Numerical Analysis and Programming

جامعة المنارة
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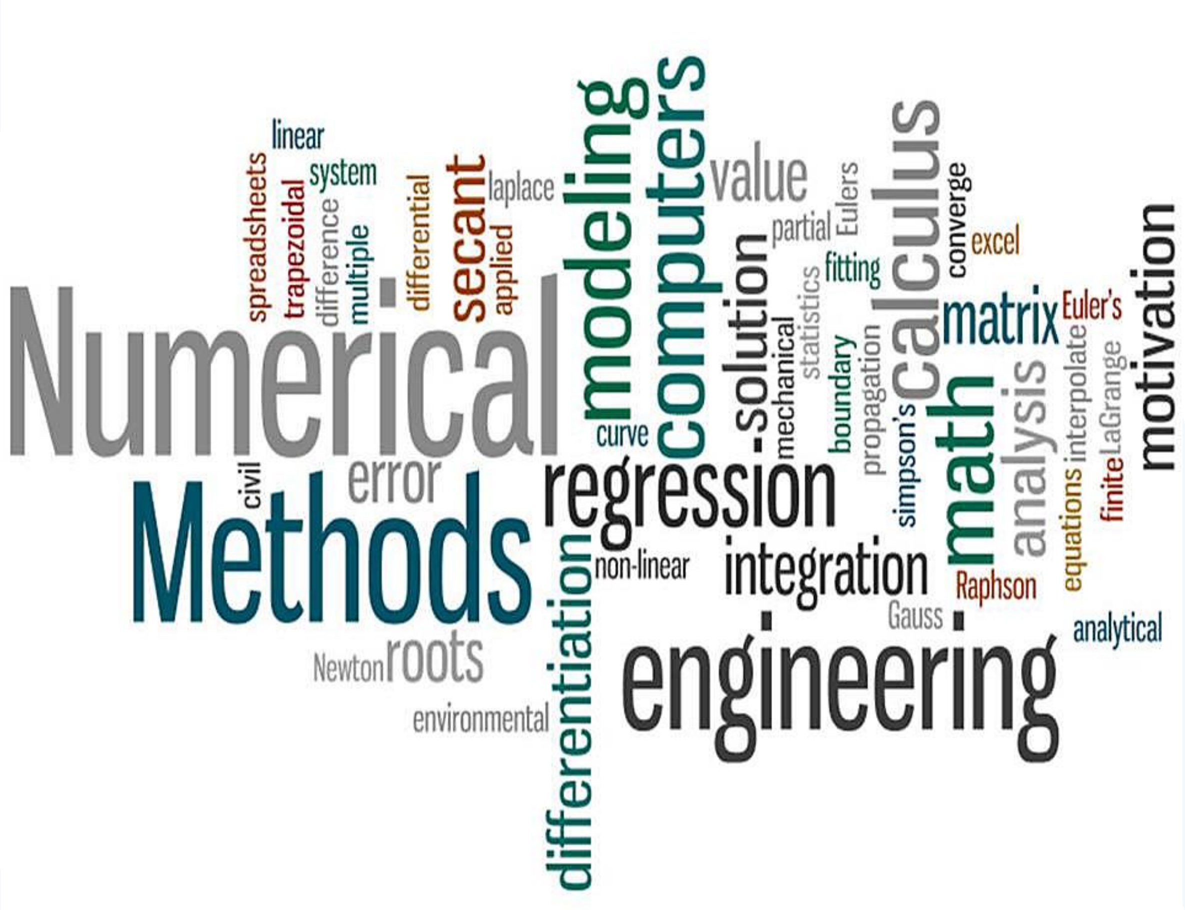


Linear Algebraic Equations

Part-01

Noncomputer Methods for Solving System of Equations

- For small number of equations (**$n \leq 3$**) linear equations can be solved readily by simple techniques such as "**method of elimination**."
- Linear algebra provides the tools to solve such systems of linear equations.
- There are many ways to solve a system of linear equations:
  - Graphical Methods.
  - Cramer's Rule.
  - Method of Elimination.

