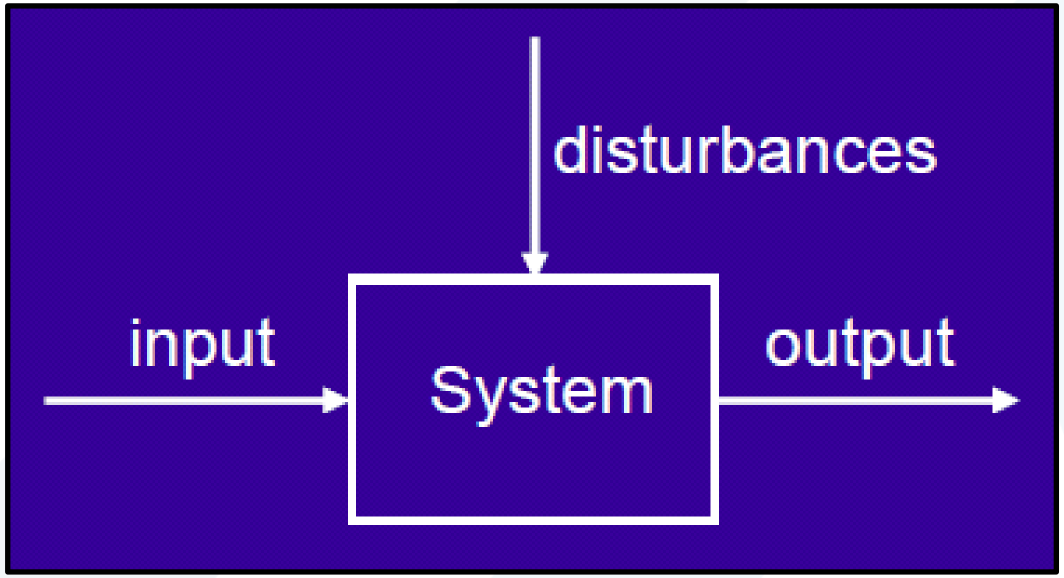
جامعة المنارة

كلية الهندسة

قسم الروبوتيك و الأنظمة الذكية

مقرر النمذجة و المطابقة

Introduction to
Estimation of System Output using State Observer



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Controllability and Observability

The concepts of controllability and observability were introduced by Kalman (1960) and play an important role in the control of multivariable systems.

A system is said to be controllable if a control vector $\mathbf{u}(t)$ exists that will transfer the system from any initial state $\mathbf{x}(t_0)$ to some final state $\mathbf{x}(t)$ in a finite time interval.

A system is said to be observable if at time t_0, the system state $\mathbf{x}(t_0)$ can be exactly determined from observation of the output $\mathbf{y}(t)$ over a finite time interval.

If a system is described by equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

then a sufficient condition for complete state controllability

$$\mathbf{M} = [\mathbf{B} : \mathbf{AB} : \ldots : \mathbf{A}^{n-1}\mathbf{B}]$$

contains n linearly independent row or column vectors, i.e. is of rank n (that is, the matrix is non-singular, i.e. the determinant is non-zero).

