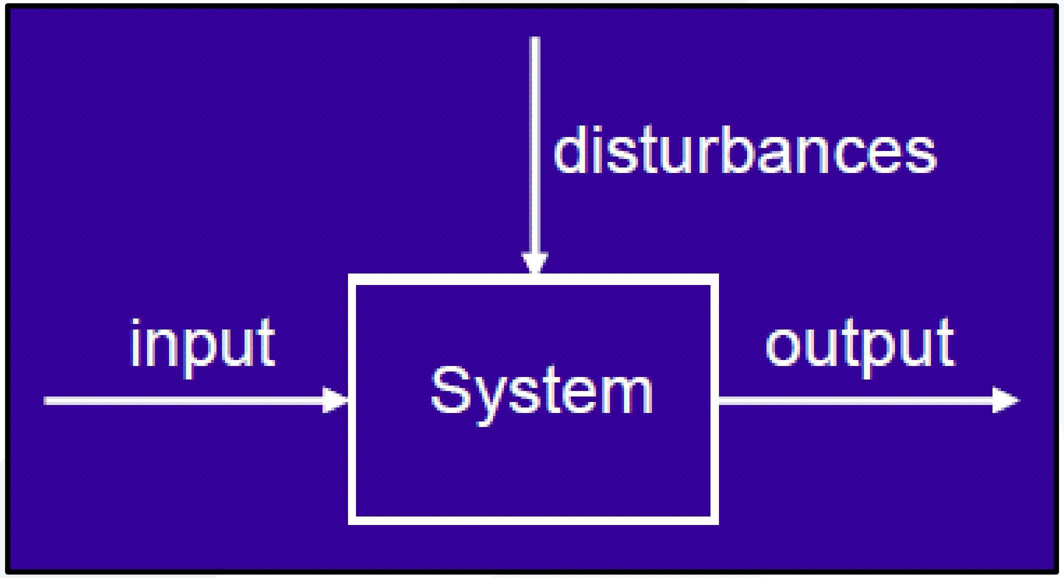
جامعة المنارة
كلية الهندسة
قسم الروبوتيك و الأنظمة الذكية
مقرر النمذجة و المطابقة

Mathematical Techniques to Analyze System Responses



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INTRODUCTION

An input that changes at a constant rate is modeled by the ramp function. The step function models an input that rapidly reaches a constant value, while the rectangular pulse function models a constant input that is suddenly removed. The *impulse* is similar to the pulse function, but it models an input that is suddenly applied and removed after a *very short* time. The impulse, which is a mathematical function only and has no physical counterpart, has an infinite magnitude for an infinitesimal time.

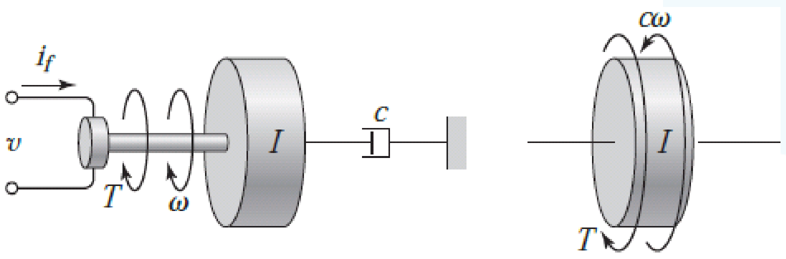
THE STEP FUNCTION APPROXIMATION

The step function is an approximate description of an input that can be switched on in a time interval that is very short compared to the time constant of the system. A good example of a step input is the voltage applied to a circuit due to the sudden closure of a switch.

Example

A certain rotational system has an inertia $I = 50\ \mathrm{kg \cdot m^2}$ and a viscous damping constant $c = 10\ \mathrm{N \cdot m \cdot s/rad}$. The torque $T(t)$ is applied by an electric motor . From the free body diagram shown in part (b) of the figure, the equation of motion is

$$50\frac{d\omega}{dt} + 10\omega = T(t) = K_f i_f \quad (1)$$



(a) (b)

The model of the motor's field current i_f in amperes is

$$0.001\frac{di_f}{dt} + 5i_f = v(t) \qquad (2)$$

where $v(t)$ is the voltage applied to the motor. The motor torque constant is $K_T = 25\ \mathrm{N \cdot m/A}$.

