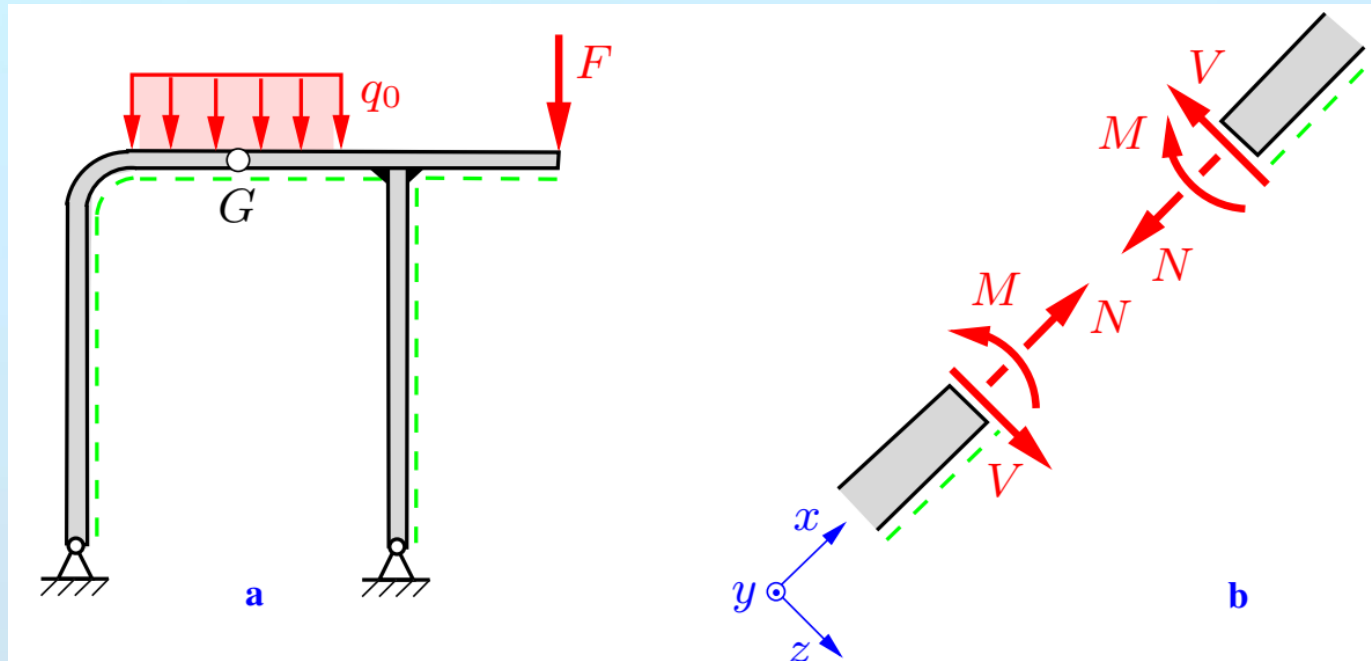
The above figure shows the stress resultants with their positive directions. In the following examples, we shall strictly adhere to this sign convention.

*It should be noted, however, that different sign conventions exist.*

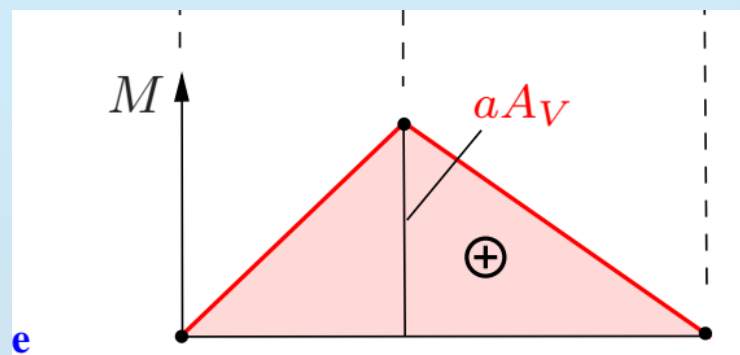
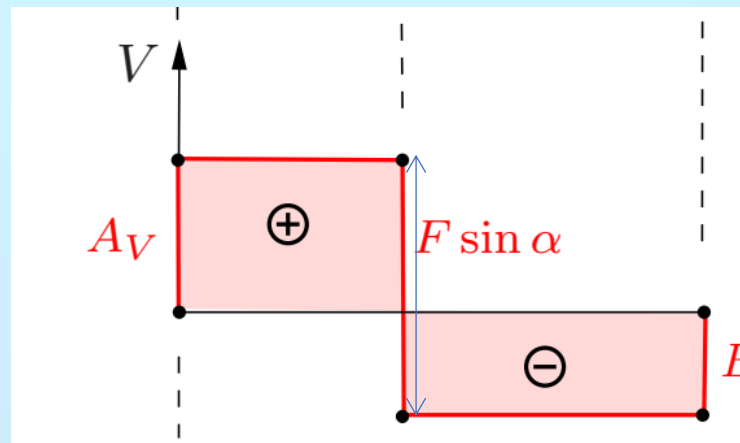
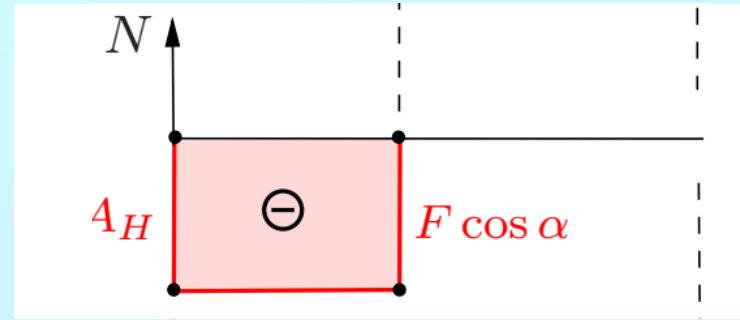
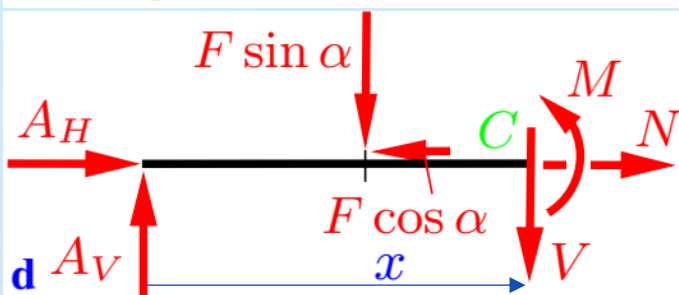
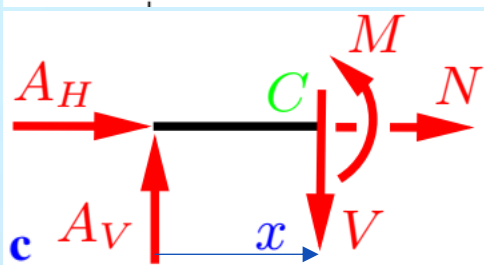
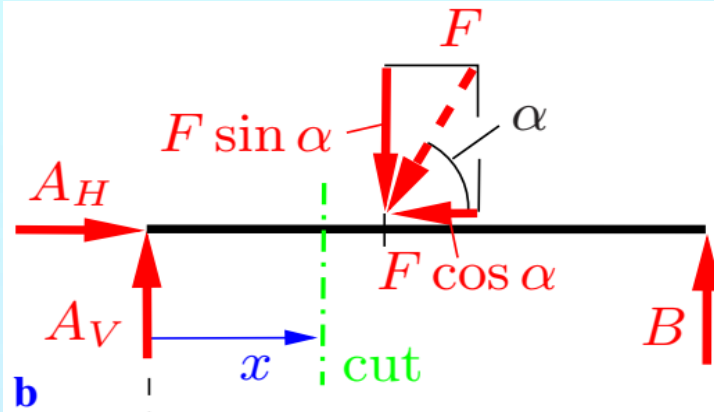
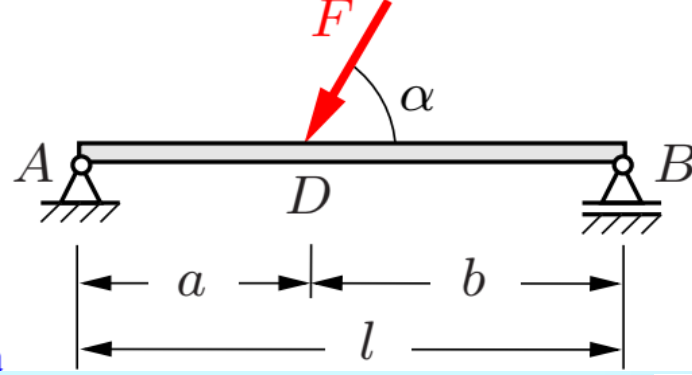