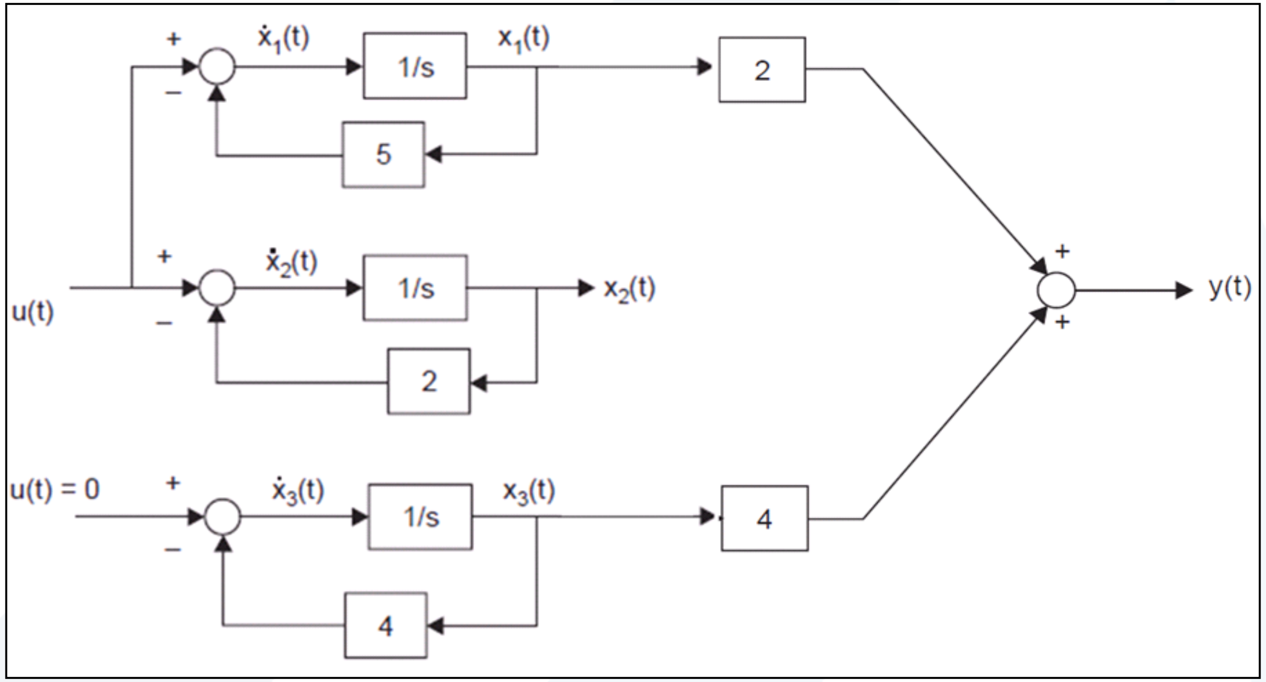
The system is completely observable if the matrix

$$\mathbf{N} = \left[\mathbf{C}^{\mathrm{T}} : \mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} : \ldots : (\mathbf{A}^{\mathrm{T}})^{n-1}\mathbf{C}^{\mathrm{T}}\right]$$

is of rank n, i.e. is non-singular having a non-zero determinant.

Example

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \mathbf{u}(t); \quad \mathbf{y}(t) = [2 \quad 0 \quad 4] \times (t)$$



Solution

```
A=[-5 0 0;0 -2 0;0 0 -4];
B=[1;1;0];
C=[2 0 4];
M=ctrb(A,B)
rank_of_M=rank(M)
System_order=length(A)
N=(obsv(A,C))'
Rank_of_N=rank(N)
```

```
M =
    1   -5   25
    1   -2    4
    0    0    0
rank_of_M =
    2
System_order =
    3
```

```
N =
    2  -10   50
    0    0    0
    4  -16   64
Rank_of_N =
    2
```

State Observers

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables. Estimation of unmeasurable state variables is commonly called *observation*. A device (or a computer program) that estimates or observes the state variables is called a *state observer*, or simply an *observer*. If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a *full-order state observer*. There are times when this will not be necessary, when we will need observation of only the unmeasurable state variables, but not of those that are directly measurable as well. For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only $n - m$ state variables, where n is the dimension of the state vector and m is the dimension of the output vector.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a *reduced-order state observer* or, simply, a *reduced-order observer*. If the order of the reduced-order state observer is the minimum possible, the observer is called a *minimum-order state observer* or *minimum-order observer*. In this section, we shall discuss both the full-order state observer and the minimum-order state observer.

State Observer. A state observer estimates the state variables based on the measurements of the output and control variables. Here the concept of observability plays an important role. As we shall see later, state observers can be designed if and only if the observability condition is satisfied.

In the following discussions of state observers, we shall use the notation $\tilde{\mathbf{x}}$ to designate the observed state vector. In many practical cases, the observed state vector $\tilde{\mathbf{x}}$ is used in the state feedback to generate the desired control vector.