Suppose the applied voltage is 10 V. Determine the steady-state speed of the inertia and estimate the time required to reach that speed.

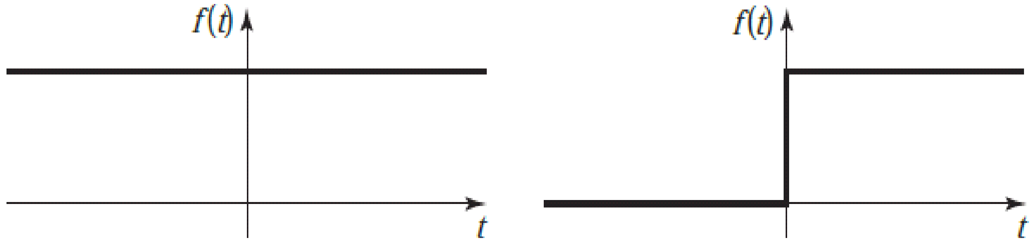
Solution

From equation (2) we see that the time constant of the motor circuit is $0.001/5 = 2 \times 10^{-4}$ s. Thus the current will reach a steady-state value of $10/5 = 2$ A in approximately $4(2 \times 10^{-4}) = 8 \times 10^{-4}$ s. The resulting steady-state torque is $K_T(2) = 25(2) = 50$ N · m.

From equation (1) we find the time constant of the rotational system to be $50/10 = 5$ s. Since this is much larger than the time constant of the circuit (2×10^{-4} s), we conclude that the motor torque may be modeled as a step function. The magnitude of the step function is 50 N · m. The steady-state speed is $\omega = 50/10 = 5$ rad/s, and therefore it will take approximately $4(5) = 20$ s to reach this speed.

STEP INPUTS VERSUS CONSTANT INPUTS

Consider the two input functions shown in Figure . The one shown in part (a) is constant, whereas in part (b) the input is a step function. For the constant input, the derivative is always zero, but for the step input the derivative is an impulse function, which is infinite at $t = 0$ and zero elsewhere. So if we are interested in finding the response of a model only for $t \geq 0$, then we can conclude that it does not matter whether we model the input as a constant or as a step, *as long as the model does not contain any time derivatives of the input.*



(a) (b)

STEP RESPONSE WITH AN INPUT DERIVATIVE

The general form of the first-order linear model is

$$m\dot{v} + cv = b\dot{f}(t) + f(t)$$

whose transfer function is of the form

$$\frac{V(s)}{F(s)} = \frac{bs + 1}{ms + c}$$

The presence of an input derivative is indicated by an s in the numerator of the transfer function. Such a transfer function is said to have *numerator dynamics*.

If $f(t)$ is a step function of magnitude F, then $F(s) = F/s$ and $f(0) = 0$,

$$V(s) = \frac{mv(0) + bF}{ms + c} + \frac{F}{s(ms + c)}$$

which gives

$$v(t) = \left[v(0) + \frac{bF}{m} \right] e^{-ct/m} + \frac{F}{c}\left(1 - e^{-ct/m}\right)$$

we see that the effect of the $b\dot{f}$ term is to increase the effective initial value of $v(t)$ by the amount bF/m.

IMPULSE RESPONSE

The impulse response of $m\dot{v} + cv = f(t)$ is found as follows, where A is the impulse *strength*, or the area under the impulse versus time curve.

$$msV(s) - mv(0) + cV(s) = A$$

$$V(s) = \frac{mv(0) + A}{ms + c} = \frac{v(0) + A/m}{s + c/m}$$

$$v(t) = \left[v(0) + \frac{A}{m}\right] e^{-ct/m}$$

We can see that the effect of the impulse is to increase the effective initial condition by A/m.

