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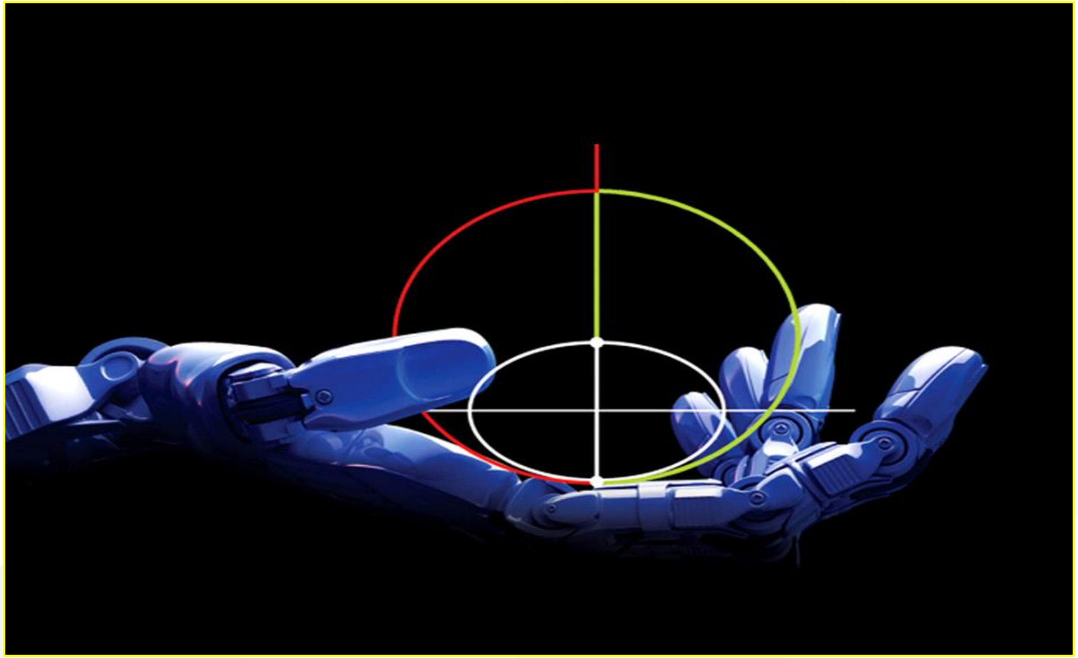
# Planar Kinematics of a Rigid Body

## Motion Analysis :Acceleration



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# Relative-Motion Analysis : Acceleration

An equation that relates the accelerations of two points on a bar (rigid body) subjected to general plane motion may be determined by differentiating $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ with respect to time. This yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms $d\mathbf{v}_B/dt = \mathbf{a}_B$ and $d\mathbf{v}_A/dt = \mathbf{a}_A$ represent the *absolute accelerations* of points $B$ and $A$.

The last term represents the acceleration of $B$ with respect to $A$ as measured by an observer fixed to translating $x'$, $y'$ axes which have their origin at the base point $A$. It was shown that to this observer point $B$ appears to move along a *circular arc* that has a radius of curvature

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$r_{B/A}$. Consequently, $\mathbf{a}_{B/A}$ can be expressed in terms of its tangential and normal components; i.e., $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$, where $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Hence, the relative-acceleration equation can be written in the form

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

where

$\mathbf{a}_B$ = acceleration of point $B$

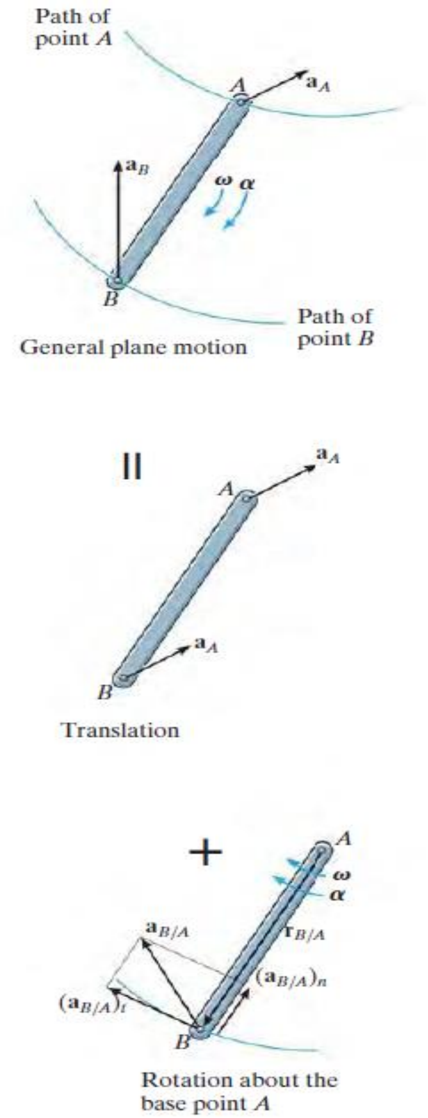
$\mathbf{a}_A$ = acceleration of point $A$