In the case of a horizontal beam, very often only the  $x$  coordinate is given. Then it is understood that the  $z$ -axis points downward.

The sign convention for frames and arches may be introduced by drawing a dashed line at one side of each part of the system. The side with the dashed line can then be interpreted as the “underneath side” of the respective part and the coordinate system can be chosen as the one for a beam:  $x$ -axis in the direction of the dashed line,  $z$ -axis toward the dashed line (“downward”).



**Example 1.** We will now determine the stress resultants for the simply supported beam shown in Fig.a.



Solution:

1. Reactions, Fig.b:

$$A_H = F \cos \alpha,$$

$$A_V = (b/l)F \sin \alpha, \quad B = (a/l)F \sin \alpha$$

2. Cut, Fig.c,  $0 < x < a$ :

$$N = -A_H = -F \cos \alpha$$

$$V = A_V = (b/l)F \sin \alpha,$$

$$M = xA_V = [(b/l)F \sin \alpha]x.$$

$$x = 0 \Rightarrow M = 0, \quad x = a \Rightarrow M = aA_V$$

3. Cut, Fig.d,  $a < x < l$ :

$$N = A_H - F \cos \alpha = 0$$

$$V = A_V - F \sin \alpha = -B,$$

$$M = xA_V - (x - a)F \sin \alpha$$

$$x = a \Rightarrow M = aA_V$$

$$x = l \Rightarrow M = lA_V - bF \sin \alpha = 0$$

**Example 2.** Draw the diagrams of the stress resultants for the beam shown in figure.

**Solution:**

1. Reactions:

$$\rightarrow: A_x = 0$$

$$\curvearrowright_B: A_z = (b/l)F, \curvearrowleft_A: B_z = (a/l)F, \uparrow: A_z + B_z = F \text{ (yes)}$$

1. Cut: A...D,  $0 < x < a$ :

$$\rightarrow: N = 0;$$

$$\uparrow: (b/l)F - V = 0 \Rightarrow V = (b/l)F;$$

$$\curvearrowright_x: M - x(b/l)F = 0 \Rightarrow M = x(b/l)F$$

$$x = 0: M = 0; x = a: M = (ab/l)F$$

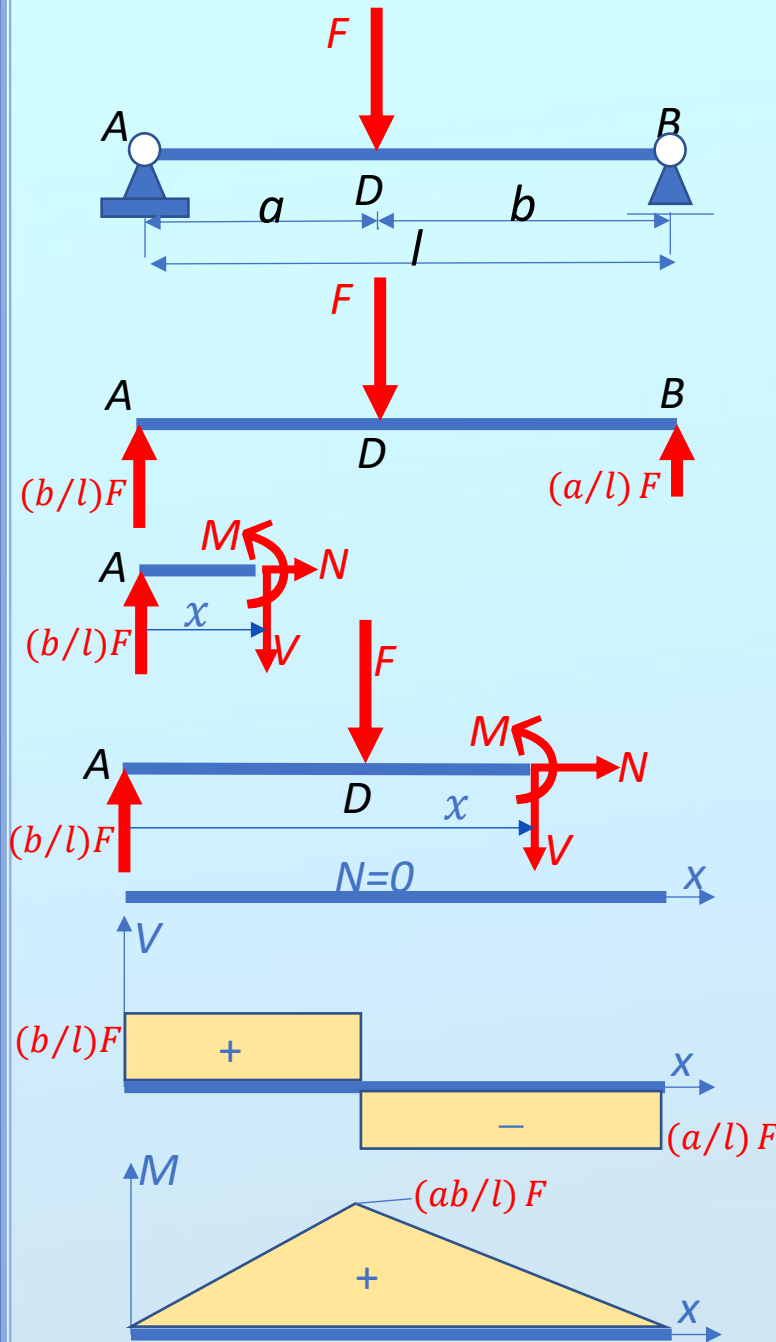
2. Cut: D...B,  $a < x < l$ :