Because $\dot{f}(t)$ is undefined if $f(t)$ is an impulse, we do not consider the impulse response of the equation $m\dot{v} + cv = f(t) + b\dot{f}(t)$.

RAMP RESPONSE AND THE TIME CONSTANT

We can obtain the response of the general equation $\tau \dot{v} + v = f(t)$ to the ramp input $f(t) = mt$ as follows. Setting $v(0) = 0$ and transforming the equation with $F(s) = m/s^2$ gives

$$\tau s V(s) + V(s) = F(s) = \frac{m}{s^2}$$

or

$$V(s) = \frac{m}{s^2(\tau s + 1)} = \frac{m}{s^2} - \frac{m\tau}{s} + \frac{m\tau}{s + 1/\tau}$$

The inverse transforms give

$$v(t) = m(t - \tau) + m\tau e^{-t/\tau}$$

The response is in steady state after approximately $t = 4\tau$. At steady state, $v(t) = m(t - \tau)$, so the response is parallel to the input but lags behind it by a time τ

Obtaining the ramp response can be tedious for higher-order systems, but sometimes we only need to find the steady-state *difference* between the input and the output.

Example

Obtain the steady-state difference $f(\infty) - v(\infty)$ between the input and output of the following model: $\tau\dot{v} + v = bf(t)$, where b is a constant and $f(t) = mt$. Assume that $v(0) = 0$ and that the model is stable ($\tau > 0$).

Solution

The transform of the response is

$$V(s) = \frac{b}{\tau s + 1} F(s) = \frac{b}{\tau s + 1} \frac{m}{s^2}$$

Use this with the final value theorem to find the steady-state difference:

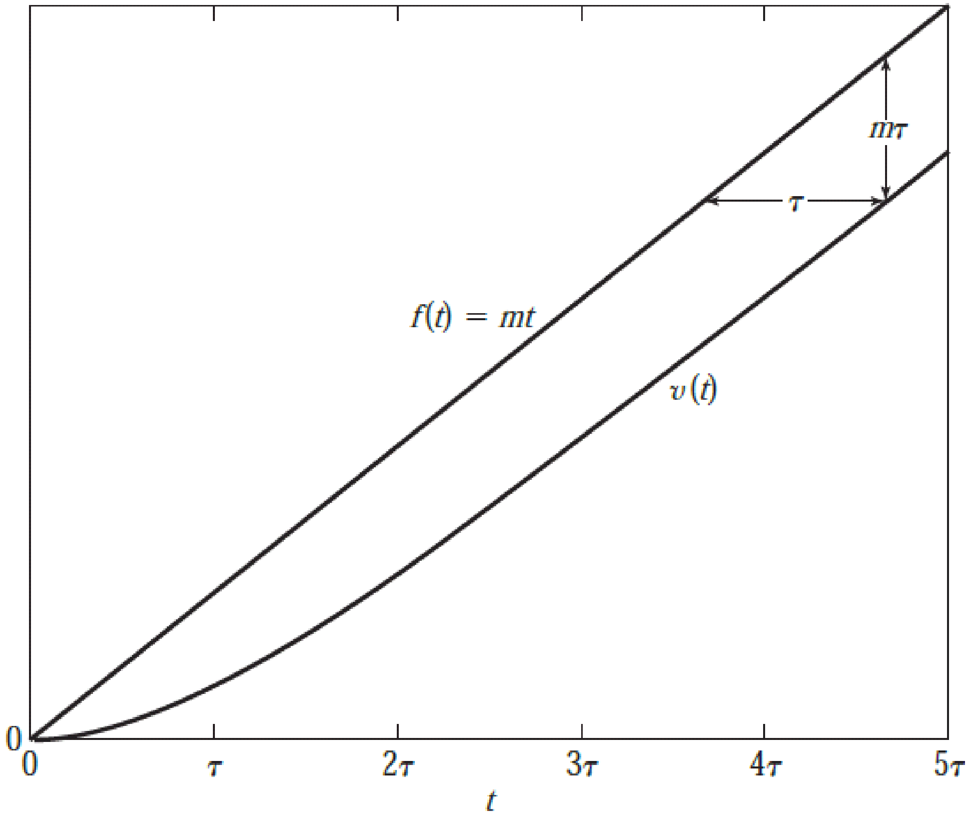
$$f(\infty) - v(\infty) = \lim_{s\to 0}[sF(s)] - \lim_{s\to 0}[sV(s)] = \lim_{s\to 0} s[F(s) - V(s)]$$