  For $n \leq 3$
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

Graphical Methods

- Consider a set of two equations:

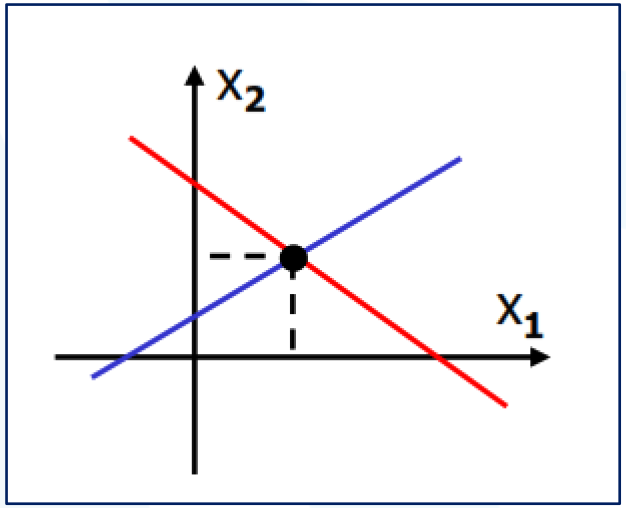
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Plot these on the Cartesian coordinate system with axes x_1 and x_2.

Solve for (x_2)

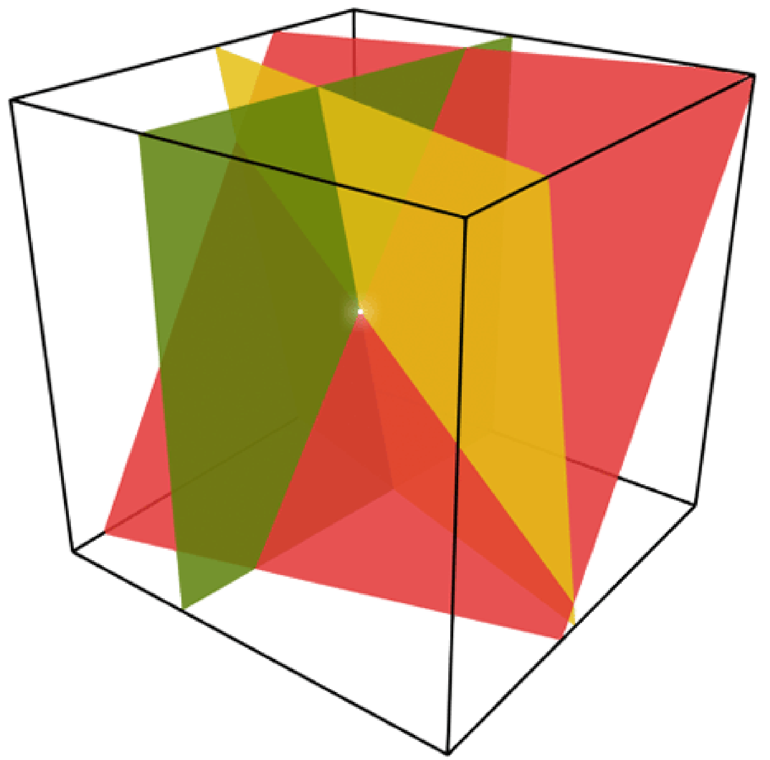
$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



Graphical Methods

- For n=3, each equation will be a plane on a 3D coordinate system. Solution is the point where these planes intersect.



- For n>3, graphical solution is not practical.

