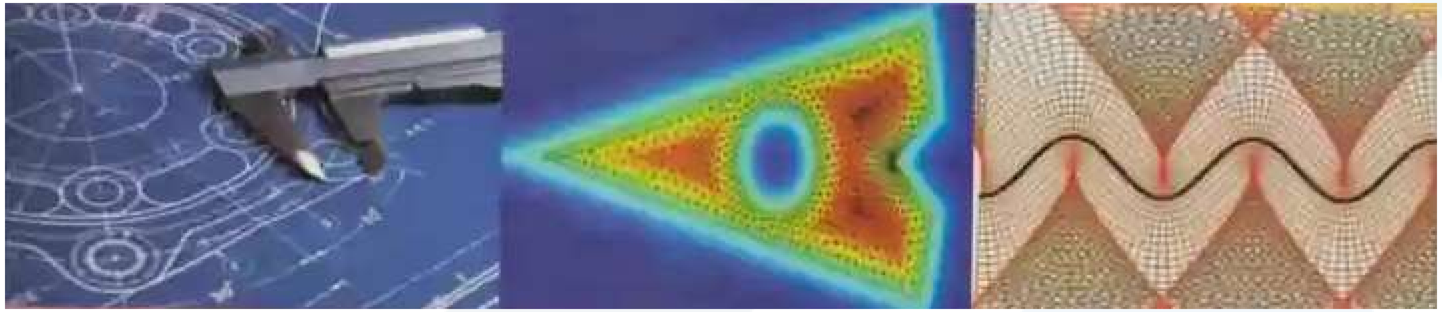
CEDC301: Engineering Mathematics

Lecture Notes 4 & 5: Series and Residues



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Chapter 3

Series and Residues

1. Sequences and Series

Sequences

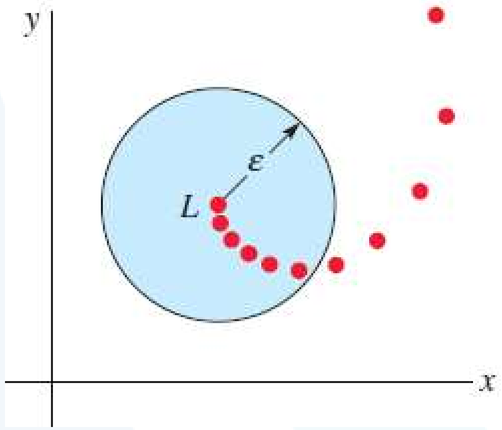
- A sequence $\{z_n\}$ is a function whose domain is the set of positive integers; in other words, to each integer $n = 1, 2, 3, \ldots$, we assign a complex number z_n. For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{ccccc} 1+i, & 0, & 1-i, & 2, & 1+i, \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ n=1, & n=2, & n=3, & n=4, & n=5, \ldots \end{array}$$

- If $\lim_{n\to\infty} z_n = L$ we say the sequence $\{z_n\}$ is convergent.

  $\{z_n\}$ converges to the number L if, for each positive number ε, an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.

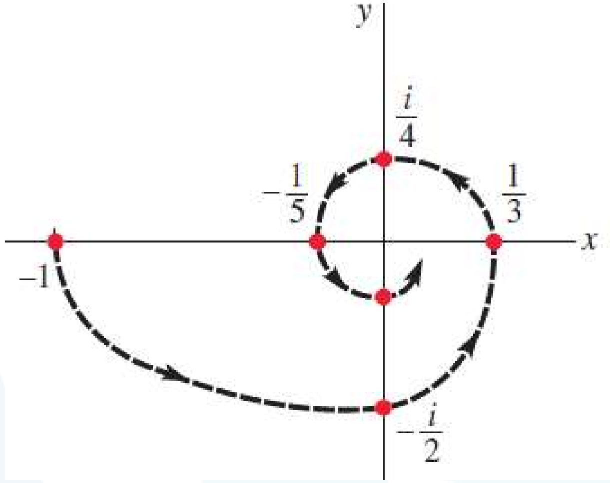
  The sequence $\{1 + i^n\}$ is divergent.



- Example 1: A Convergent Sequence

The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ converges, since $\lim\limits_{n\to\infty} \frac{i^{n+1}}{n} = 0$

$$-1,\ -\frac{i}{2},\ \frac{1}{3},\ \frac{i}{4},\ -\frac{1}{5},\ \ldots$$



- **Theorem 1 (Criterion for Convergence):** A sequence $\{z_n\}$ converges to a complex number L if and only if $Re(z_n)$ converges to $Re(L)$ and $Im(z_n)$ converges to $Im(L)$.

- **Example 2:** The sequence $\left\{\frac{ni}{n+2i}\right\}$ converges to i. since

$Re(z_n) = 2n/(n^2 + 4) \to 0$ and $Im(z_n) = n^2/(n^2 + 4) \to 1$ as $n \to \infty$

Series

- An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is convergent if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges. If $S_n \to L$ as $n \to \infty$, we say that the sum of the series is L.

Geometric Series

$$\sum_{k=0}^{\infty} az^k = a + az + az^2 + \cdots + az^{n-1} + \cdots$$

$$S_n = a + az + az^2 + \cdots + az^{n-1} = \frac{a(1 - z^n)}{1 - z} \underset{n\to\infty}{\to} \frac{a}{1 - z} \text{ when } |z| < 1$$

$\sum_{k=1}^{\infty} az^k$ converges when $|z| < 1$, and diverges when $|z| > 1$.

$1/(1-z) = 1 + z + z^2 + z^3 + \cdots,$ $1/(1+z) = 1 - z + z^2 - z^3 + \cdots$ valid for $|z| < 1$

$(1-z^n)/(1-z) = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$

- Example 3: Convergent Geometric Series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series with $a = (1+2i)/5$ and $z = (1+2i)/5$.