$$\rightarrow: N = 0;$$

$$\uparrow: (b/l)F - F - V = 0 \Rightarrow V = (b/l)F - F = [(b - l)/l]F = -(a/l)F;$$

$$\curvearrowright_x: M + (x - a)F - x(b/l)F = 0 \Rightarrow M = (l - x)(a/l)F;$$

$$x = a: M = (ab/l)F; x = l: M = 0$$



**Example 3.** Draw the diagrams of the stress resultants for the beam shown in figure.

**Solution:**

0. Reactions:

$$\hat{\curvearrowright}_B: -lA_z + M_0 = 0 \Rightarrow A_z = M_0/l;$$

$$\hat{\curvearrowleft}_A: +lB_z + M_0 = 0 \Rightarrow B_z = -M_0/l (\downarrow) \Rightarrow B_x = -M_0/l (\rightarrow)$$

$$\rightarrow: -A_x + B_x = 0 \Rightarrow A_x = M_0/l (\leftarrow)$$

1. Cut: A...D,  $0 < x < a$  :

$$\rightarrow: N = M_0/l;$$

$$\uparrow: M_0/l - V = 0 \Rightarrow V = M_0/l$$

$$\hat{\curvearrowleft}_x: M - x(M_0/l) = 0 \Rightarrow M = (x/l)M_0,$$

$$x = 0: M = 0; x = a: M = (a/l)M_0.$$

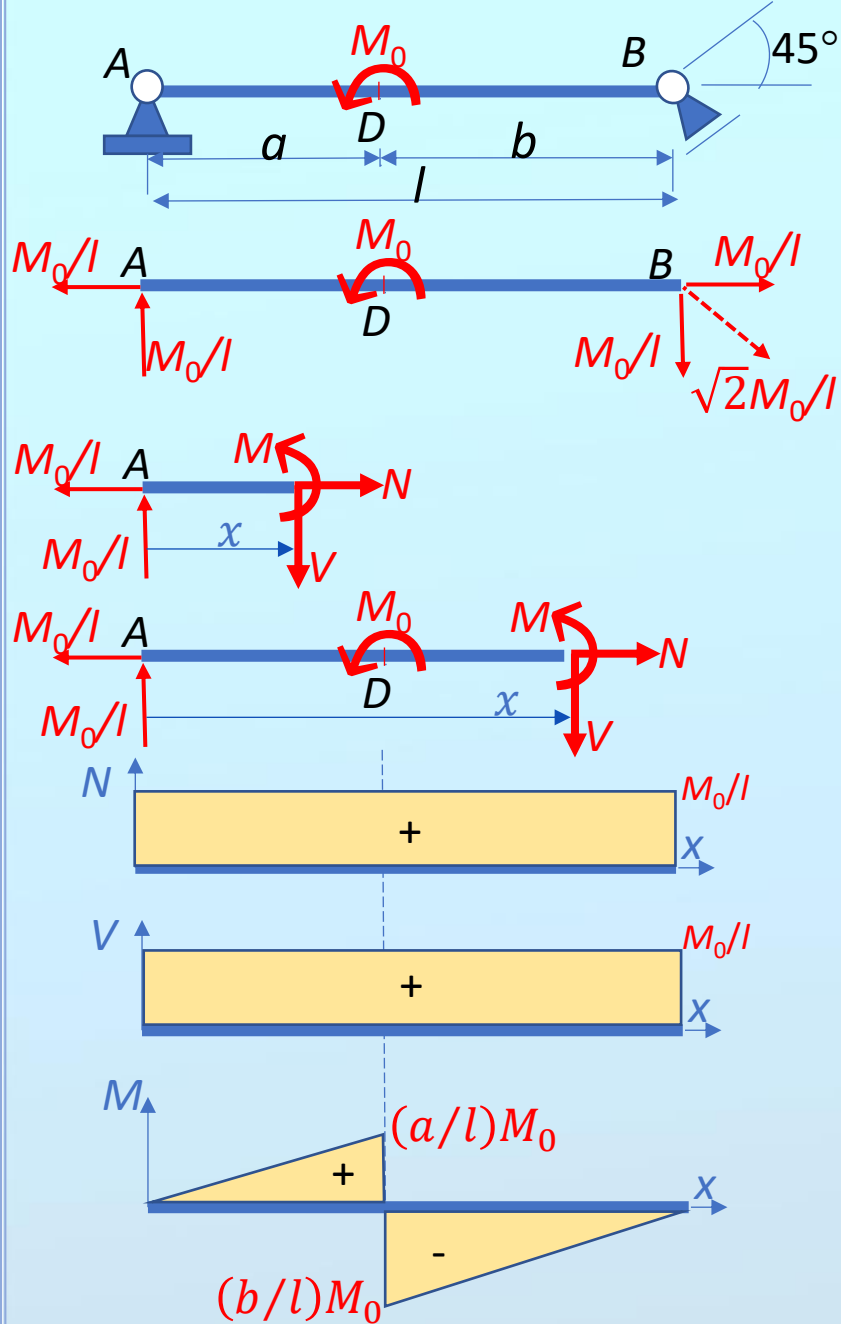
2. Cut: D...B,  $a < x < l$  :