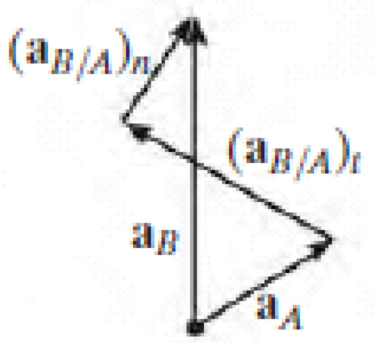
$(\mathbf{a}_{B/A})_t$ = tangential acceleration component of $B$ with respect to $A$. The *magnitude* is $(a_{B/A})_t = \alpha r_{B/A}$, and the *direction* is perpendicular to $\mathbf{r}_{B/A}$.

$(\mathbf{a}_{B/A})_n$ = normal acceleration component of $B$ with respect to $A$. The *magnitude* is $(a_{B/A})_n = \omega^2 r_{B/A}$, and the *direction* is always from $B$ toward $A$.

It is seen that at a given instant the acceleration of ***B*** *is* determined by considering the bar to translate with an acceleration $\mathbf{a}_A$ *, and simultaneously rotate about the base point* ***A*** *with an* instantaneous angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ .

Vector addition of these two effects, applied to ***B****, yields* $\mathbf{a}_B$*. It should be noted that since points* ***A*** *and* ***B*** move along *curved paths, the accelerations of these points will have both tangential and normal components.*



Since the relative-acceleration components represent the effect of *circular motion* observed from translating axes having their origin at the base point $A$, these terms can be expressed as $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$ and $(\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A}$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

where

$\mathbf{a}_B$ = acceleration of point $B$

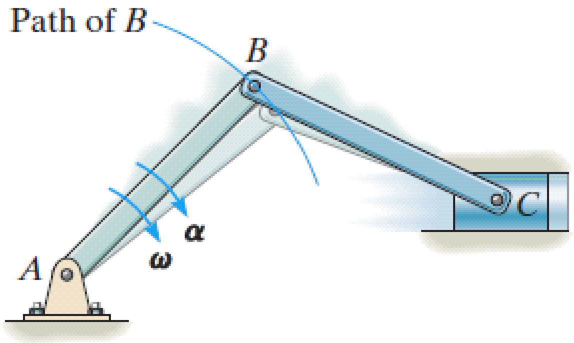
$\mathbf{a}_A$ = acceleration of the base point $A$

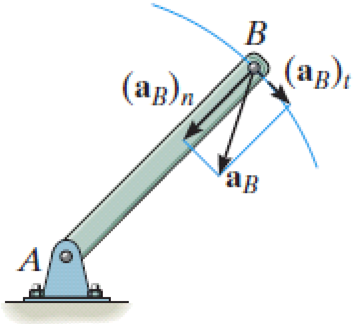
$\boldsymbol{\alpha}$ = angular acceleration of the body

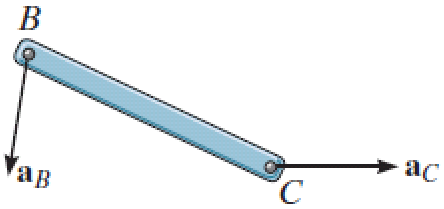
$\boldsymbol{\omega}$ = angular velocity of the body

$\mathbf{r}_{B/A}$ = position vector directed from $A$ to $B$

If equations are applied in a practical manner to study the accelerated motion of a rigid body which is *pin connected to two other* bodies, it should be realized that points which are *coincident at the pin* move with the *same acceleration, since the path of motion over which* they travel is the *same. For example, point **B** lying on either rod **BA** or **BC** of the crank mechanism shown has the same* acceleration, since the rods are pin connected at ***B***. *Here the motion of **B*** is along a *circular path, so that* **$a_B$** ***can be expressed in terms of its*** tangential and normal components. At the other end of rod ***BC*** *point **C*** moves along a *straight-lined path, which is defined by the piston. Hence,* **$a_C$** ***is horizontal***



Finally, consider a disk that rolls without slipping as shown

As a result, $v_A = 0$ and so from the kinematic diagram, the velocity of the mass center $G$ is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\omega\mathbf{k}) \times (r\mathbf{j})$$