Graphical Methods

- For example : solve

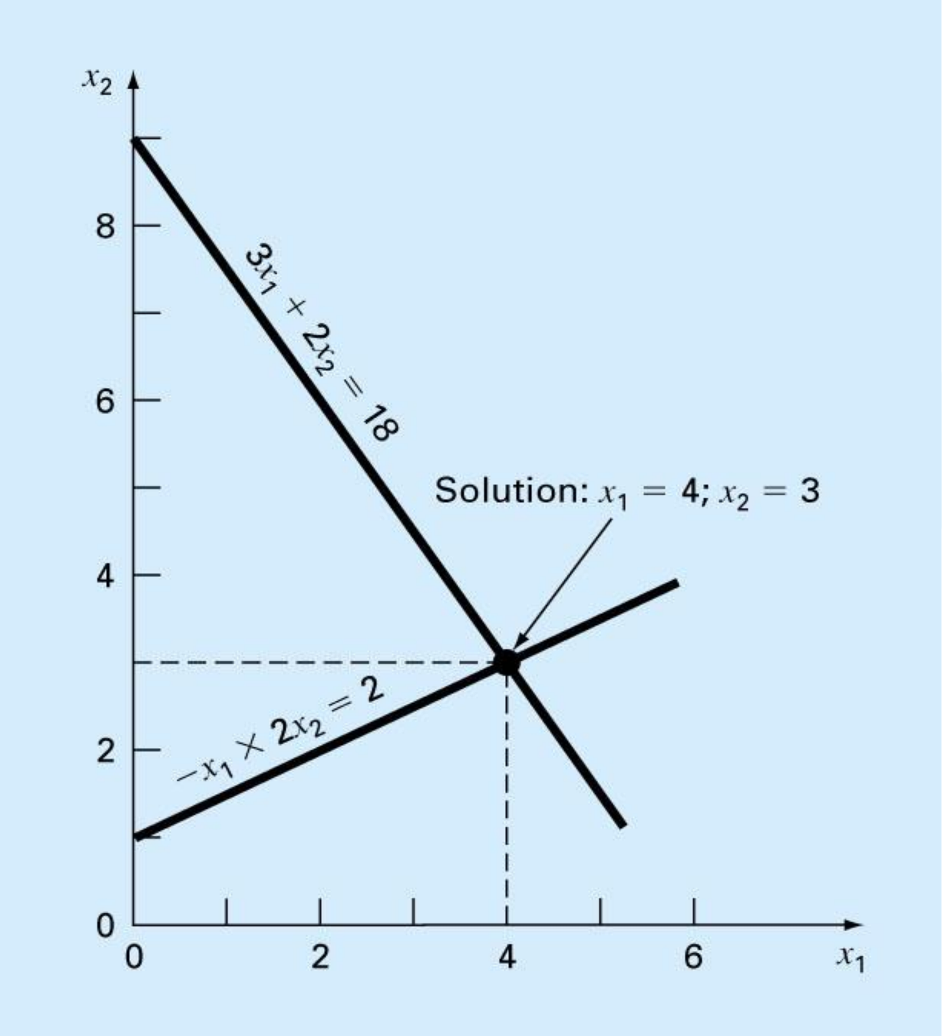
$$3x_1 + 2x_2 = 18$$

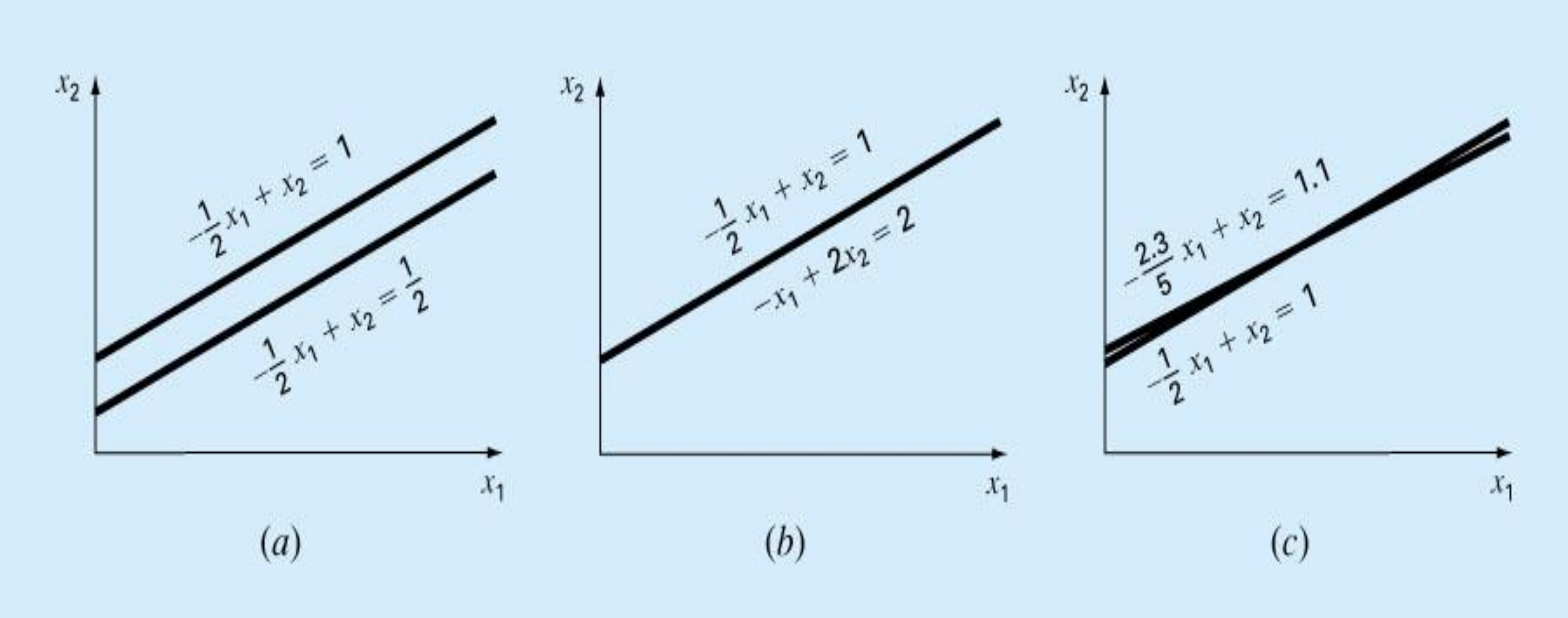
$$-x_1 + 2x_2 = 2$$

- The solution is the intersection of the two lines at $x_1=4$ and $x_2=3$.
- This result can be checked by substituting these values into the original equations to yield

$$3(4)+2(3)=18$$

$$-(4)+2(3)=2$$





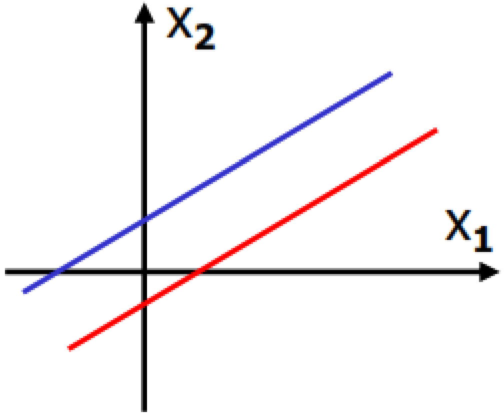