$|z| = \sqrt{5}/5 < 1 \Rightarrow$ the series converges $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}} = \frac{i}{2}$

- **Theorem 2 (Necessary Condition for Convergence):** If $\sum_{k=1}^{\infty} z_k$ converges, then:

$$\lim_{n\to\infty} z_n = 0$$

- **Theorem 3 (The nth Term Test for Divergence):** If $\lim_{n\to\infty} z_n \neq 0$, then the series: $\sum_{k=1}^{\infty} z_k$ diverges.

  For example, the series $\sum_{k=1}^{\infty} \frac{k+5i}{k}$ diverges since $z_n = (n+5i)/n \to 1$ as $n \to \infty$

- **Definition:** An infinite series $\sum_{k=1}^{\infty} z_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges.

  For example, the series $\sum_{k=1}^{\infty} (i^k)/k^2$ is absolutely convergent $\left|(i^k)/k^2\right| = 1/k^2$

- Note: Absolute convergence implies convergence.

$\sum_{k=1}^{\infty} (i^k)/k^2$ is convergent

- Theorem 4 (Ratio Test): Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$$

(i) If $L < 1$, then the series converges absolutely.
(ii) If $L > 1$ or $L = \infty$, then the series diverges.
(iii) If $L = 1$, the test is inconclusive.

- Theorem 5 (Root Test): Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that:

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$$

(i) If $L < 1$, then the series converges absolutely.
(ii) If $L > 1$ or $L = \infty$, then the series diverges.
(iii) If $L = 1$, the test is inconclusive.

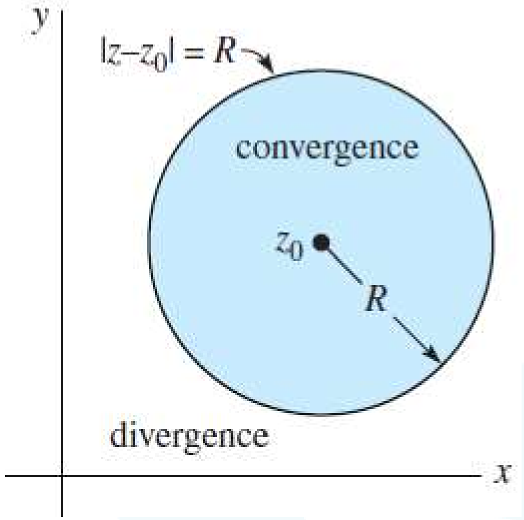
Power Series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

where the coefficients a_k are complex constants, is called a power series in $z - z_0$, centered at z_0,

Circle of Convergence

- Every complex power series has radius of convergence R, where R is a real number.
- When $0 < R < \infty$, a complex power series has a circle of convergence defined by $|z - z_0| = R$.
- The power series converges absolutely for all z satisfying $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The radius R of convergence can be:



(i) zero (the power series converges at only $z = z_0$),

(ii) a finite number (the power series converges at all interior points of the circle $|z - z_0| = R$), or

(iii) ∞ (the power series converges for all z).

- Example 4: Circle of Convergence

Consider the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}|z| = |z|$$

Thus the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$.

On the circle of convergence, the series does not converge absolutely.

It can be shown that the series converges at all points on the circle $|z| = 1$ except at $z = 1$.

- Note: the radius of convergence is $R = 1/L$. $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ or $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$

- Example 5: Radius of Convergence

  Consider the power series $\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{6n+1}{2n+5} = 3$$

The radius of convergence of the series is $R = 1/3$. The circle of convergence is $|z-2i| = 1/3$, the series converges absolutely for $|z-2i| < 1/3$.

2. Taylor Series

- A power series defines or represents a function f; for a specified z within the circle of convergence, the number L to which the power series converges is defined to be the value of f at z; that is, $f(z) = L$.

- Theorem 6 (Continuity): A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R$, $R \neq 0$.
- Theorem 7 (Term-by-Term Integration): A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$, for every contour C lying entirely within the circle of convergence.
- Theorem 8 (Term-by-Term Differentiation): A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$.

Taylor Series

- A power series represents an analytic function within its circle of convergence.

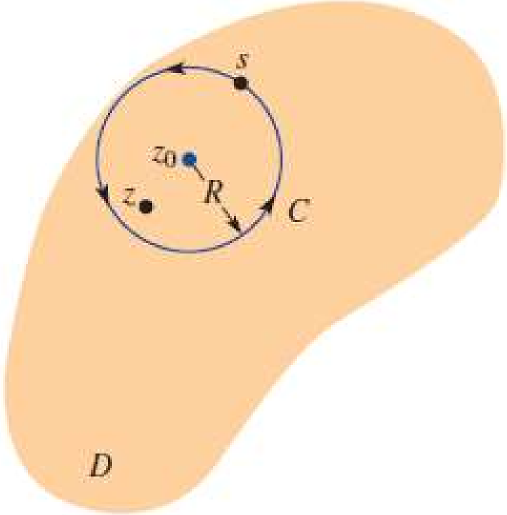
$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$ Taylor series for f centered at z_0.

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k$$ Maclaurin series for f.

- Theorem 9 (Taylor's Theorem): Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.



Some Important Maclaurin Series

$$e^{z} = 1 + \frac{z}{1!} + \frac{z^{2}}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{k}}{k!}, \qquad R = \infty$$

$$\sin z = z - \frac{z^{3}}{3!} + \frac{z^{5}}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k+1}}{(2k+1)!}, \qquad R = \infty$$

$$\cos z = 1 - \frac{z^{2}}{2!} + \frac{z^{4}}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k}}{(2k)!}, \qquad R = \infty$$

- Note: the radius of convergence R is the distance from the center z_0 of the series to the nearest isolated singularity of f. An isolated singularity is a point at which f fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

- **Example 6:** Radius of Convergence

Suppose the function $f(z) = \dfrac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4 - 2i$. What is its radius of convergence R?