So that

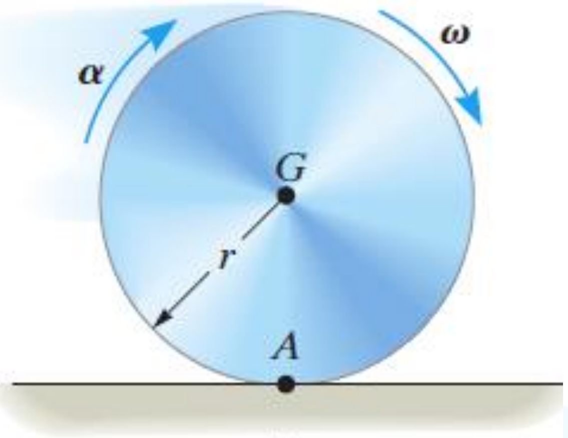
$$v_G = \omega r$$

This same result can also be determined using the IC method where point $A$ is the $IC$.

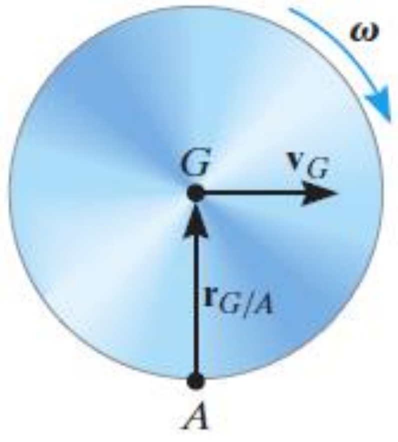
Since $G$ moves along a *straight line*, its acceleration in this case can be determined from the time derivative of its velocity.

$$\frac{dv_G}{dt} = \frac{d\omega}{dt} r$$

$$a_G = \alpha r$$



(a)



(b)

## Procedure for Analysis

The relative acceleration equation can be applied between any two points $A$ and $B$ on a body either by using a Cartesian vector analysis, or by writing the $x$ and $y$ scalar component equations directly.

### Velocity Analysis.

- Determine the angular velocity $\boldsymbol{\omega}$ of the body by using a velocity analysis . Also, determine velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points $A$ and $B$ *if these points move along curved paths.*

## Vector Analysis

### Kinematic Diagram.

- Establish the directions of the fixed *x*, *y* coordinates and draw the kinematic diagram of the body. Indicate on it $\mathbf{a}_A$, $\mathbf{a}_B$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{r}_{B/A}$.
- If points *A* and *B* move along *curved paths*, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\mathbf{a}_A = (\mathbf{a}_A)_t + (\mathbf{a}_A)_n$ and $\mathbf{a}_B = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n$.

### Acceleration Equation.

- To apply $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective **i** and **j** components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates that the sense of direction of the vector is opposite to that shown on the kinematic diagram.

## Scalar Analysis

### Kinematic Diagram.

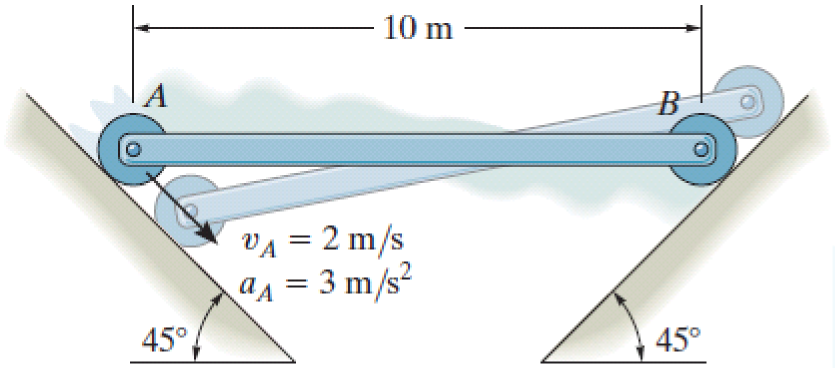
- If the acceleration equation is applied in scalar form, then the magnitudes and directions of the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$ must be established. To do this draw a kinematic diagram such as shown. Since the body is considered to be momentarily "pinned" at the base point $A$, the *magnitudes* of these components are $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Their *sense of direction* is established from the diagram such that $(\mathbf{a}_{B/A})_t$ acts perpendicular to $\mathbf{r}_{B/A}$, in accordance with the rotational motion $\boldsymbol{\alpha}$ of the body, and $(\mathbf{a}_{B/A})_n$ is directed from $B$ toward $A$.