No solution

Infinite solutions

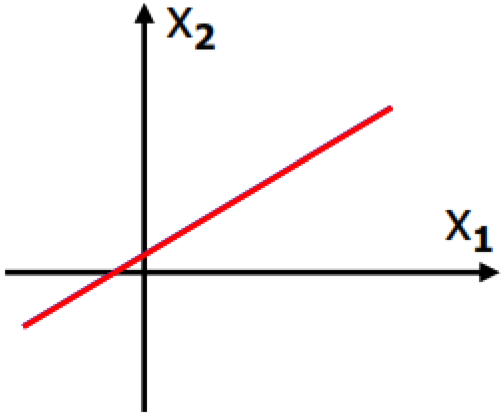
Ill-conditioned
(Slopes are too close)

Graphical Methods

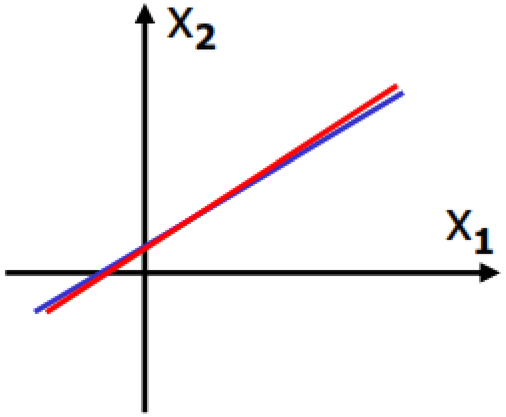
(1) no solution



(2) infinitely many solutions



(3) ill-conditioned system
(Slopes are too close)



- In system (1) and (2), equations are linearly dependent.
- In system (3), the slopes of the lines are very close to each other.

