Tension and Compression in Bars

1. Stress الإجهاد
2. Strain التشوه (الانفعال)
3. Constitutive Law قانون السلوك
4 .Single Bar under Tension or Compression قضيب مفرد: شد أو ضغط
5. Systems of Bars جمل القضبان
6. Supplementary Examples أمثلة إضافية

Objectives: *Mechanics of Materials* investigates the stressing and the deformations of structures subjected to applied loads, starting by the simplest structural members, namely, bars in tension or compression.

يدرس ميكانيك المواد إجهادات وتشوهات الجمل الإنشائية (الهياكل الحاملة) الناتجة عن الحمولات الخارجية، مبتدأً بالعناصر الأبسط أي القضبان (العناصر الطولية) المشدودة أو المضغوطة.

In order to treat such problems, the kinematic relations and a constitutive law are needed to complement the equilibrium conditions which are known from Engineering Mechanics (Statics).

تقوم هذه الدراسة على:

(1) معادلات التوازن التي دُرست في الميكانيك الهندسي (علم السكون)

(2) العلاقات الكينماتيكية التي ستدرس وهي تصف التشوهات كمياً أي تحدد شكل ومقدار تغيرات الشكل الجيومتري.

(3) قوانين سلوك مادة الجملة وهي كما ستُعرض لاحقاً، قوانين تجريبية تعرّف السلوك الميكانيكي لمادة الهيكل الحامل.

The kinematic relations represent the geometry of the deformation, whereas the behavior of the material is described by the constitutive law. The students will learn how to apply these equations and how to solve determinate as well as statically indeterminate problems. يعالج الطلبة مسائل مقررة سكونياً وأخرى غير مقررة سكونياً

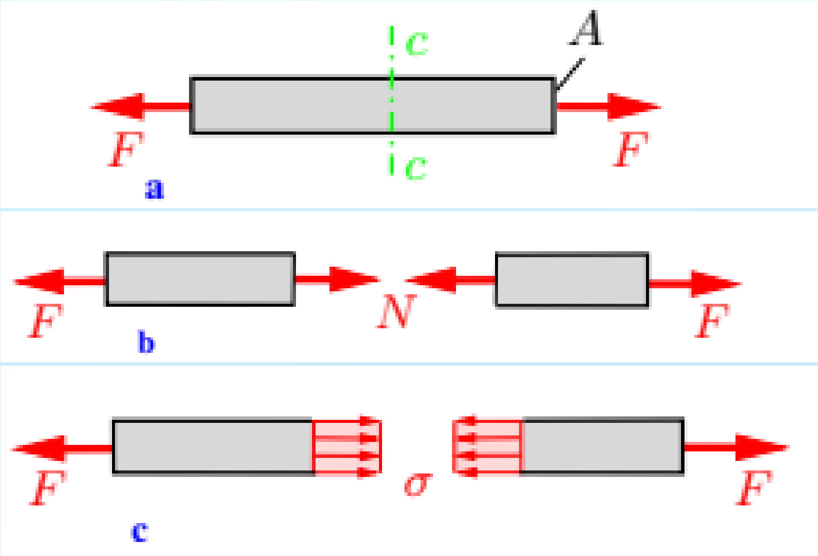
؟؟

1. Stress
الإجهاد

Consider a straight bar with a constant cross-sectional area A.
Its *axis* is connecting the centroids of the cross sections.
Its ends are subjected to the forces F acting on the axis (Fig. a).

The *external* load causes *internal* forces, which can be visualized by an imaginary cut of the bar (Fig. b). They are distributed over the cross section and called *stresses* (Fig. c).

They have the dimension force/area, for example, as multiples of MPa (1MPa=1N/mm^2). The "Pascal" (1 Pa=1 N/m^2) after the mathematician & physicist Blaise Pascal (1623–1662). The notion of "stress" was introduced by Augustin Louis Cauchy (1789–1857).