### Acceleration Equation.

- Represent the vectors in $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$ graphically by showing their magnitudes and directions underneath each term. The scalar equations are determined from the $x$ and $y$ components of these vectors.

## EXAMPLE

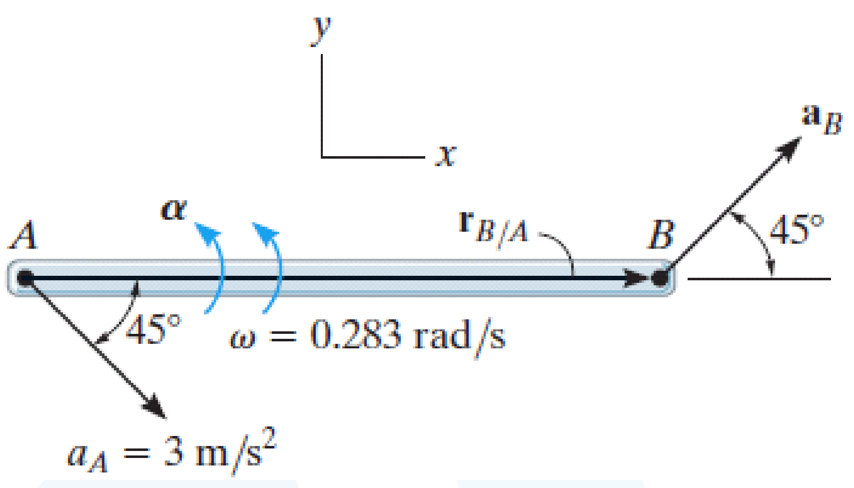
The rod $AB$ shown is confined to move along the inclined planes at $A$ and $B$. If point $A$ has an acceleration of $3\ \text{m/s}^2$ and a velocity of $2\ \text{m/s}$, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.




### SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points $A$ and $B$ on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283\ \text{rad/s}$ ↺ using either the velocity equation or the method of instantaneous centers.

**Kinematic Diagram.** Since points $A$ and $B$ both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns , namely, $a_B$ and $\alpha$.




**Acceleration Equation.**

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

$$a_B \cos 45°\mathbf{i} + a_B \sin 45°\mathbf{j} = 3 \cos 45°\mathbf{i} - 3 \sin 45°\mathbf{j} + (\alpha\mathbf{k}) \times (10\mathbf{i}) - (0.283)^2(10\mathbf{i})$$

Carrying out the cross product and equating the **i** and **j** components yields

$$a_B \cos 45° = 3 \cos 45° - (0.283)^2(10) \qquad (1)$$

$$a_B \sin 45° = -3 \sin 45° + \alpha(10) \qquad (2)$$

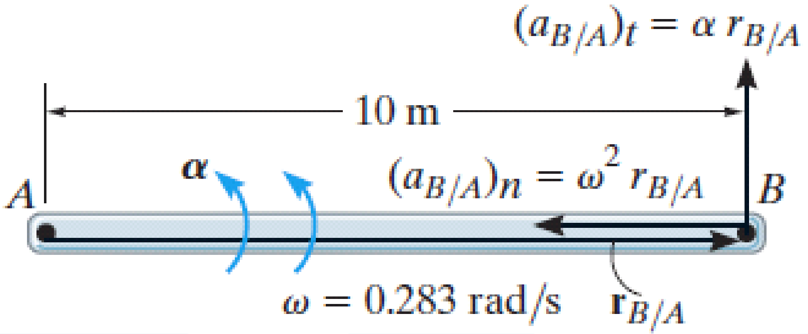
Solving, we have

$$a_B = 1.87 \text{ m/s}^2 \measuredangle 45°$$

$$\alpha = 0.344 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

# SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$, we have



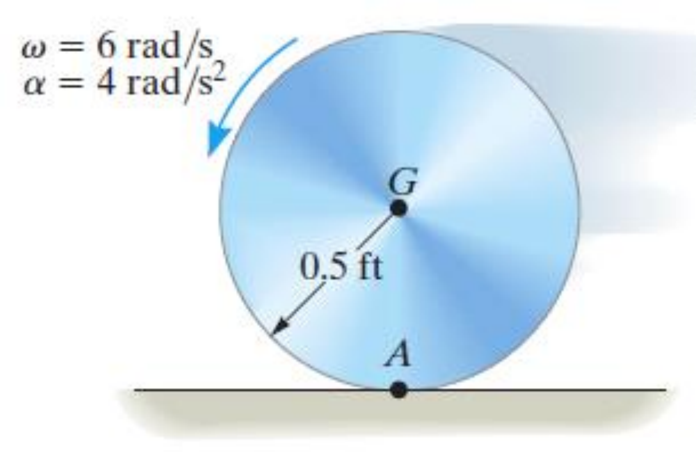

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$\begin{bmatrix} a_B \\ \measuredangle 45^\circ \end{bmatrix} = \begin{bmatrix} 3\text{ m/s}^2 \\ \triangleleft 45^\circ \end{bmatrix} + \begin{bmatrix} \alpha(10\text{ m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283\text{ rad/s})^2(10\text{ m}) \\ \leftarrow \end{bmatrix}$$

Equating the $x$ and $y$ components yields Eqs. 1 and 2, and the solution proceeds as before.