$$= \lim_{s\to 0} s\left(\frac{m}{s^2} - \frac{b}{\tau s + 1}\frac{m}{s^2}\right)$$

$$= \lim_{s\to 0} \frac{m}{s}\left(\frac{\tau s + 1 - b}{\tau s + 1}\right)$$

$$= \begin{cases} \infty & b \neq 1 \\ m\tau & b = 1 \end{cases}$$

Thus, the steady-state difference is infinite unless $b = 1$. Both the input and output approach straight lines at steady state. The preceding result shows that the lines diverge unless $b = 1$.



NUMERATOR DYNAMICS AND SECOND-ORDER SYSTEM RESPONSE

For the model $m\ddot{x} + c\dot{x} + kx = a\dot{g}(t) + bg(t)$, the effect on the step response of the numerator dynamics coefficient a is to change the formulas for the constants A_1 and A_2 in Table for the real roots cases, and the formulas for the amplitude B and angle ϕ for the complex roots case.

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Unit step response of a stable second-order model

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. Overdamped case ($\zeta > 1$): distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2} e^{-r_1 t} - \frac{r_1}{r_1 - r_2} e^{-r_2 t} + 1\right)$$

2. Critically damped case ($\zeta = 1$): repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t)e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1)e^{-r_1 t} + 1]$$

3. Underdamped case ($0 \le \zeta < 1$): complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$x(t) = Be^{-t/\tau} \sin\left(\omega_n\sqrt{1 - \zeta^2}\,t + \phi\right) + \frac{1}{k}$$

$$= \frac{1}{k}\left[\frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1 - \zeta^2}\,t + \phi\right) + 1\right]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right) + \pi \qquad \text{(third quadrant)}$$

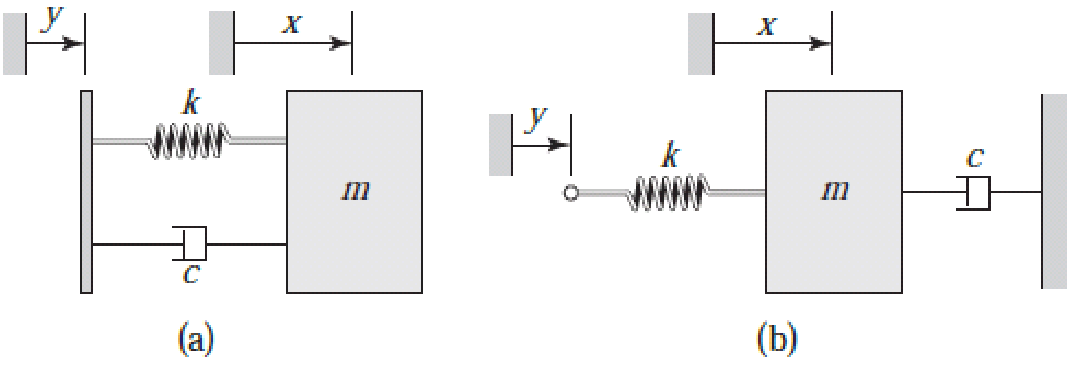
Time constant: $\tau = 1/\zeta\omega_n$

For example, consider the two systems shown in Figure for $m = 1$, $c = 6$, and $k = 8$. The model for part (a) of the figure is $\ddot{x} + 6\dot{x} + 8x = 6\dot{y}(t) + 8y(t)$, and the unit-step response is obtained from the Laplace transform.

$$X(s) = \frac{6s+8}{s(s^2+6s+8)} = \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+4}$$