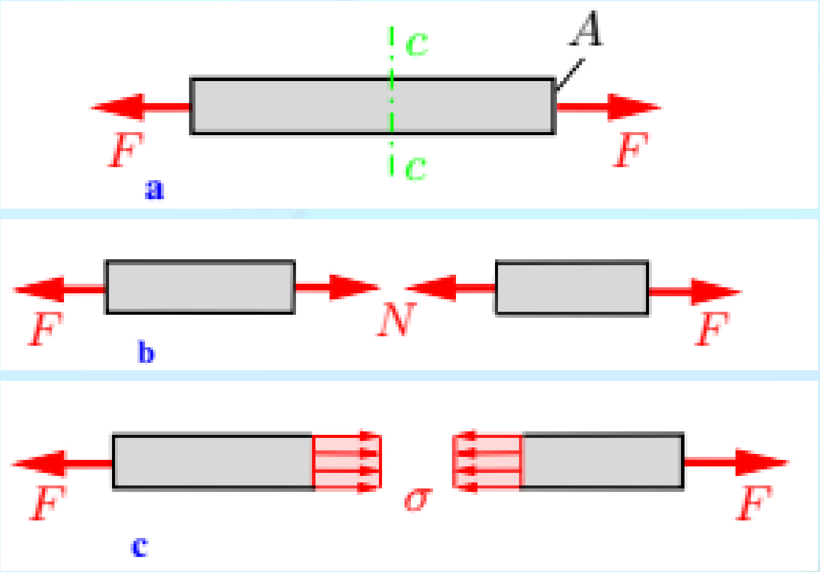


In (Statics) we only dealt with the resultant of the stresses : The internal forces.

To determine the stresses we make an imaginary cut *c-c* perpendicular to the bar axis The stresses are shown in the free-body diagram (Fig. c); they are denoted by σ.

We assume that they act perpendicularly to the exposed surface *A* of the cross section and that they are uniformly distributed.

Since they are normal to the cross section they are called *normal stresses*. Their resultant is the normal force *N* shown in (Fig. b) .



Therefore we have $N=\sigma A$ and the stresses σ can be calculated from the normal force *N*:

$$\sigma = \frac{N}{A}$$

In the present example the normal force *N* is equal to the applied force *F*. Thus, we write the last equation as