## EXAMPLE

The disk rolls without slipping and has the angular motion shown Determine the acceleration of point $A$ at this instant.



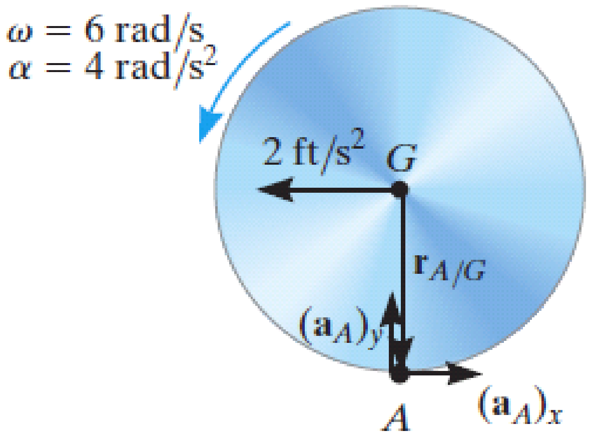

## SOLUTION I (VECTOR ANALYSIS)

**Kinematic Diagram.** Since no slipping occurs,

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

**Acceleration Equation.**

We will apply the acceleration equation to points $G$ and $A$



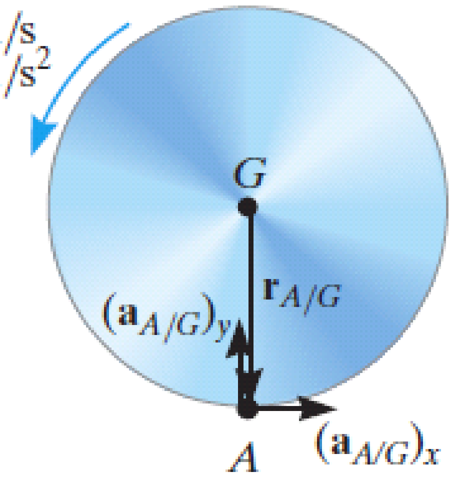

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} - \omega^2 \mathbf{r}_{A/G}$$

$$\mathbf{a}_A = -2\mathbf{i} + (4\mathbf{k}) \times (-0.5\mathbf{j}) - (6)^2(-0.5\mathbf{j})$$

$$= \{18\mathbf{j}\} \text{ ft/s}^2$$

## SOLUTION II (SCALAR ANALYSIS)

Using the result for $a_G = 2\ \text{ft/s}^2$ determined above, and from the kinematic diagram, showing the relative motion $\mathbf{a}_{A/G}$, we have




$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_x + (\mathbf{a}_{A/G})_y$$

$$\begin{bmatrix}(a_A)_x\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_A)_y\\ \uparrow\end{bmatrix} = \begin{bmatrix}2\ \text{ft/s}^2\\ \leftarrow\end{bmatrix} + \begin{bmatrix}(4\ \text{rad/s}^2)(0.5\ \text{ft})\\ \rightarrow\end{bmatrix} + \begin{bmatrix}(6\ \text{rad/s})^2(0.5\ \text{ft})\\ \uparrow\end{bmatrix}$$

$\overset{+}{\rightarrow}$ $\qquad (a_A)_x = -2 + 2 = 0$

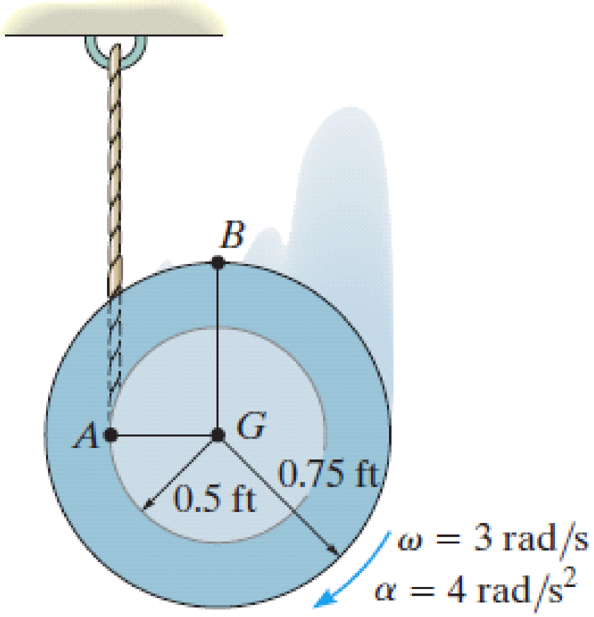
$+\uparrow$ $\qquad (a_A)_y = 18\ \text{ft/s}^2$