The function is analytic at every point except at $z = -1 + i$, which is an isolated singularity of f. The distance from $z = -1 + i$ to $z_0 = 4 - 2i$ is:

$$|z - z_0| = \sqrt{(-1-4)^2 + (1-(-2))^2} = \sqrt{34} = R$$

- **Example 7:** Maclaurin Series

Find the Maclaurin expansion of $f(z) = \dfrac{1}{(1-z)^2}$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1$$

Differentiating both sides

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}, \quad |z| < 1$$

- Example 8: Taylor Series

Expand $f(z) = \dfrac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

First Method:

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow f^{(n)}(2i) = \frac{n!}{(1-2i)^{n+1}}$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{1}{(1-2i)^{n+1}}(z-2i)^k$$

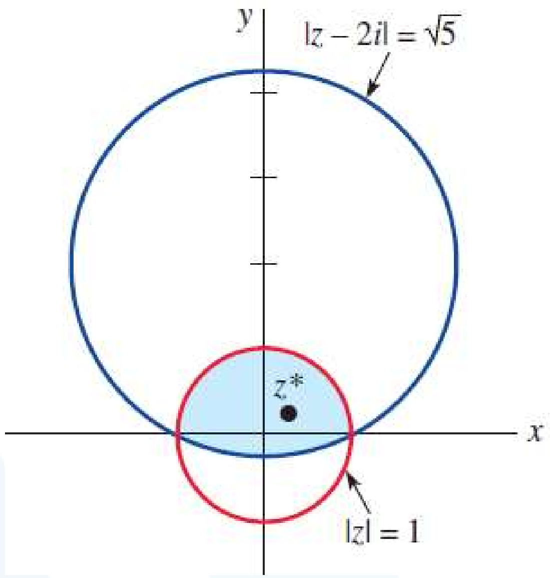
circle of convergence $|z-2i| = \sqrt{5}$ (using ratio test)

Second Method:

$$\frac{1}{1-z}=\frac{1}{1-z+2i-2i}=\frac{1}{1-2i-(z-2i)}=\frac{1}{1-2i}\frac{1}{1-\dfrac{z-2i}{1-2i}}$$

$$\frac{1}{1-z}=\frac{1}{1-2i}\left[1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\cdots\right]$$

$$\frac{1}{1-z}=\frac{1}{1-2i}+\frac{1}{(1-2i)^2}(z-2i)+\frac{1}{(1-2i)^3}(z-2i)^2+\cdots$$



- Note: we represented the same function $1/(1-z)$ by 2 different power series. The first has center 0 and radius of convergence (ROC) 1. The second has center $2i$ and ROC $\sqrt{5}$. The shaded region is where both series converge.

3. Laurent Series

- If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a singularity or a singular point of the function.
- For example, the complex numbers $z = 2i$ and $z = -2i$ are singularities of the function $f(z) = z/(z^2 + 4)$ because f is discontinuous at each of these points.

Isolated Singularities

- Suppose that $z = z_0$ is a singularity of a complex function f. The point $z = z_0$ is said to be an isolated singularity of the function f if there exists some deleted neighborhood of z_0, $0 < |z - z_0| < R$ throughout which f is analytic.
- For example, $z = \pm 2i$ are isolated singularities of $f(z) = z/(z^2 + 4)$ since f is analytic at every point in the neighborhood $|z - 2i| < 1$ except at $z = 2i$ and at every point in the neighborhood $|z - (-2i)| < 1$ except at $z = -2i$.

- On the other hand, the branch point $z = 0$ is not an isolated singularity of $\text{Log } z$ since every neighborhood of $z = 0$ must contain points on the negative x-axis.
- We say that a singular point $z = z_0$ of a function f is nonisolated if every neighborhood of z_0 contains at least one singularity of f other than z_0.
- For example, the branch point $z = 0$ is a nonisolated singularity of $\text{Log } z$ since every neighborhood of $z = 0$ contains points on the negative real axis.

A New Kind of Series

- If $z = z_0$ is a singularity of a function f, then certainly f cannot be expanded in a power series with z_0 as its center. However, about an isolated singularity $z = z_0$ it is possible to represent f by a new kind of series:

$$f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k=\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}+\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

Such series representation is called a Laurent series or a Laurent expansion of f.

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}=\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

is called the principal part and will converge for $|1/(z-z_0)|<r^*$ or equivalently for $|z-z_0|>1/r^*=r$.

$\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is called the analytic part and will converge for $|z-z_0|<R$.

$$f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k \quad \text{will converge for } r<|z-z_0|<R$$

- Example 9: A New Kind of Series

The function $f(z) = (\sin z)/z^4$ is not analytic at $z = 0$ and hence cannot be expanded in a Maclaurin series.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

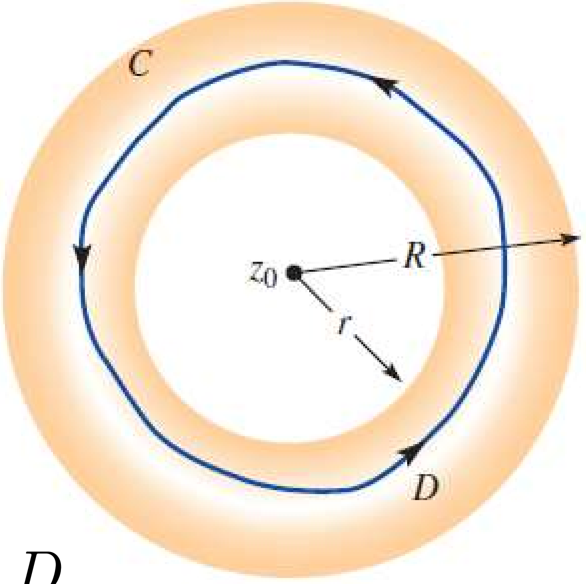
converges for all $|z| < \infty$

$$f(z) = \frac{\sin z}{z^4} = \overbrace{\frac{1}{z^3} - \frac{1}{3!z}}^{\text{principal part}} + \overbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}^{\text{analytic part}}$$

The analytic part of the series converges for $|z| < \infty$. The principal part is valid for $|z| > 0 \Rightarrow$ the series converges for all z except at $z = 0$ $(0 < |z| < \infty)$.