$$\sigma = \frac{F}{A}$$

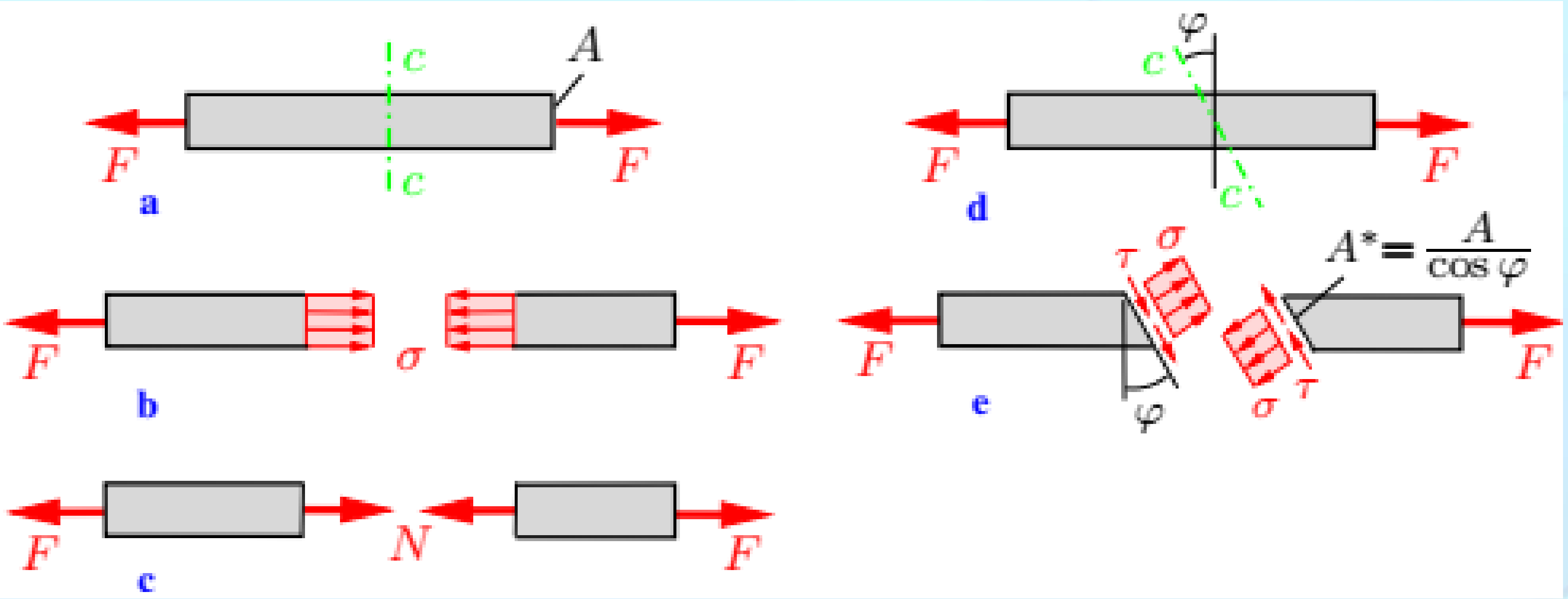
For a positive normal force *N* (tension) the stress σ is then positive (tensile stress). Reversely, if the normal force is negative (compression) the stress is also negative (compressive stress)

Let us now imagine the bar being sectioned by a cut which is not orthogonal to the axis of the bar so that its direction is given by the angle ϕ (Fig. d).

$$A^* = A/\cos\varphi.$$

Again we assume that they are uniformly distributed.

Resolve the stresses into a component σ perpendicular to the surface (normal stress) & a component τ tangential to the surface (shear stress) (Fig. e).



Equilibrium of the forces acting on the left portion of the bar (see Fig. e) yields:

$$\rightarrow: \ \sigma A^* \cos\varphi + \tau A^* \sin\varphi - F = 0$$
$$\uparrow: \quad \sigma A^* \sin\varphi - \tau A^* \cos\varphi = 0$$