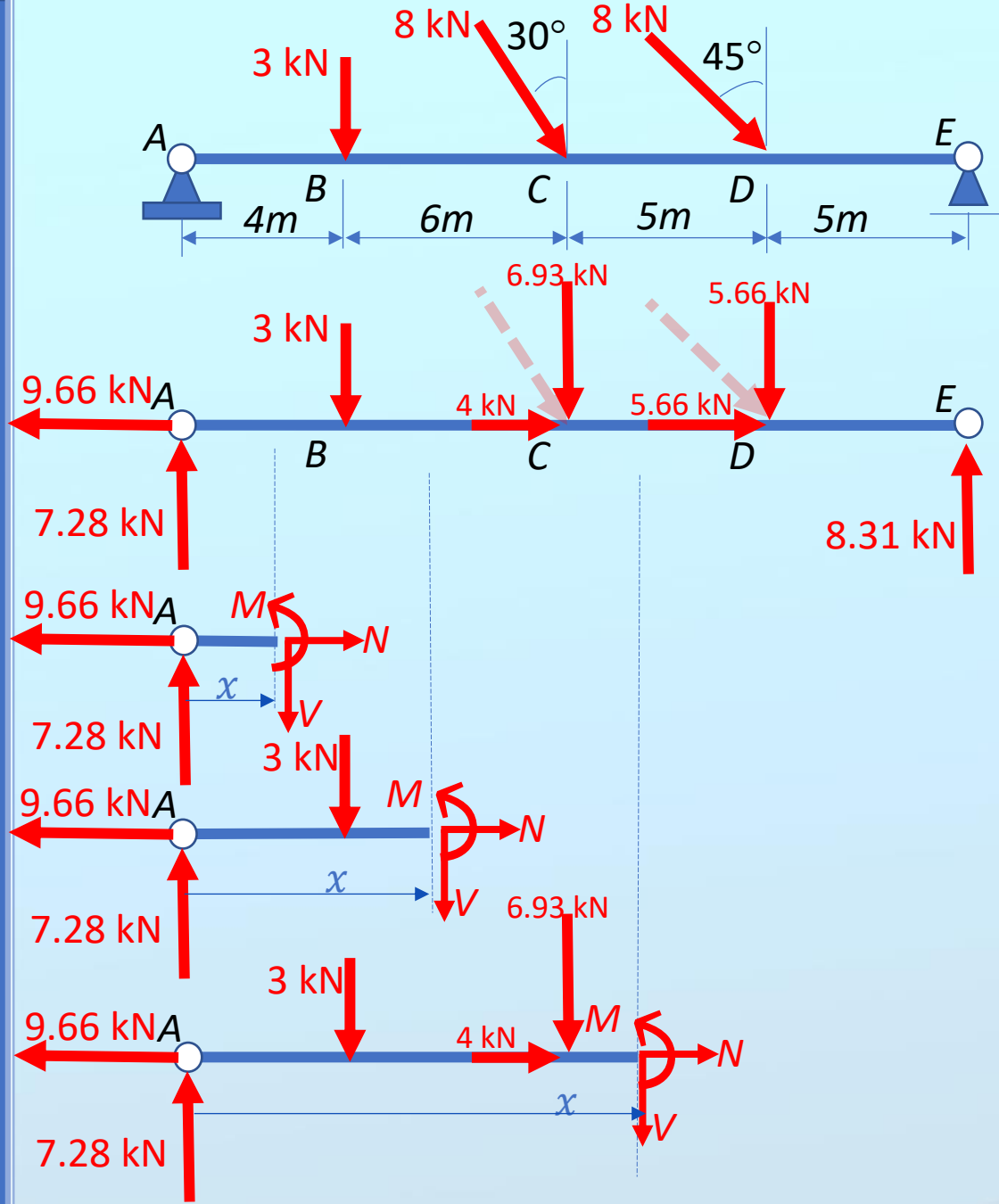
$$\rightarrow: N = M_0/l;$$

$$\uparrow: M_0/l - V = 0 \Rightarrow V = M_0/l$$

$$\hat{\curvearrowleft}_x: M + M_0 - x(M_0/l) = 0 \Rightarrow M = -M_0 + (x/l)M_0 = \left(\frac{x-l}{l}\right)M_0;$$

$$x = a: M = -(b/l)M_0; x = l: M = 0.$$





**Problem 1.** Draw the diagrams of the stress resultants for the beam shown in figure.

**Solution:**

0. Reactions:

$$\rightarrow: -A_x + 4 + 5.66 = 0 \Rightarrow A_x = 9.66 \text{ kN} (\leftarrow)$$

$$\hat{\curvearrowright}_E: +5(5.66) + 10(6.93) + 16(3) - 20A_z = 0 \Rightarrow A_z = 7.28 \text{ kN}$$

$$\hat{\curvearrowright}_A: -4(3) - 10(6.93) - 15(5.66) + 20E_z = 0 \Rightarrow E_z = 8.31 \text{ kN}$$

1. Cut: A...B,  $0 < x < 4\text{m}$  :

$$\rightarrow: N = 9.66 \text{ kN}; \quad \uparrow: V = 7.28 \text{ kN}$$

$$\hat{\curvearrowright}_X: M - x(7.28) = 0 \Rightarrow M = 7.28x,$$

$$x = 0: M = 0; \quad x = 4: M = 29.1 \text{ kNm}.$$

2. Cut: B...C,  $4 < x < 10\text{m}$  :

$$\rightarrow: N = 9.66 \text{ kN}; \quad \uparrow: V = 7.28 - 3 = 4.28 \text{ kN}$$

$$\hat{\curvearrowright}_X: M - x(7.28) + (x - 4)(3) = 0 \Rightarrow M = 4.28x + 12,$$