Therefore,

$$a_A = \sqrt{(0)^2 + (18\ \text{ft/s}^2)^2} = 18\ \text{ft/s}^2 \qquad \textit{Ans.}$$

EXAMPLE

The spool shown unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s$^2$. Determine the acceleration of point $B$.




## SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point $A$. Therefore, we can determine the acceleration of point $G$

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

We will apply the acceleration equation to points $G$ and $B$.

**Kinematic Diagram.** Point $B$ moves along a *curved path* having an *unknown* radius of curvature. Its acceleration will be represented by its unknown $x$ and $y$ components as shown

**Acceleration Equation.**

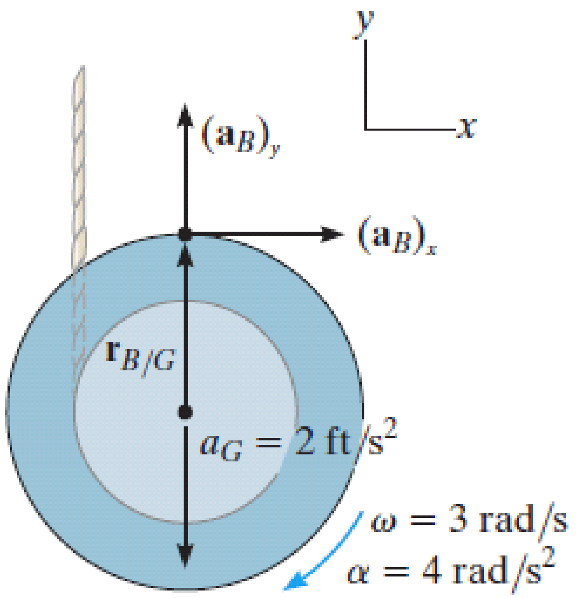
$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \omega^2 \mathbf{r}_{B/G}$$

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -2\mathbf{j} + (-4\mathbf{k}) \times (0.75\mathbf{j}) - (3)^2(0.75\mathbf{j})$$

Equating the **i** and **j** terms, the component equations are

$$(a_B)_x = 4(0.75) = 3 \text{ ft/s}^2 \rightarrow \qquad (1)$$

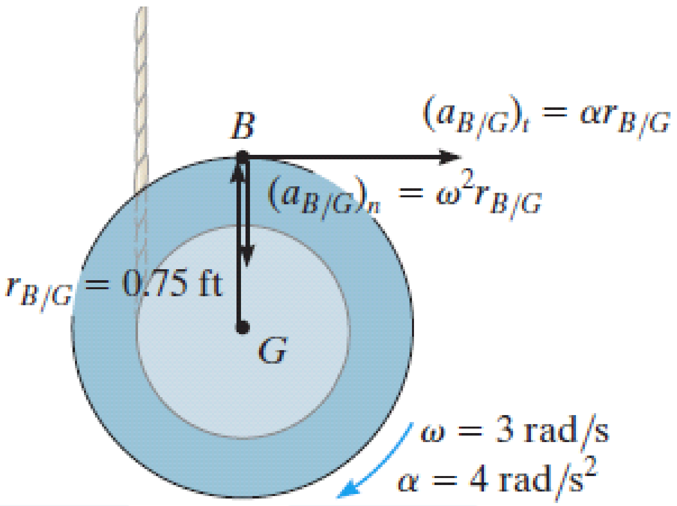
$$(a_B)_y = -2 - 6.75 = -8.75 \text{ ft/s}^2 = 8.75 \text{ ft/s}^2 \downarrow \qquad (2)$$



The magnitude and direction of $\mathbf{a}_B$ are therefore

$$a_B = \sqrt{(3)^2 + (8.75)^2} = 9.25 \text{ ft/s}^2 \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{8.75}{3} = 71.1° \qquad \textit{Ans.}$$