- **Theorem 10 (Laurent's Theorem):** Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. and let z_0 be a point in D. Then f has the series representation:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$



valid for $r < |z - z_0| < R$. The coefficients a_k are given by:

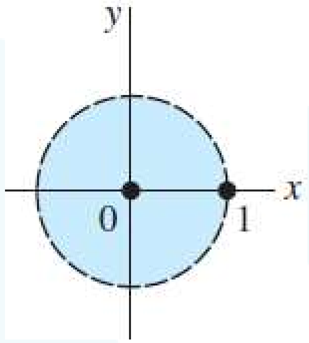
$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \ldots$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior.

- Example 10: Laurent Expansions

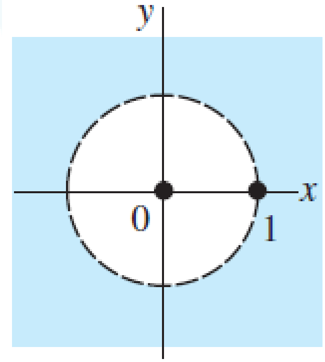
Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for (a) $0 < |z| < 1$, (b) $1 < |z|$, (c) $0 < |z-1| < 1$, and (d) $1 < |z-1|$.

(a) $$f(z) = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\underbrace{[1 + z + z^2 + z^3 + \cdots]}_{|z|<1}$$



$$= -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$ converges for $0 < |z| < 1$

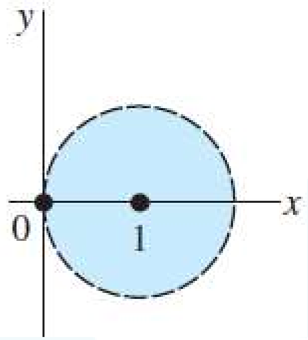
(b) $$f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\underbrace{\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]}_{\left|\frac{1}{z}\right|<1}$$



$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots \quad \text{converges for } 1 < |z|$$

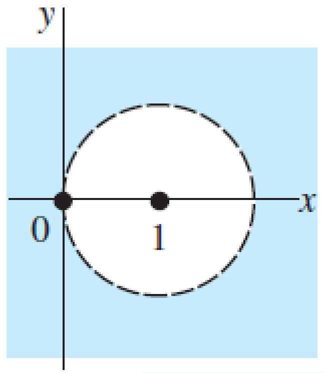
(c) $$f(z) = \frac{1}{1-1+z}\frac{1}{z-1} = \frac{1}{z-1}\frac{1}{1+(z-1)}$$

$$= \frac{1}{z-1}\underbrace{[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]}_{|z-1|<1}$$



$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots \quad \text{converges for } 0 < |z-1| < 1$$

(d) $$f(z) = \frac{1}{z-1}\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\frac{1}{1+\dfrac{1}{z-1}}$$



$$f(z)=\frac{1}{(z-1)^2}\underbrace{\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right]}_{\left|\frac{1}{z-1}\right|<1}$$

$$f(z)=\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots$$ converges for $1<|z-1|$

- **Example 11:** Laurent Expansions

Expand $f(z)=\frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for

(a) $0<|z-1|<2$,

(b) $0<|z-3|<2$.

(a) $$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\frac{1}{1-\dfrac{z-1}{2}}$$

$$f(z) = \frac{-1}{2(z-1)^2}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \cdots\right]$$

$$= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

valid for $0 < |z-1| < 2$

(b) $$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3}\frac{1}{[2+(z-3)]^2} = \frac{1}{4(z-3)}\left[1 + \frac{z-3}{2}\right]^{-2}$$

using the general binomial theorem:

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{2!}z^2 + \frac{m(m-1)(m-2)}{3!}z^3 + \cdots, \quad |z| < 1,\ m \in Q$$

$$f(z) = \frac{1}{4(z-3)}\left[1 + \frac{(-2)}{1!}\left(\frac{z-3}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2 + \cdots\right]$$

$$f(z) = \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \cdots$$

valid for $0 < |z-3| < 2$

- Example 12: Laurent Expansions

Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

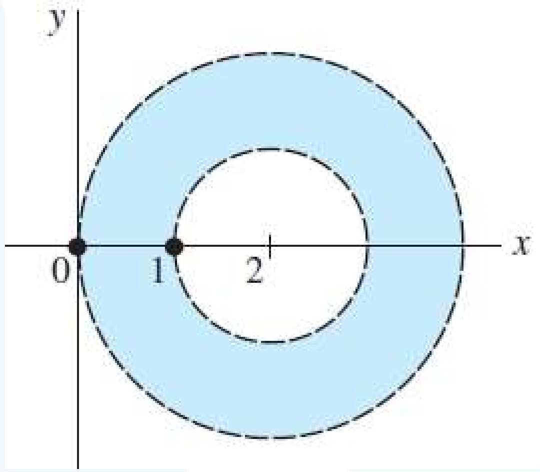
$$f(z)=\frac{8z+1}{z(1-z)}=\frac{8z+1}{z}\frac{1}{1-z}=\left(8+\frac{1}{z}\right)(1+z+z^2+z^3+\cdots)$$

$$f(z)=\frac{1}{z}+9+9z+9z^2+\cdots$$ valid for $0<|z|<1$

- Example 13: Laurent Expansions

Expand the function $f(z)=\dfrac{1}{z(z-1)}$ in a Laurent series valid for $1<|z-2|<2$.

Find two series involving integer powers of $z-2$: one converging for $1<|z-2|$ and the other converging for $|z-2|>2$.