$$x = 4: M = 29.1 \text{ kNm}; \quad x = 10: M = 54.8 \text{ kNm}.$$

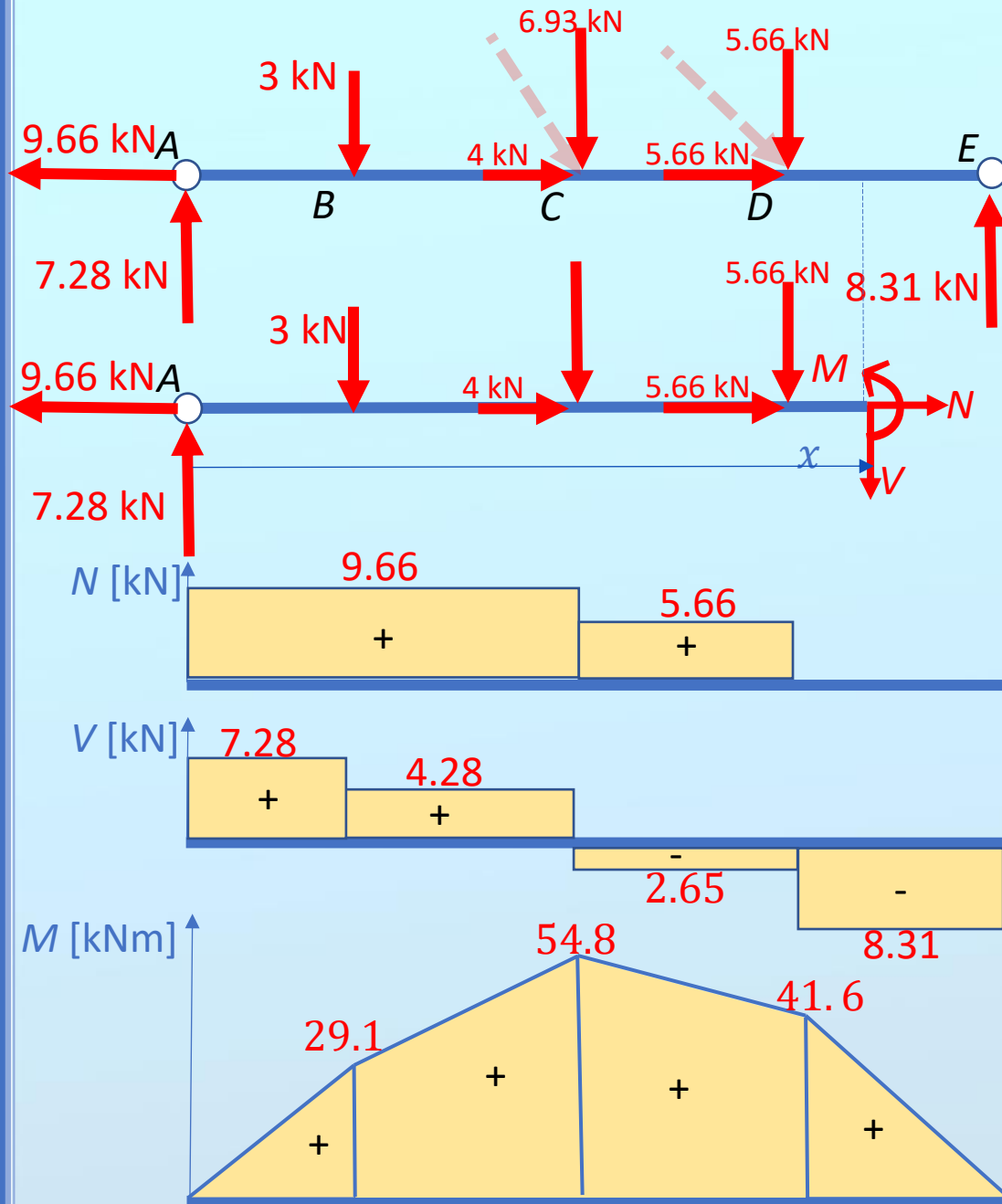
3. Cut: C...D,  $10 < x < 15\text{m}$  :

$$\rightarrow: N = 5.66 \text{ kN}; \quad \uparrow: V = 7.28 - 3 - 6.93 = -2.65 \text{ kN}$$

$$\hat{\curvearrowright}_X: M - x(7.28) + (x - 4)(3) + (x - 10)(6.93) = 0$$

$$\Rightarrow M = -2.65x + 81.3,$$

$$x = 10: M = 54.8 \text{ kNm}; \quad x = 15: M = 41.6 \text{ kNm}.$$



4. Cut: D...E,  $15 < x < 20\text{m}$  :

$$\rightarrow: N = 0; \uparrow: V = 7.28 - 3 - 6.93 - 5.66 = -8.31\text{kN}$$

$$\curvearrow: M - x(7.28) + (x - 4)(3) + (x - 10)(6.93) + (x - 15)(5.66) = 0 \Rightarrow M = -8.31x + 166.2,$$

$$x = 15: M = 41.6\text{kNm}; x = 20: M = 0.$$

# Stress Resultants in Straight Beams

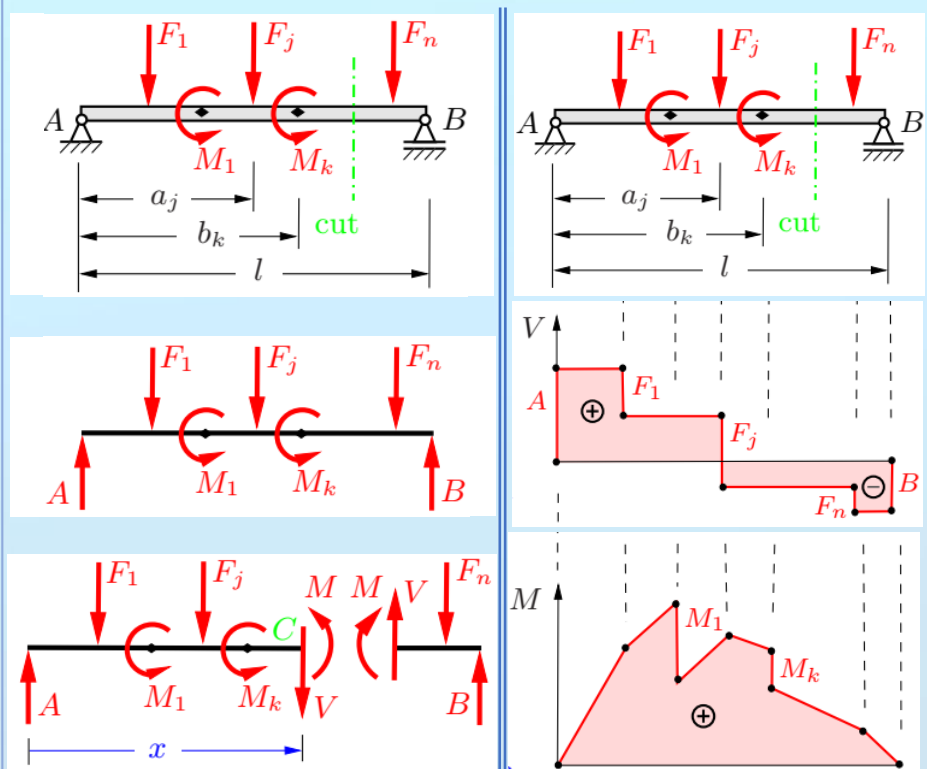
Beams are usually subjected to forces perpendicular to their axes. If there is no loading (external or reactions) in the direction of the beam axis, the normal force vanishes  $N=0$ .

تندعم القوة الناعظمية في الجيزان المستقيمة المحملة عمودياً على محاورها.

## Beams under Concentrated Loads

To determine  $V$  &  $M$  choose a coordinate system and cut at an arbitrary  $x$ . Represent  $V$  &  $M$  with their positive directions in the F. B. Ds.; use Eq. Eqs. for either portion of the beam. Results are a shear-force and a bending-moment diagram.