Thus

$$x(t) = 1 + e^{-2t} - 2e^{-4t}$$

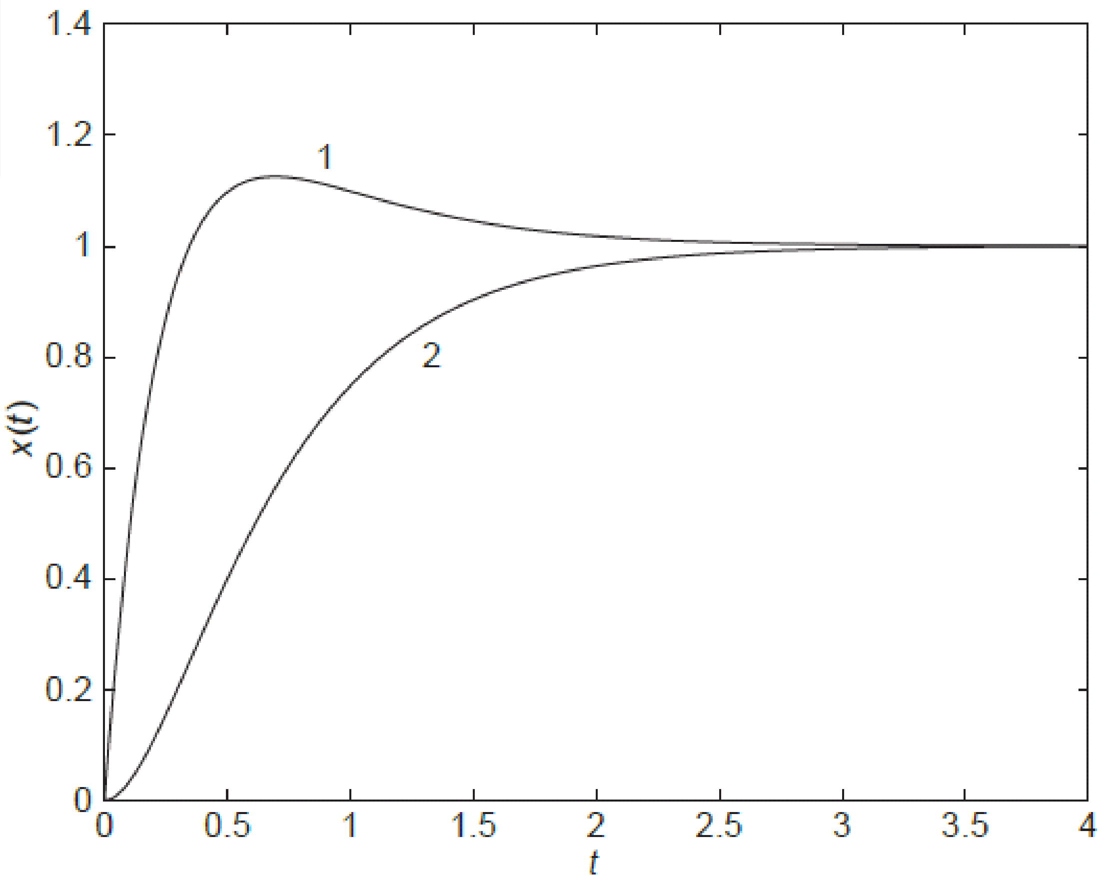


The model for part (b) of the figure is $\ddot{x} + 6\dot{x} + 8x = 8y(t)$, and the unit-step response is given by

$$X(s) = \frac{8}{s(s^2 + 6s + 8)} = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+4}$$

$$x(t) = 1 - 2e^{-2t} + e^{-4t}$$

The responses are shown in Figure . Curve (1) corresponds to the case with numerator dynamics. We see that the numerator dynamics produces a smaller 10%–90% rise time, and causes an overshoot here, even though $\zeta > 1$.


1.4
1.2
1
0.8
0.6
0.4
0.2
0
x(t)
0
0.5
1
1.5
2
2.5
3
3.5
4
t
1
2

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