$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$$

$$f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\frac{1}{1+\frac{z-2}{2}}$$

$$f_1(z) = -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right]$$

$$f_1(z) = -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$ converges for $|z-2| < 2$

$$f_2(z) = \frac{1}{1+z-2} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right]$$

$$f_2(z) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$$ converges for $1 < |z-2|$

$$f(z) = \cdots + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$

converges for $1 < |z-2| < 2$

- Example 14: Laurent Expansions

Expand $f(z) = e^{3/z}$ in a Laurent series valid for $|z| > 0$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots$$ valid for $|z| > 0$

4. Zeros and Poles

Classification of Isolated Singular Points

- A classification is given depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms.

$z = z_0$	**Laurent Series**
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\dfrac{a_{-n}}{(z - z_0)^n} + \dfrac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z - z_0)^2} + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

- Example 14: Removable Singularity

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$z = 0$ is a removable singularity of $f(z) = (\sin z)/z$.

- Example 15: Poles and Essential Singularity

$$\frac{\sin z}{z^2} = \underset{\text{principal part}}{\frac{1}{z}} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

$|z| > 0$, we see that $a_{-1} \neq 0$, and so $z = 0$ is a simple pole of the function $f(z) = (\sin z)/z^2$.

The Laurent expansion of $f(z) = 1/(z-1)^2(z-3)$ valid for $0 < |z-1| < 2$

$$f(z) = \overbrace{-\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)}}^{\text{principal part}} - \frac{1}{8} - \frac{z-1}{16} - \cdots$$

since $a_{-2} \neq 0$, we conclude that $z = 1$ is a pole of order 2.

The principal part of Laurent series of the function $f(z) = e^{3/z}$ contains an infinite number of terms. Thus $z = 0$ is an essential singularity.

Zeros

- z_0 is a zero of a function f if $f(z_0) = 0$. An analytic function f has a zero of order n at $z = z_0$ if

$$f(z_0) = 0, \quad f'(z_0) = 0, \quad f''(z_0) = 0, \ \cdots, \quad f^{(n-1)}(z_0) = 0, \text{ but } f^{(n)}(z_0) \neq 0$$

- If an analytic function f has a zero of order n at $z = z_0$, it follows that the Taylor series expansion of f centered at z_0 must have the form:

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + a_{n+2}(z - z_0)^{n+2} + \cdots \\ &= (z - z_0)^n[a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots] \end{aligned}$$

- **Theorem 11 (Zero of Order n):** A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- **Example 16:** Order of a Zero

  The analytic function $f(z) = z \sin z^2$ has a zero of order 3 at $z = 0$.

$$z \sin z^2 = z\left[z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots\right] = z^3\left[1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots\right]$$

Poles

- **Theorem 12 (Pole of Order n):** A function f that is analytic in a deleted neighborhood of z_0, $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written $f(z) = \phi(z)/(z - z_0)^n$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- Theorem 13 (Pole of Order n): If the functions f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order n at $z = z_0$.

- Example 17: Order of Poles

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

The denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at any of these points, it follows that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.

$z = 0$ is a zero of order 3 of $f(z) = z\sin z^2 \Rightarrow F(z) = 1/(z\sin z^2)$ has a pole of order 3 at $z = 0$.

- If a function has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ from any direction.

5. Residues and Residue Theorem

- If the complex function f has an isolated singularity at the point z_0, then f has a Laurent series representation:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

which converges for all z near z_0. More precisely, the representation is valid in some deleted neighborhood of z_0, $0 < |z - z_0| < R$.

Residue

- The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series given above is called the residue of the function f at the isolated singularity z_0.

$$a_{-1} = Res\ (f(z), z_0)$$

- Example 18: Residues

  $z = 1$ is a pole of order 2 of the function $f(z) = 1/(z-1)^2(z-3)$. From the Laurent series we see that the coefficient of $1/(z-1)$ is $a_{-1} = Res\,(f(z), 1) = -¼$.

  $z = 0$ is an essential singularity of $f(z) = e^{3/z}$. From the Laurent series we see that the coefficient of $1/z$ is $a_{-1} = Res\,(f(z), 0) = 3$.

- Theorem 14 (Residue at a Simple Pole): If f has a simple pole at $z = z_0$, then:

$$Res\,(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

- Theorem 15 (Residue at a Pole of Order n): If f has a pole of order n at $z = z_0$, then

$$Res\,(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

- **Example 19:** Residue at a Pole

  The function $f(z) = 1/(z - 1)^2(z - 3)$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$

$$Res\ (f(z), 3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4}$$

$$Res\ (f(z), 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3} = -\frac{1}{4}$$

- Suppose a function f can be written as a quotient $f(z) = g(z)/h(z)$, where g and h are analytic at $z = z_0$. If $g(z_0) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and

$$Res\ (f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

- Example 20: Residue at a Pole

  The function $1/(z^4+1)$ has four simple poles

$$z_1 = e^{\pi i/4},\ z_2 = e^{3\pi i/4},\ z_3 = e^{5\pi i/4},\ z_4 = e^{7\pi i/4}$$

$$Res\,(f(z),\, z_1) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$Res\,(f(z),\, z_2) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

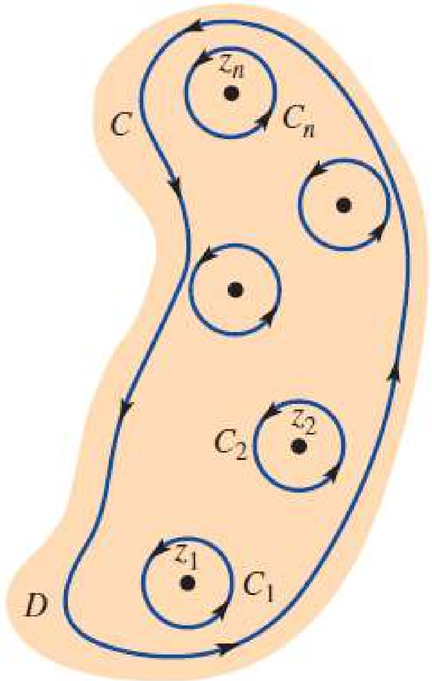
$$Res\,(f(z),\, z_3) = \frac{1}{4z_3^3} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$Res\,(f(z),\, z_4) = \frac{1}{4z_4^3} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

Residue Theorem

- **Theorem 16 (Cauchy's Residue Theorem):** Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of singular points z_1, z_2, ..., z_n within C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res\ (f(z),\ z_k)$$



- **Example 21:** Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$, where

(a) C is the rectangle defined by $x = 0$, $x = 4$, $y = -1$, $y = 1$, and

(b) C is the circle $|z| = 2$.