جملة احداثيات، قطع، معادلات توازن لأي من الجزئين



### 1. Reactions

$$\curvearrowleft_A: lB - \sum a_i F_i + \sum M_i = 0 \rightarrow B = \frac{1}{l} [\sum a_i F_i - \sum M_i]$$

$$\curvearrowleft_B: -lA + \sum (l - a_i) F_i + \sum M_i = 0 \rightarrow A = \frac{1}{l} [\sum (l - a_i) F_i + \sum M_i]$$

### 2. Cut at $X$

$$\uparrow: -V + A - \sum F_i = 0 \rightarrow V = A - \sum F_i$$

$$\curvearrowleft_X: M - xA + \sum (x - a_i) F_i + \sum M_i = 0$$

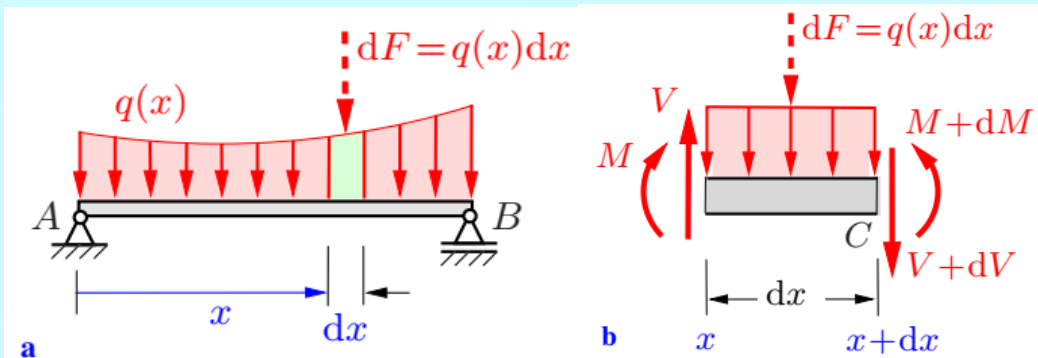
$$\rightarrow M = xA - \sum (x - a_i) F_i - \sum M_i$$

$$\frac{dM}{dx} = A - \sum F_i = V$$

مشتق تابع عزم الانعطاف يساوي تابع قوة القص

عند القوى أو العزوم المركزة توجد قفزة مساوية عكسا في المخطط المقابل

# Relationship between distributed Loading and Stress Resultants (General case)



**Any part of the beam is in Equilibrium**

Eq. Eqs. of  $[dx]$ :

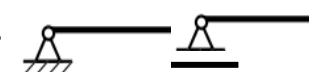


$$\uparrow: V - q(x)dx - (V + dV) = 0 \Rightarrow \frac{dV}{dx} = -q(x)$$

$$\curvearrowright: (M + dM) + \left(\frac{dx}{2}\right) q(x)dx - dxV - M = 0$$

with  $dx \rightarrow 0, \Rightarrow \frac{dM}{dx} = V(x)$  &  $\frac{d^2M}{dx^2} = -q(x)$

$q$	$V$	$M$
0	constant	linear
constant	linear	quadratic parabola
linear	quadratic parabola	cubic parabola

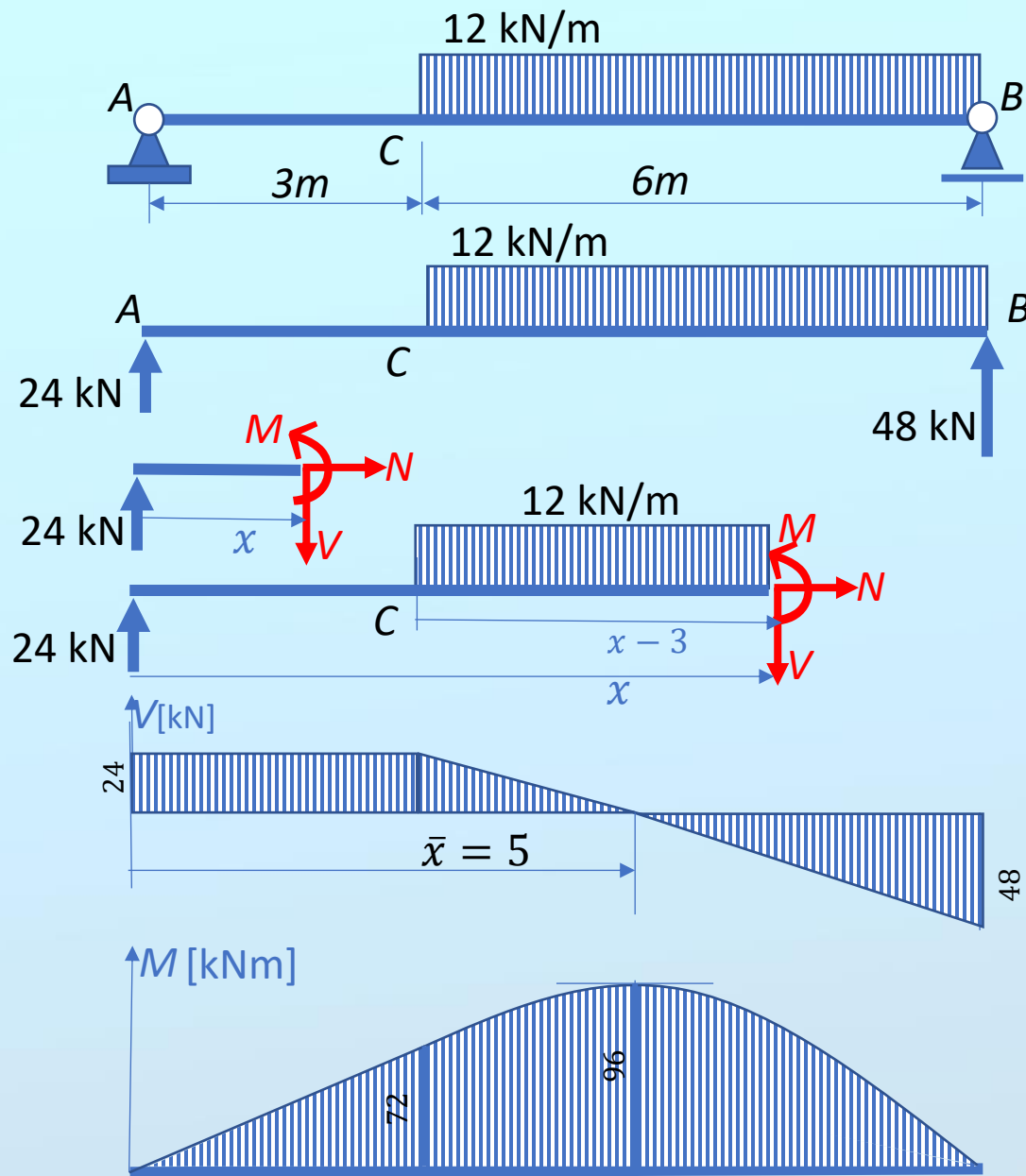
القص موجب فالعزم متزايد، القص سالب فالعزم متناقص،  
القص معدوم فالعزم عند نهاية حدية (كبرى أو صغرى).

المساند	support	$V$	$M$
مفصل	pin / roller 	$\neq 0$	0
وثاقة	fixed end 	$\neq 0$	$\neq 0$
طرف حر	free end 	0	0

إذا كانت الحمولة كثيرة حدود (معدومة، ثابتة، خطية....)  
فالقص كثيرة حدود (ثابتة، خطية، درجة ثانية: قطع مكافئ...)  
والعزم كثيرة حدود (خطية، درجة ثانية: مكافئ، درجة ثالثة...)