Consider the plant defined by

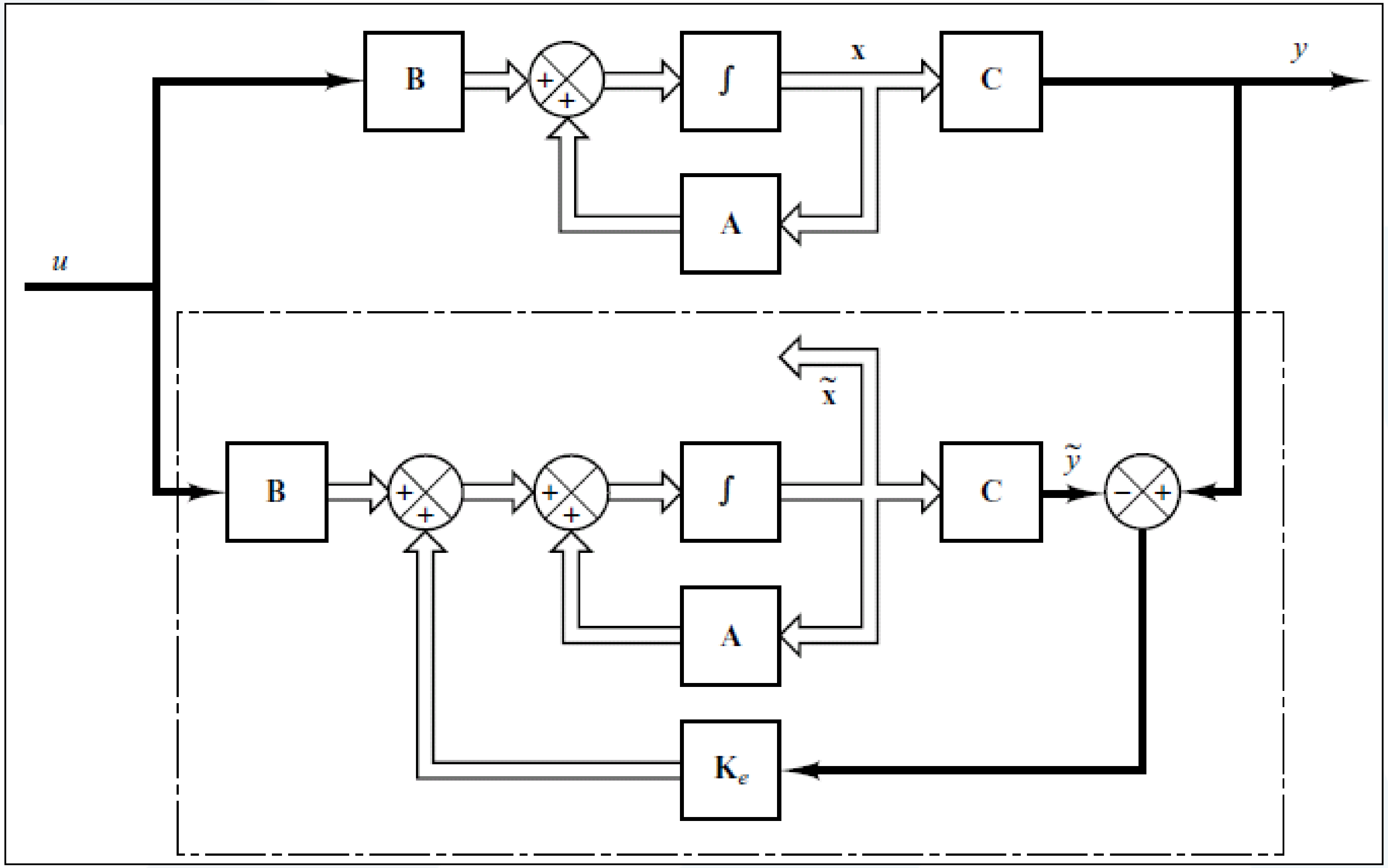
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices **A** and **B** and the lack of the initial error. The estimation error or observation error is the difference between the measured output and the estimated output. The initial error is the difference between the initial state and the initial estimated state. Thus, we define the mathematical model of the observer to be

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

where $\tilde{\mathbf{x}}$ is the estimated state and $\mathbf{C}\tilde{\mathbf{x}}$ is the estimated output. The inputs to the observer are the output y and the control input u. Matrix $\mathbf{K}_e$, which is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output y and the estimated output $\mathbf{C}\tilde{\mathbf{x}}$. This term continuously corrects the model output and improves the performance of the observer. Figure shows the block diagram of the system and the full-order state observer.


B
∫
x
C
y
A
u
x̃
B
∫
C
ỹ
A
K_e

Full-Order State Observer. The order of the state observer that will be discussed here is the same as that of the plant.

To obtain the observer error equation,

$$\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K}_e(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}})$$

Define the difference between $\mathbf{x}$ and $\tilde{\mathbf{x}}$ as the error vector $\mathbf{e}$, or

$$\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

From Equation , we see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$. If matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is a stable matrix, the error vector will converge to zero for any initial error vector $\mathbf{e}(0)$. That is, $\tilde{\mathbf{x}}(t)$ will converge to $\mathbf{x}(t)$ regardless of the values of $\mathbf{x}(0)$ and $\tilde{\mathbf{x}}(0)$. If the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.

If the plant is completely observable, then it can be proved that it is possible to choose matrix $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has arbitrarily desired eigenvalues. That is, the observer gain matrix $\mathbf{K}_e$ can be determined to yield the desired matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.

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