(a) $\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i[Res\ (f(z), 1) + Res\ (f(z), 3)] = 2\pi i\left[-\frac{1}{4}+\frac{1}{4}\right] = 0$

(b) $\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i\, Res\ (f(z), 1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$

- Example 22: Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}dz$, where C is the circle $|z - i| = 2$

$$\oint_C \frac{2z+6}{z^2+4}dz = 2\pi i\, Res\ (f(z), 2i) = 2\pi i \frac{3+2i}{2i} = \pi(3+2i)$$

- Example 23: Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z dz$, where C is the circle $|z| = 2$

$\tan z$ has simple poles at the points where $\cos z = 0$. $z = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$. Since only $-\pi/2$ and $\pi/2$ are within the circle $|z| = 2$,

$$\oint_C \tan z dz = 2\pi i[Res(f(z), -\pi/2) + Res(f(z), \pi/2)] = 2\pi i[-1-1] = -4\pi i$$

- Example 24: Evaluation by the Residue Theorem

  Evaluate $\oint_C e^{3/z} dz$, where C is the circle $|z| = 1$

  $$\oint_C e^{3/z} dz = 2\pi i\, Res(f(z), 0) = 6\pi i$$

- Note: L'Hôpital's rule is valid in complex analysis. If $f(z) = g(z)/h(z)$, where g and h are analytic at $z = z_0$, $g(z_0) = h(z_0) = 0$, and $h'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}$$

6. Evaluation of Real Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$

- The basic idea here is to convert this integral into a complex integral where the contour C is the unit circle centered at the origin. $z = \cos\theta + i\sin\theta = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$dz = ie^{i\theta}d\theta, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\theta = \frac{1}{2i}(z - z^{-1})$$

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz}$$

where C is $|z| = 1$.

- Example 25: A Real Trigonometric Integral

Evaluate $\int_0^{2\pi} \frac{d\theta}{(2+\cos\,\theta)^2}$

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\,dz$$

$$f(z) = \frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_0)^2(z-z_1)^2}$$

$z_0 = -2-\sqrt{3},\ z_1 = -2+\sqrt{3}$ only z_1 is inside the unit circle C,

$$\oint_C \frac{z}{(z^2+4z+1)^2}\,dz = 2\pi i\,Res\,(f(z), z_1)$$

$$Res\,(f(z),\,z_1) = \lim_{z\to z_1}\frac{d}{dz}(z-z_1)^2 f(z) = \lim_{z\to z_1}\frac{d}{dz}\frac{z}{(z-z_0)^2} = \frac{1}{6\sqrt{3}}$$

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}dz = \frac{4}{i}2\pi i\frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

$$\int_0^{2\pi}\frac{d\theta}{(2+\cos\,\theta)^2} = \frac{4\pi}{3\sqrt{3}}$$

Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

- When f is continuous on (–∞, ∞), $\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{R\to\infty}\int_{0}^{R} f(x)dx$
- If both limits exist, the integral is said to be convergent; if one or both of the limits fail to exist, the integral is divergent.
- In the event that we know (a priori) that an integral $\int_{-\infty}^{\infty} f(x)dx$ converges:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

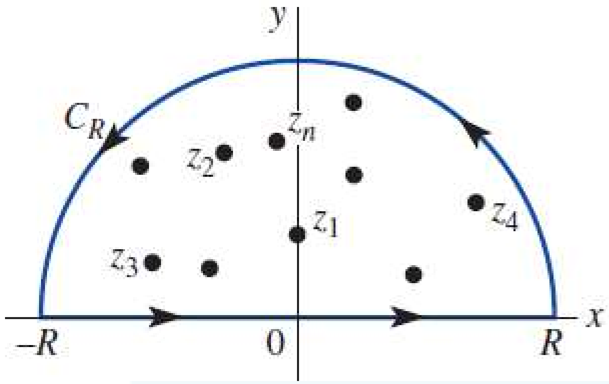
- This limit is called the Cauchy principal value of the integral and is written:

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

- When an integral of the form $\int_{-\infty}^{\infty} f(x)dx$ converges, its Cauchy principal value is the same as the value of the integral. If the integral diverges, it may still possess a Cauchy principal value. For ex., the integral $\int_{-\infty}^{\infty} xdx$ diverge, but:

$$P.V.\int_{-\infty}^{\infty} xdx = \lim_{R\to\infty}\int_{-R}^{R} xdx = \lim_{R\to\infty}\left[\frac{R^2}{2} - \frac{(-R)^2}{2}\right] = 0$$

- To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = P(z)/Q(z)$ in the upper half-plane $Re(z) > 0$.



$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

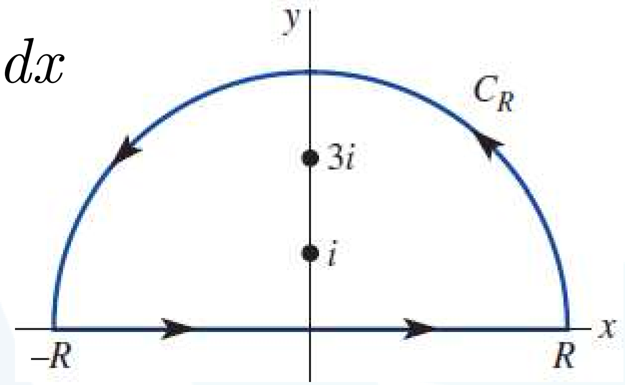
where z_k, $k = 1, 2, ..., n$, denotes poles in the upper half-plane.