$\rightarrow$: $\sigma A^* \cos\varphi + \tau A^* \sin\varphi - F = 0$

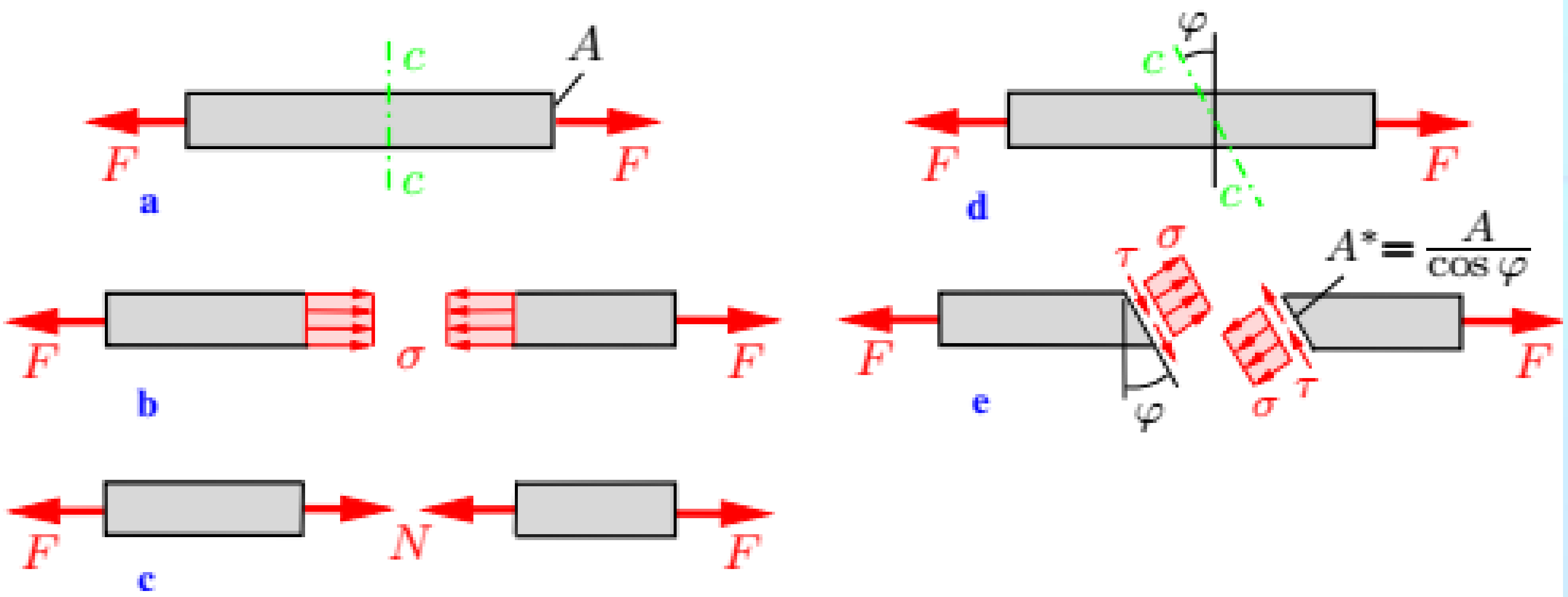
$\uparrow$: $\sigma A^* \sin\varphi + \tau A^* \cos\varphi = 0$

These Eq. Eqs. are written for the *forces, not* for the *stresses*. With $A^* = A/\cos\varphi$ we obtain

$$\begin{cases} \sigma + \tau \tan\varphi = \frac{F}{A} \\ \sigma \tan\varphi - \tau = 0 \end{cases}$$

Solving yields

$$\begin{cases} \sigma = \frac{1}{1+\tan^2\varphi}\frac{F}{A} \\ \tau = \frac{\tan\varphi}{1+\tan^2\varphi}\frac{F}{A} \end{cases}$$



It is practical to write these equations in a different form. Using the trigonometric relations

$$\frac{1}{1+\tan^2\varphi} = \cos^2\varphi = \tfrac{1}{2}(1+\cos 2\varphi), \quad \frac{\tan\varphi}{1+\tan^2\varphi} = \sin\varphi\cos\varphi\,, \quad \sin 2\varphi = \frac{2\tan\varphi}{1+\tan^2\varphi}, \quad \cos 2\varphi = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

and the abbreviation $\sigma_0 = F/A$ (normal stress in a section perpendicular to the axis) we get

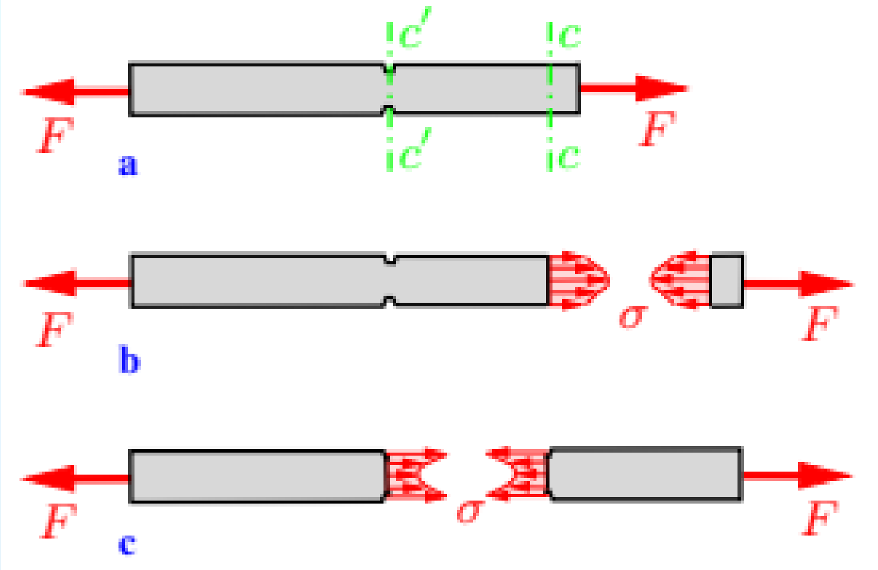
$$\sigma = \frac{\sigma_0}{2}(1+\cos 2\varphi), \qquad \tau = \frac{\sigma_0}{2}\sin 2\varphi$$