## Problem 2. Draw the shear force and bending moment diagrams for the shown beam.



### 0. Reactions:

$$\rightarrow: A_x = 0$$

$$\curvearrowright_B: +3(6)(12) - 9A_z = 0 \Rightarrow A_z = 24 \text{ kN}$$

$$\curvearrowright_A: -6(6)(12) + 9B_z = 0 \Rightarrow B_z = 48 \text{ kN}$$

### 1. Cut: A...C, $0 < x < 3\text{m}$ :

$$\rightarrow: N = 0; \quad \uparrow: V = 24 \text{ kN}$$

$$\curvearrowright_x: M - x(24) = 0 \Rightarrow M = 24x,$$

$$x = 0: M = 0; \quad x = 3: M = 72 \text{ kNm.}$$

### 2. Cut: C...B, $3 < x < 9\text{m}$ :

$$\rightarrow: N = 0;$$

$$\uparrow: V = 24 - 12(x - 3) = -12x + 60$$

$$x = 3: V = 24 \text{ kN}, \quad x = 9: V = -48 \text{ kN};$$

$$V = 0: \bar{x} = 60/12 = 5 \text{ m}$$

$$\curvearrowright_x: M - x(24) + \frac{1}{2}(x - 3)(12)(x - 3) = 0$$

$$\Rightarrow M = -6x^2 + 60x - 54,$$

$$x = 3: M = 72 \text{ kNm}; \quad x = 9: M = 0.$$

$$M_{max} = M|_{V=0} = M|_{\bar{x}=5} = 96 \text{ kNm.}$$

### Problem 3. For the shown beam find

- Reactions in supports A and B.
- Shear force and bending moment diagrams.
- Determine the value and the location of the maximum bending moment.

#### Solution:

##### (a) Reactions:

$$\sum M_A = 0: -22.5(4)(4) + 6B_y = 0 \Rightarrow B_y = 60 \text{ kN}(\uparrow)$$

$$\sum F_y = 0: A_y + 60 - 22.5(4) = 0 \Rightarrow A_y = 30 \text{ kN}(\uparrow)$$

##### (b) Shear force and bending moment diagrams

$$\text{For: } 0 < x < 2: V(x) = 30,$$

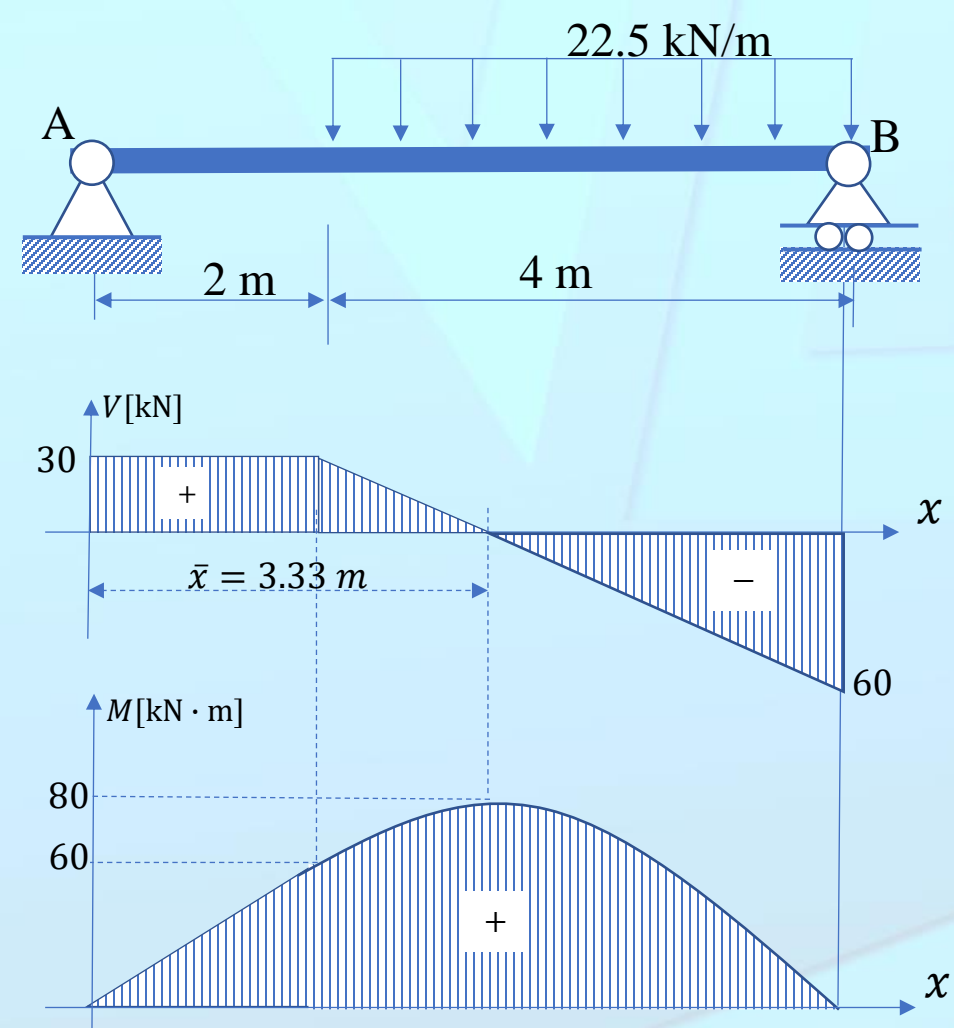
$$M = 30x, M(0) = 0, M(2) = 60 \text{ kNm}$$

$$\text{For: } 2 < x < 6: V(x) = 30 - 22.5(x - 2) = -22.5x + 75, V(2) = 30 \text{ kN}, V(6) = -60 \text{ kN}$$

$$M(x) = 30x - 22.5 \frac{(x-2)^2}{2} = -11.25x^2 + 75x - 45, M(2) = 60 \text{ kNm}, M(6) = 0$$