If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have:

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

- Example 26: Cauchy $P.V.$ of an Improper Integral

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$



$$f(z) = \frac{1}{(z^2+1)(z^2+9)} = \frac{1}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz = I_1 + I_2$$

$$I_1 + I_2 = 2\pi i[Res(f(z), i) + Res(f(z), 3i)] = 2\pi i\left[\frac{1}{16i} + \left(-\frac{1}{48i}\right)\right] = \frac{\pi}{12}$$

On C_R, $\left|(z^2+1)(z^2+9)\right| = \left|z^2+1\right|\left|z^2+9\right| \ge \left||z|^2-1\right|\left||z|^2-9\right| = (R^2-1)(R^2-9)$

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ML-inequality

$$|I_2| = \left| \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz \right| \le \frac{\pi R}{(R^2-1)(R^2-9)} \underset{R\to\infty}{\to} 0$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)} dx = P.V. \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}$$

- Theorem 17 (Behavior of Integral as $R \to \infty$): Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \ge n + 2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

- Example 27: Cauchy $P.V.$ of an Improper Integral

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$

$$Res\,(f(z),\, z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i \qquad Res\,(f(z),\, z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$P.V.\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i[Res\,(f(z),\, z_1) + Res\,(f(z),\, z_2)] = \frac{\pi}{\sqrt{2}}$$

Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$

Using Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number

$$\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

whenever both integrals on the right side converge. When $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$ we can evaluate both integrals at the same time by considering the integral $\oint_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$ and C again consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z)$ in the upper half-plane $Re(z) > 0$.

- **Theorem 18 (Behavior of Integral as $R \to \infty$):** Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 1$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then:

$$\int_{C_R} f(z)e^{i\alpha z}dz \to 0 \text{ as } R \to \infty$$

- **Example 28:** Using Symmetry

Evaluate the Cauchy principal value of $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + 9} dx$

$$\int_0^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9} dx$$

With $\alpha = 1$, we now form the contour integral $\oint_C \frac{z}{z^2+9} e^{iz} dz$ where C is the same contour as example 26

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx = 2\pi i \, Res\left(f(z)e^{iz}, 3i\right) = \frac{\pi}{e^3} i$$

$$\int_{C_R} f(z) e^{iz} dz \to 0 \text{ as } R \to \infty \Rightarrow P.V. \int_{-\infty}^{\infty} \frac{x}{x^4+9} e^{ix} dx = \frac{\pi}{e^3} i$$

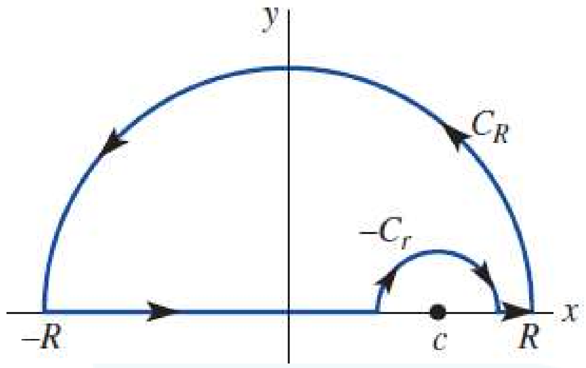
$$\int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} e^{ix} dx = \frac{\pi}{e^3} i$$

$$P.V. \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx = 0, \qquad P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3}$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{2e^3}$$

Indented Contours

- When $f(x) = P(x)/Q(x)$ have poles on the real axis, we must modify the procedure used in previous Examples. For example, to evaluate $\int_{-\infty}^{\infty} f(x)dx$ by residues when $f(z)$ has a pole at $z = c$, where c is a real number, we use an indented contour.

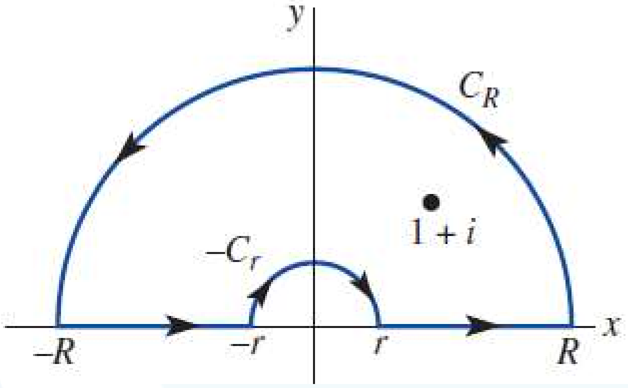


- Theorem 19 (Behavior of Integral as $r \to 0$): Suppose f has a simple pole at $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then:

$$\lim_{r \to 0} \int_{C_r} f(z)dz = \pi i \, Res(f(z), c)$$

- **Example 29:** Using an Indented Contour

Evaluate the Cauchy $P.V.$ of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\,dx$



The function $f(z) = 1/z(z^2 - 2z + 2)$ has simple poles at $z = 0$ and at $z = 1 + i$ in the upper half-plane.

$$\oint_C \frac{e^{iz}}{z(z^2-2z+2)}\,dz = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \, Res\,(f(z)e^{iz}, 1+i)$$

where $\int_{-C_r} = -\int_{C_r}$

Taking the limits $R \to \infty$ and $r \to 0$, we find

$$P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}\,dx - \pi i \, Res\,(f(z)e^{iz}, 0) = 2\pi i \, Res\,(f(z)e^{iz}, 1+i)$$

$$P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx = \pi i\left(\frac{1}{2}\right) + 2\pi i\left(\frac{e^{-1+i}}{4}(1+i)\right)$$

$$P.V.\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)}dx = \frac{\pi}{2}e^{-1}(\sin 1 + \cos 1)$$

$$P.V.\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}dx = \frac{\pi}{2}[1 + e^{-1}(\sin 1 - \cos 1)]$$

Integration along a Branch Cut

Branch Point at $z = 0$ We will examine integrals of the form $\int_0^{\infty} f(x)dx$.
These integrals require a special type of contour because when $f(x)$ is converted to a complex function, the resulting integrand $f(z)$ has, in addition to poles, a nonisolated singularity at $z = 0$.