## SOLUTION II (SCALAR ANALYSIS)

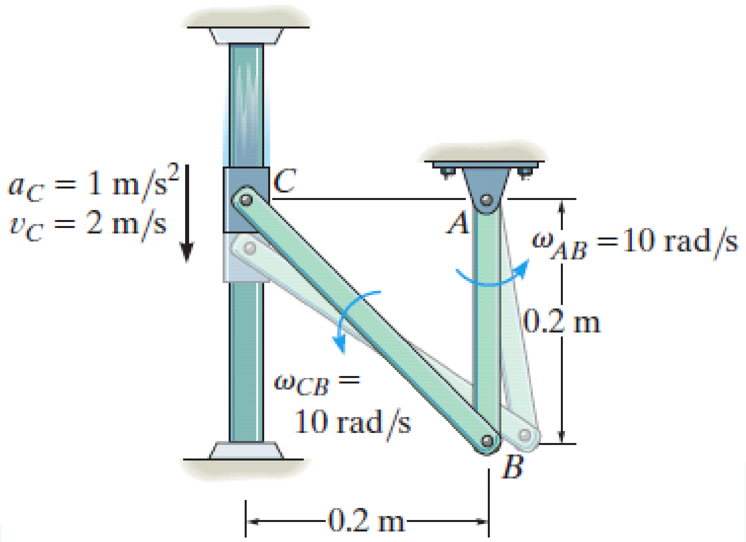
This problem may be solved by writing the scalar component equations directly. The kinematic diagram shows the relative-acceleration components $(\mathbf{a}_{B/G})_t$ and $(\mathbf{a}_{B/G})_n$. Thus,

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$

$$\begin{bmatrix}(a_B)_x \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(a_B)_y \\ \uparrow\end{bmatrix} = \begin{bmatrix}2\ \text{ft/s}^2 \\ \downarrow\end{bmatrix} + \begin{bmatrix}4\ \text{rad/s}^2\ (0.75\ \text{ft}) \\ \rightarrow\end{bmatrix} + \begin{bmatrix}(3\ \text{rad/s})^2(0.75\ \text{ft}) \\ \downarrow\end{bmatrix}$$

The $x$ and $y$ components yield Eqs. 1 and 2 above.

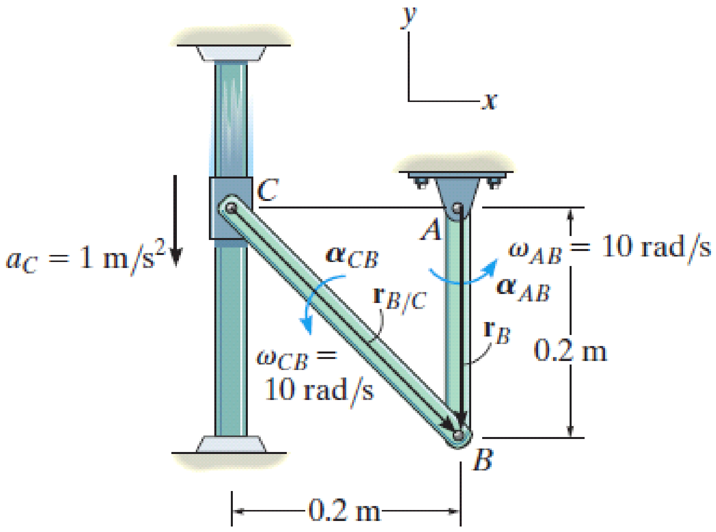
## EXAMPLE




The collar $C$ moves downward with an acceleration of $1\ \text{m/s}^2$. At the instant shown, it has a speed of $2\ \text{m/s}$ which gives links $CB$ and $AB$ an angular velocity $\omega_{AB} = \omega_{CB} = 10\ \text{rad/s}$. Determine the angular accelerations of $CB$ and $AB$ at this instant.

## SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams of *both* links $AB$ and $CB$ are shown . To solve, we will apply the appropriate kinematic equation to each link.




**Acceleration Equation.**

Link $AB$ (rotation about a fixed axis):

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$

$$\mathbf{a}_B = (\alpha_{AB}\mathbf{k}) \times (-0.2\mathbf{j}) - (10)^2(-0.2\mathbf{j})$$

$$\mathbf{a}_B = 0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j}$$

Note that $\mathbf{a}_B$ has $n$ and $t$ components since it moves along a *circular path*.