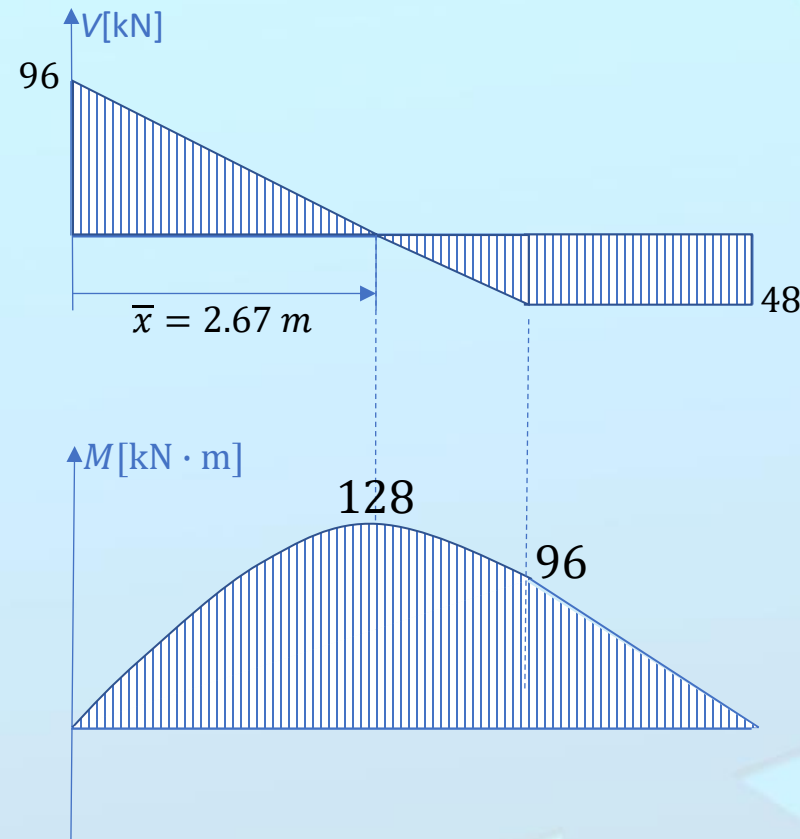
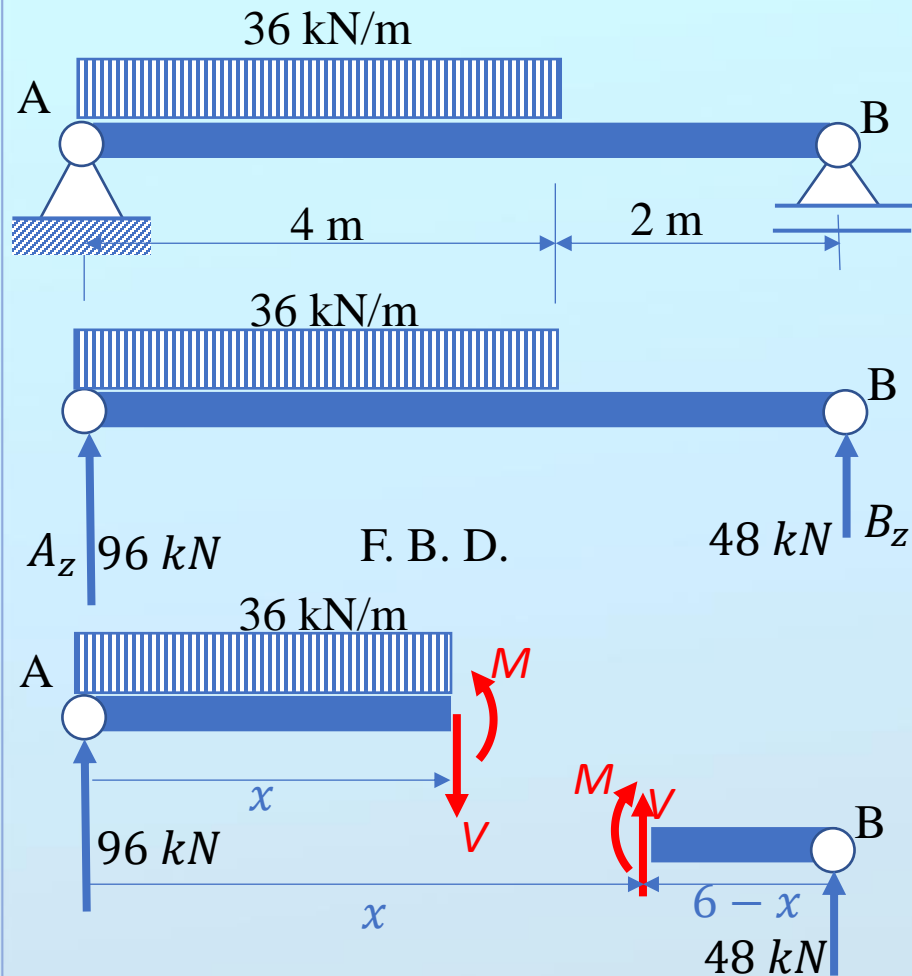
##### (c) Maximum bending moment:

$$V(x) = 0 \text{ if } x = 75 \div 22.5 = 10/3 = \bar{x}, \quad M_{max} = M(10/3) = 80 \text{ kNm}$$



### Problem 4. For the shown beam find

- Reactions in supports A and B.
- Shear force and bending moment diagrams.
- Determine the value and the location of the maximum bending moment.



Cut 1:  $0 < x < 4$ :

$$V(x) = 96 - 36x.$$

$$x = 0: V = 96 \text{ kN};$$

$$x = 4: V = -48 \text{ kN};$$

$$V = 0: \bar{x} = 2.67 \text{ m}$$

$$M(x) = 96x - 18x^2.$$

$$x = 0: M = 0;$$

$$x = 4: M = 96 \text{ kN} \cdot \text{m};$$

$$\bar{x} = 2.67 \text{ m}: M_{\text{max}} = 128 \text{ kN} \cdot \text{m}$$

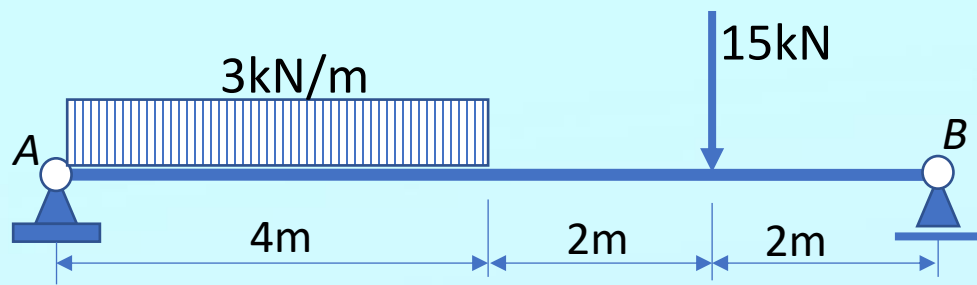
Cut 2:  $4 < x < 6$ :

$$V = -48 \text{ kN}.$$

$$M(x) = 48(6 - x).$$

$$x = 4: M = 96 \text{ kN} \cdot \text{m}.$$

$$x = 6: M = 0.$$



**Problem 5.** Determine the shear-force and bending-moment diagrams for the simple beam shown in Fig. using the section method.

**Problem 6.** Determine the shear-force and bending-moment diagrams for the simple beam shown in Fig. using the section method. Then determine the location and the value of the maximum bending moment.