Stresses depend on the direction of the cut. If σ_0 is known, σ & τ can be calculated for any ϕ. The maximum value of σ is obtained for $\phi = 0$, where $\sigma_{max} = \sigma_0$; the maximum value of τ is found for $\phi = \pi/4$ where $\tau_{max} = \sigma_0/2$.

Two dangerous cuts:

If we section a bar near an end which is subjected to a concentrated force F (Fig. a, section *c-c*) we find that the normal stress is not distributed uniformly over area.

The concentrated force produces high stresses near it (Fig.b).



It can be shown that this *stress concentration* is restricted to sections close to the end concentrated force: the high stresses decay rapidly towards the average value σ_0 far from the end of the bar. This fact is referred to as *Saint-Venant's principle* (Adhémar Jean Claude Barré de Saint-Venant, 1797-1886).

The uniform distribution of the stress is also disturbed by holes, notches or any abrupt changes (discontinuities) of the geometry. If, for example, a bar has notches the remaining cross-sectional area (section *c-c*) is also subjected to a stress concentration (Fig. c). The determination of these stresses is not possible with the elementary analysis presented in this course.

General Case Let us now consider a bar with only a *slight* taper (See Example 1). In this case the normal stress may be not calculated from $\sigma = \frac{N}{A}$, with a sufficient accuracy.

In the General case the area A and the stress σ depend on the location along the axis. If volume forces act in the direction of the axis in addition to the concentrated forces, then the normal force N also depends on the location. Introducing the coordinate x in the direction of the axis we can write:

Here it is also assumed that the stress is uniformly distributed over the cross section at x.

$$\sigma(x) = \frac{N(x)}{A(x)}$$

Design Criteria (Requirement) معيار أو متطلبات التصميم

In applications structures have to be designed in such a way that a given maximum stressing is not exceeded.

In the case of a bar this requirement means that the absolute value of the stress σ must not exceed a given *allowable stress* σ_{allow}: $|\sigma| \leq \sigma_{allow}$. The required cross section A_{req} of a bar for a given load and thus a known normal force N can then be determined from :

$$A_{req} = |N|/\sigma_{allow}$$

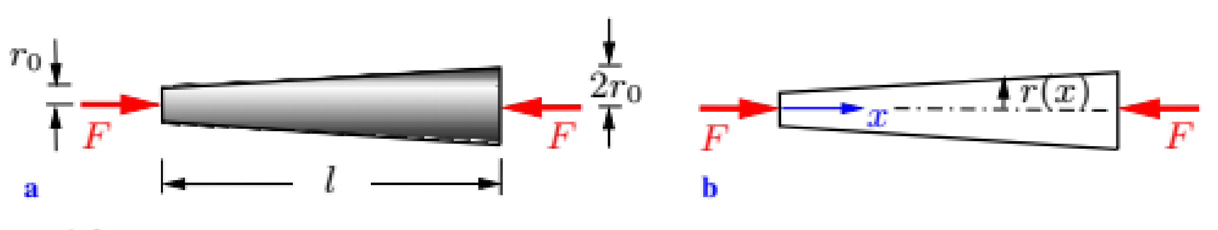
This is referred to as *dimensioning* of the bar. Alternatively, the allowable load can be calculated from $|N| \leq A\sigma_{allow}$ in the case of a given cross-sectional area A.