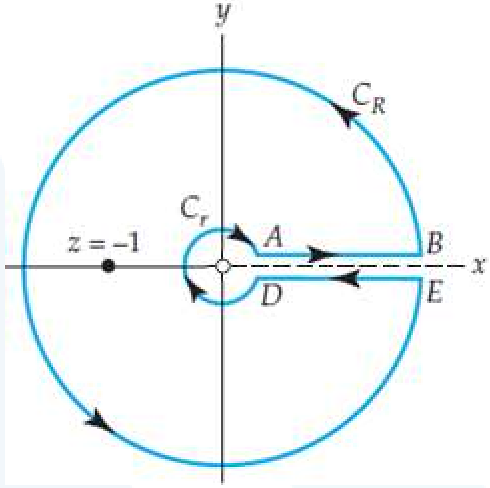
- Example 30: Integration along a Branch Cut

Evaluate $\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx$

$f(z) = \frac{1}{\sqrt{z}(z+1)}$ The origin is a branch point since $z^{1/2}$ has two values for any $z \neq 0$.

We can force $z^{1/2}$ to be single valued by choosing the positive x-axis as a branch cut ($0 < \theta < 2\pi$).

The integrand $f(z)$ is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{\pi i}$.



$$\oint_C \frac{1}{z^{1/2}(z+1)}\,dz = \int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i\, Res\,(f(z), -1)$$

On AB, $z = xe^{0i}$, and on ED, $z = xe^{(0+2\pi)i} = xe^{2\pi i}$

$$\int_{ED} f(z)dz = \int_R^r \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1} e^{2\pi i}dx = -\int_R^r \frac{x^{-1/2}}{x+1}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx$$

$$\int_{AB} f(z)dz = \int_r^R \frac{(xe^{0i})^{-1/2}}{xe^{0i}+1} e^{0i}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx$$

$z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R, $\Rightarrow \left|\int_{C_r} f(z)dz\right| \le \frac{r^{-1/2}}{1-r}2\pi r = \frac{2\pi}{1-r} r^{1/2} \underset{r\to 0}{\to} 0$

and $\left|\int_{C_R} f(z)dz\right| \le \frac{R^{-1/2}}{R-1}2\pi R = \frac{2\pi R}{R-1}\frac{1}{R^{1/2}} \underset{R\to\infty}{\to} 0$

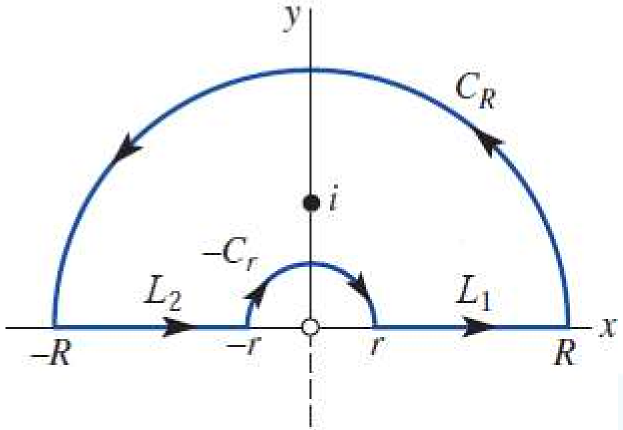
$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = 2\pi i\, Res\,(f(z),-1) = 2\pi i(-i) = 2\pi$$

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = \pi$$

- Example 31: Integration around a Point Cut

Evaluate $\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$

$$f(z) = \frac{\log z}{(z^2+1)^2}, \qquad |z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$



The branch cut consists of the origin and the negative imaginary axis.

In order that the isolated singularity $z = i$ be inside the closed path, we require that $r < 1 < R$.

$$\oint_C \frac{\log z}{(z^2+1)^2}\,dz = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R \quad = 2\pi i\, Res\,(f(z), i)$$

$$f(z) = \frac{\ln r + i\theta}{(r^2 e^{i2\theta}+1)^2}, \qquad (z = re^{i\theta})$$

On L_1, $z = xe^{0i} = x$, and on L_2, $z = xe^{\pi i} = -x$

$$\int_{L_2} f(z)dz = \int_{-R}^{-r} \frac{\ln(-x) + i\pi}{(x^2+1)^2}dx = -\int_R^r \frac{\ln x + i\pi}{(x^2+1)^2}dx = \int_r^R \frac{\ln x + i\pi}{(x^2+1)^2}dx$$

$$\int_{L_1} f(z)dz = \int_r^R \frac{\ln x}{(x^2+1)^2}dx$$

$z = re^{i\theta}$ & $z = Re^{i\theta}$ on C_r and C_R, $\Rightarrow \left|\int_{C_r} f(z)dz\right| \le \dfrac{-\ln r + \pi}{(1-r^2)^2}\pi r = \pi\dfrac{\pi r - r\ln r}{(1-r^2)^2} \underset{r\to 0}{\rightarrow} 0$

and $\left|\int_{C_R} f(z)dz\right| \le \dfrac{\ln R + \pi}{(R^2-1)^2}\pi R = \underset{r\to 0}{\rightarrow} 0$

$$2\int_0^\infty \frac{\ln x}{(x^2+1)^2}dx + \int_r^R \frac{i\pi}{(x^2+1)^2}dx = 2\pi i\, Res(f(z), i) = 2\pi i\left(\frac{\pi}{8} + \frac{1}{4}i\right) = -\frac{\pi}{2} + \frac{\pi^2}{4}i$$

$$\int_0^\infty \frac{\ln x}{(x^2+1)^2}dx = -\frac{\pi}{4}, \qquad \int_0^\infty \frac{1}{(x^2+1)^2}dx = \frac{\pi}{4}$$