Link $BC$ (general plane motion):

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \omega_{CB}^2 \mathbf{r}_{B/C}$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^2(0.2\mathbf{i} - 0.2\mathbf{j})$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\alpha_{CB}\mathbf{j} + 0.2\alpha_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$

$$20 = -1 + 0.2\alpha_{CB} + 20$$

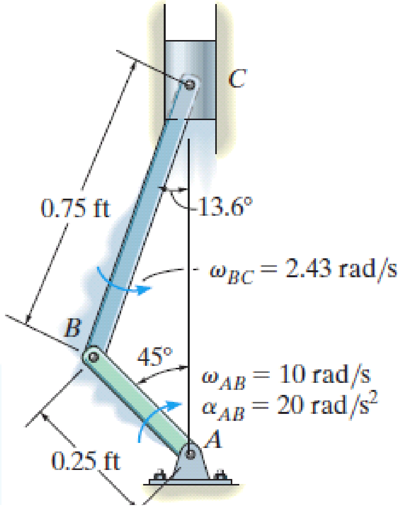
Solving,

$$\alpha_{CB} = 5 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

$$\alpha_{AB} = -95 \text{ rad/s}^2 = 95 \text{ rad/s}^2 \circlearrowright \qquad \textit{Ans.}$$

## EXAMPLE

The crankshaft $AB$ turns with a clockwise angular acceleration of $20\ \text{rad/s}^2$. Determine the acceleration of the piston at the instant $AB$ is in the position shown. At this instant $\omega_{AB} = 10\ \text{rad/s}$ and $\omega_{BC} = 2.43\ \text{rad/s}$.



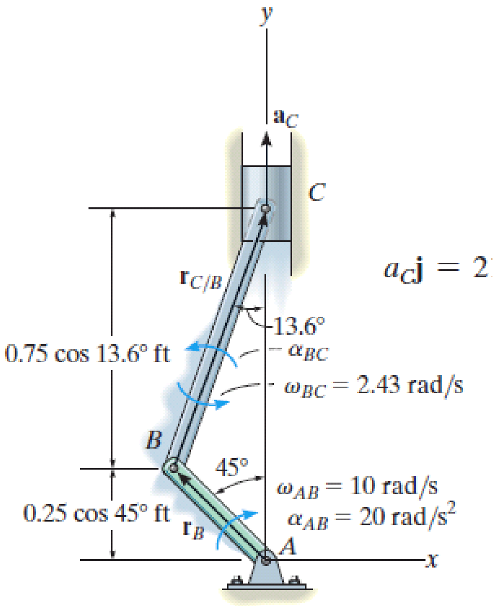

## SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams for both $AB$ and $BC$ are shown. Here $\mathbf{a}_C$ is vertical since $C$ moves along a straight-line path.

**Acceleration Equation.** Expressing each of the position vectors in Cartesian vector form

$$\mathbf{r}_B = \{-0.25 \sin 45°\mathbf{i} + 0.25 \cos 45°\mathbf{j}\}\ \text{ft} = \{-0.177\mathbf{i} + 0.177\mathbf{j}\}\ \text{ft}$$

$$\mathbf{r}_{C/B} = \{0.75 \sin 13.6°\mathbf{i} + 0.75 \cos 13.6°\mathbf{j}\}\ \text{ft} = \{0.177\mathbf{i} + 0.729\mathbf{j}\}\ \text{ft}$$



Crankshaft $AB$ (rotation about a fixed axis):

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$
$$= (-20\mathbf{k}) \times (-0.177\mathbf{i} + 0.177\mathbf{j}) - (10)^2(-0.177\mathbf{i} + 0.177\mathbf{j})$$
$$= \{21.21\mathbf{i} - 14.14\mathbf{j}\} \text{ ft/s}^2$$

Connecting Rod $BC$ (general plane motion): Using the result for $\mathbf{a}_B$ and noting that $\mathbf{a}_C$ is in the vertical direction, we have

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$
$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.177\mathbf{i} + 0.729\mathbf{j}) - (2.43)^2(0.177\mathbf{i} + 0.729\mathbf{j})$$
$$a_C\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + 0.177\alpha_{BC}\mathbf{j} - 0.729\alpha_{BC}\mathbf{i} - 1.04\mathbf{i} - 4.30\mathbf{j}$$
$$0 = 20.17 - 0.729\alpha_{BC}$$
$$a_C = 0.177\alpha_{BC} - 18.45$$

Solving yields

$$\alpha_{BC} = 27.7 \text{ rad/s}^2 \circlearrowleft$$

$$a_C = -13.5 \text{ ft/s}^2 \qquad \textit{Ans.}$$

**NOTE:** Since the piston is moving upward, the negative sign for $a_C$ indicates that the piston is decelerating, i.e., $\mathbf{a}_C = \{-13.5\mathbf{j}\}$ ft/s$^2$. This causes the speed of the piston to decrease until $AB$ becomes vertical, at which time the piston is momentarily at rest.

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