Note that a slender bar which is subjected to compression may fail due to buckling before the stress attains an inadmissibly large value. We will investigate buckling problems in Mechanics of Materials 2.

Example 1

A bar (length l) with a circular cross section and a slight taper (linearly varying from radius r_0 to $2r_0$) is subjected to the compressive forces *F* as shown in Fig.a. Determine the normal stress *σ* in an arbitrary cross section perpendicular to the axis of the bar.

Solution We introduce the coordinate *x*, see Fig.b. Then the radius of an arbitrary cross section is given by



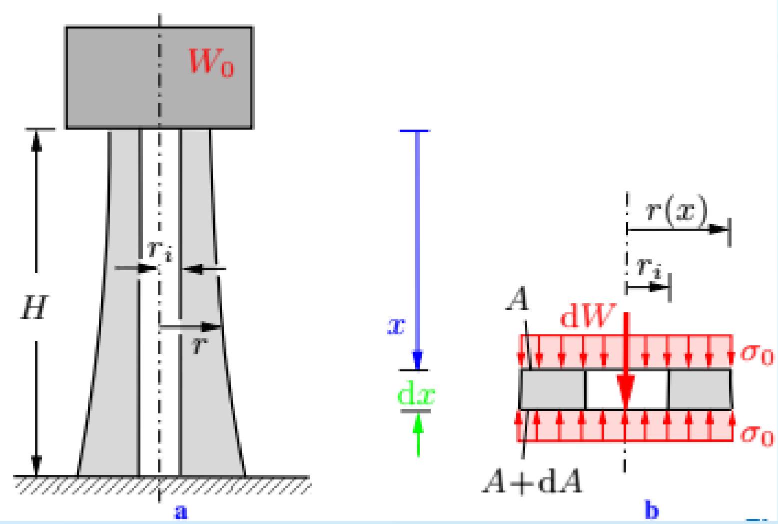
$$r(x) = r_0 + \frac{r_0}{l}x = r_0(1 + \frac{x}{l})$$

Using $\sigma = N/A$ with the cross section $A(x) = \pi\, r^2(x)$ and the constant normal force $N = -F$, yields

$$\sigma(x) = \frac{N}{A(x)} = \frac{-F}{\pi r_0^2 (1 + \frac{x}{l})^2}$$

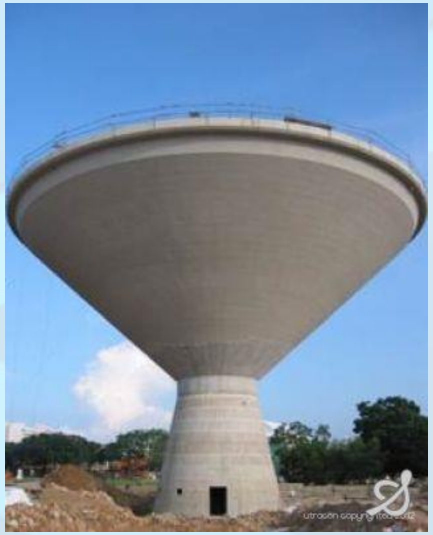
The minus sign indicates that σ is a compressive stress. Its value at the left end ($x = 0$) is four times the value at the right end ($x = l$).

Example 2 A water tower (height H, density ρ) with a cross section in the form of a circular ring carries a tank (weight W_0) as shown in Fig. a. The inner radius r_i of the ring is constant. Determine the outer radius r in such a way that the normal stress σ_0 in the tower is constant along its height. The weight of the tower cannot be neglected.



Solution: Consider the tower to be a slender bar. The relation between stress, normal force and cross-sectional area is given by $\sigma = N/A$. In this example the constant compressive stress $\sigma = \sigma_0$ is given; the normal force (here counted positive as compressive force) and the area A are unknown.