Mathematically

- Coefficient matrices of (1) & (2) are singular. Their determinants are zero and their inverse do not exist.
- Coefficient matrix of (3) is almost singular. Its inverse is difficult to take. This system has a unique solution, which is not easy to determine numerically because of its extreme sensitivity to round-off errors.

Cramer's Rule

- Cramer's rule is another solution technique that is best suited to small numbers of equations.
- This rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficients of the unknown in question by the constants $b_1, b_2, \ldots, b_n$.

For a 3x3 system

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Cramer’s Rule- Example

- Example: Cramer's Rule can be used to solve the system

$$x_1 + x_2 = 3$$
$$x_1 + 2x_2 = 5$$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

To solve N by N system requires $(N+1)(N-1)N!$ multiplications.

To solve a 30 by 30 system, 2.38×10^{35} multiplications are needed.

It can be used if the determinants are computed in efficient way

Cramer's Rule- Example

- Example:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + 2\,x_2 - 2\,x_3 &= 0 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{6}{2} = 3$$

$$\{x\} = \begin{Bmatrix} 2 \\ 2 \\ 3 \end{Bmatrix}$$

The Elimination of Unknowns

- The elimination of unknowns by combining equations is an algebraic approach that can be illustrated for a set of two equations:

$$a_{11}x_1+a_{12}x_2=b_1 \qquad (1)$$

$$a_{21}x_1+a_{22}x_2=b_2 \qquad (2)$$

- Eq. (1) might be multiplied by a_{21} and Eq. (2) by a_{11} to give

$$a_{11}a_{21}x_1+a_{12}a_{21}x_2=b_1a_{21} \qquad (3)$$

$$a_{21}a_{11}x_1+a_{22}a_{11}x_2=b_2a_{11} \qquad (4)$$

Subtracting Eq. (3) from Eq. (4) will, therefore, eliminate the x_1 term from the equations to yield

$$a_{22}a_{11}x_2-a_{12}a_{21}x_2=b_2a_{11}-b_1a_{21}$$

The Elimination of Unknows

- Which can be solved for:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

- x_2 can then be substituted into Eq. (1), which can be solved for:

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

The Elimination of Unknowns - Example

- Use the elimination to solve:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

- The elimination of unknowns can be extended to systems with more than two or three equations. However, the numerous calculations that are required for larger systems make the method extremely tedious to implement by hand. However, as described in the next section, the technique can be formalized and readily programmed for the computer.

Naïve Gauss Elimination

- In the previous section, the elimination of unknowns was used to solve a pair of simultaneous equations. The procedure consisted of two steps:

1. The equations were manipulated to eliminate one of the unknowns from the equations. The result of this elimination step was that we had one equation with one unknown.

2. Consequently, this equation could be solved directly and the result back-substituted into one of the original equations to solve for the remaining unknown.

- This basic approach can be extended to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back-substitute. Gauss elimination is the most basic of these schemes.

Naïve Gauss Elimination

- The method consists of two steps:
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward Substitution**: Solve the system starting from the last variable.

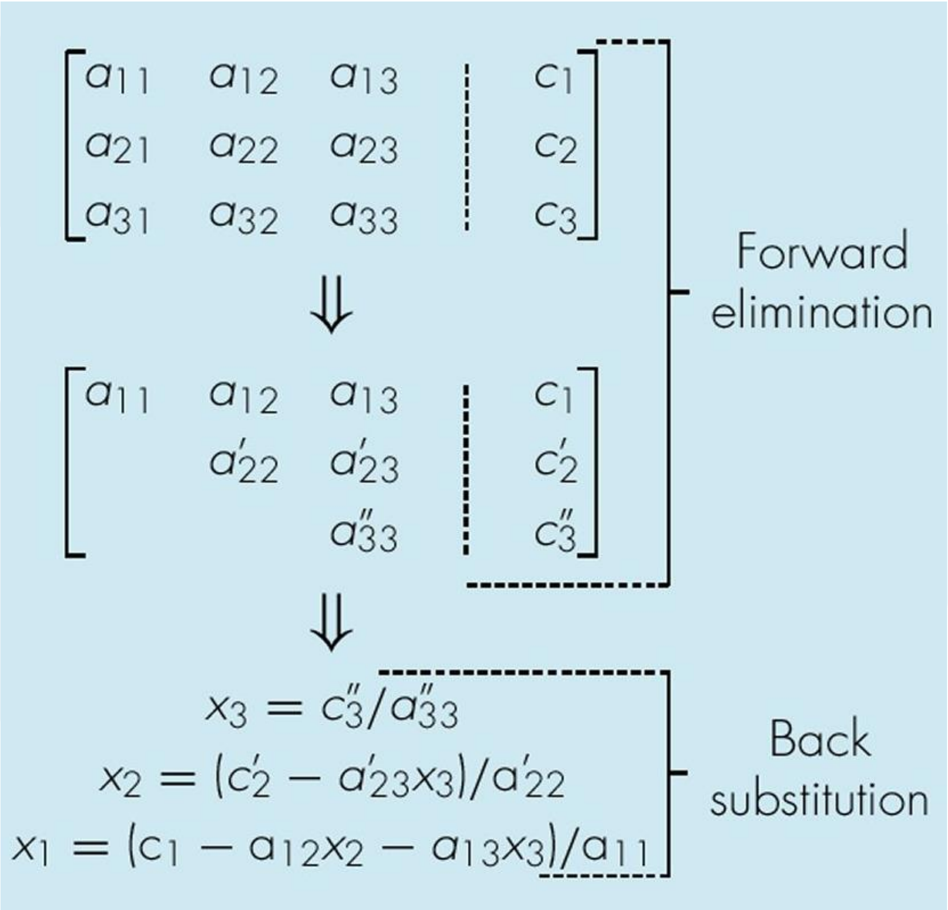
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Naïve Gauss Elimination

The two phases of Gauss elimination:

Forword elimination and back substitution.

The primes indicate the number of times that the coefficients and constants have been modified.



Naïve Gauss Elimination - Example

Example

Use Gauss elimination to solve

$$x_1 - 2x_2 + 2x_3 = 1$$
$$2x_1 + x_2 - 3x_3 = -3$$
$$-3x_1 + x_2 - x_3 = 4$$

Carry six significant figures during the computation.

1	-2	2		1
2	1	-3		-3
-3	1	-1		4
1	-2	2		1
0	5	-7		-5
0	-5	5		7
1	-2	2		1
0	5	-7		-5
0	0	-2		2
			x3	-1
			x2	-2,4
			x1	-1,8

Naïve Gauss Elimination - Homework

Example: Solve the following system using Naïve Gauss Elimination.

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \\
3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\
-6x_1 + 4x_2 + x_3 - 18x_4 &= -34
\end{aligned}
$$

Gauss Jordan

- It is a variation of Gauss elimination. The major differences are:

a. When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.

b. All rows are normalized by dividing them by their pivot elements.

c. Elimination step results in an identity matrix.

d. Consequently, it is not necessary to employ back substitution to obtain solution.

Gauss – Jordan Method

Example: **Solve the following system using Gauss - Jordan Method.**

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution. First, express the coefficients and the constants on the right-hand side as an augmented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Then normalize the first row by dividing it by the pivot element, 3, to yield

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

The x_1 term can be eliminated from the second row by subtracting 0.1 times the first row from the second row. Similarly, subtracting 0.3 times the first row from the third row will eliminate the x_1 term from the third row:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.293333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Next, normalize the second row by dividing it by 7.00333:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Reduction of the x_2 terms from the first and third equations gives

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix}$$

The third row is then normalized by dividing it by 10.0120:

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

Finally, the x_3 terms can be reduced from the first and the second equations to give

$$\begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$