The equilibrium condition furnishes a second equation. We introduce the coordinate x as shown in Fig.b and consider a slice element of length dx. The cross-sectional area of the circular ring as a function of x is: $A = \pi(r^2 - r_i^2)$ where $r = r(x)$ is the unknown outer radius. The normal force at the location x is given by $N = \sigma_0 A$. At the location $x + dx$, the area and the normal force are $A + dA$ and $N + dN = \sigma_0(A + dA)$.



The weight of the element is $dW = \rho g\, dV$ where $dV = A\, dx$ is the volume of the element. Note that terms of higher order are neglected. Equilibrium in the vertical direction yields

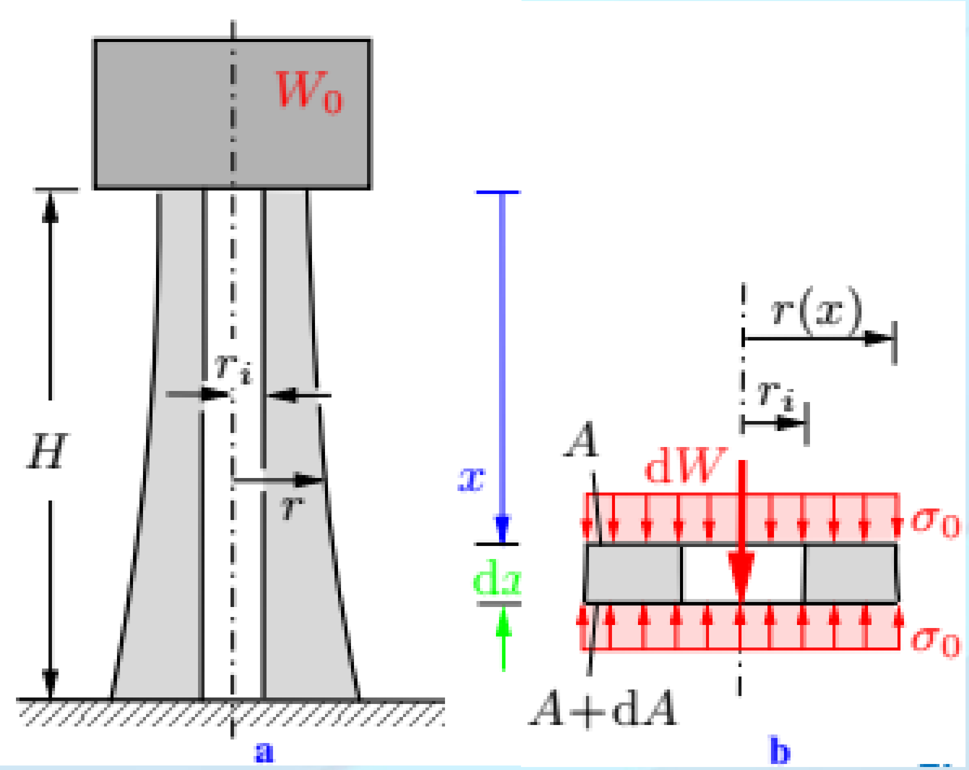
$$\uparrow:\ \sigma_0(A + dA) - \rho g dV - \sigma_0 A = 0 \ \Rightarrow\ \sigma_0 dA - \rho g A dx = 0\,.$$

Separation of variables and integration lead to

$$\int \frac{dA}{A} = \int \frac{\rho g}{\sigma_0} dx \ \Rightarrow \ln\frac{A}{A_0} = \frac{\rho g x}{\sigma_0} \ \Rightarrow A = A_0 e^{\frac{\rho g x}{\sigma_0}}.$$

The constant of integration A_0 follows from the condition that the stress at the upper end of the tower (for $x = 0 : N = W_0$) also has to be equal to σ_0:

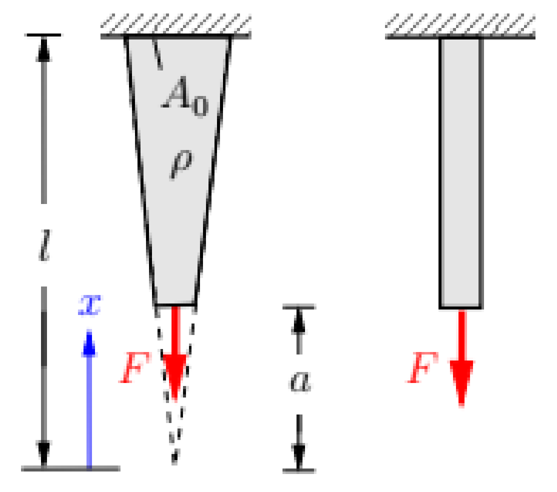
$$\frac{W_0}{A_0} = \sigma_0 \ \Rightarrow\ A_0 = \frac{W_0}{\sigma_0}.$$



Substituting this into the above two Equations yield the outer radius:

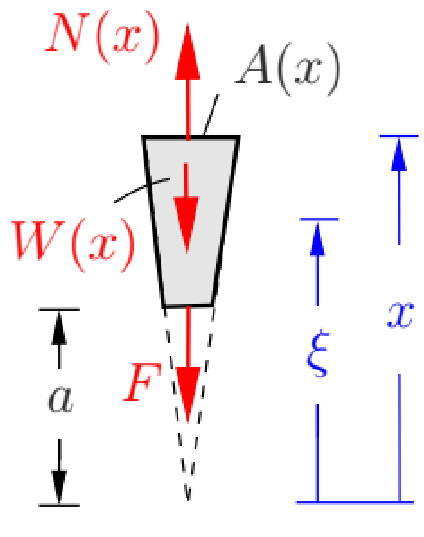
$$r^2(x) = r_i^2 + \frac{W_0}{\pi\sigma_0} e^{\frac{\rho g x}{\sigma_0}}$$

Example 3 A slender bar (density ρ) is suspended from its upper end as shown in Fig. It has a rectangular cross section with a constant depth and a linearly varying width. The cross section at the upper end is A_0.
Determine the stress $\sigma(x)$ due to the force F and the weight of the bar. Calculate the minimum stress $\sigma_{\min}$ and its location.



Solution It is reasonable to introduce the x-coordinate at the intersection of the extended edges of the trapezoid. The x dependent cross section area follows then as: $A(x) = A_0 x/h$

With the weight: $W(x) = \rho g V(x) = \rho g \int_a^x A(\xi) d\xi = \rho g A_0 \dfrac{x^2 - a^2}{2h}$



of the lower part equilibrium provides: $N(x) = F + W(x) = F + \rho g A_0 \dfrac{x^2 - a^2}{2h}$

This leads to the stress $\sigma(x) = \dfrac{N(x)}{A(x)} = \dfrac{Fh + \frac{1}{2}\rho g A_0 (x^2 - a^2)}{A_0 x}$

The location x^* of the minimum is determined by condition: $\dfrac{d\sigma}{dx} = 0$

$$\frac{d\sigma}{dx} = \frac{\rho g A_0 x (A_0 x) - A_0 F h - \frac{1}{2}\rho g A_0^2 (x^2 - a^2)}{A_0^2 x^2} = \frac{\rho g A_0 (x^2 + a^2) - 2Fh}{2 A_0 x^2}$$

$$\frac{d\sigma}{dx} = 0 \Rightarrow \frac{\rho g A_0(x^2 + a^2) - 2Fh}{2A_0 x^2} = 0 \Rightarrow x^* = \sqrt{\frac{2Fh}{\rho g A_0} - a^2}$$

Where the value of the minimum stress is

$$\sigma_{min} = \sigma(x^*) = \rho g \sqrt{\frac{2Fh}{\rho g A_0} - a^2} = \rho g x^*$$

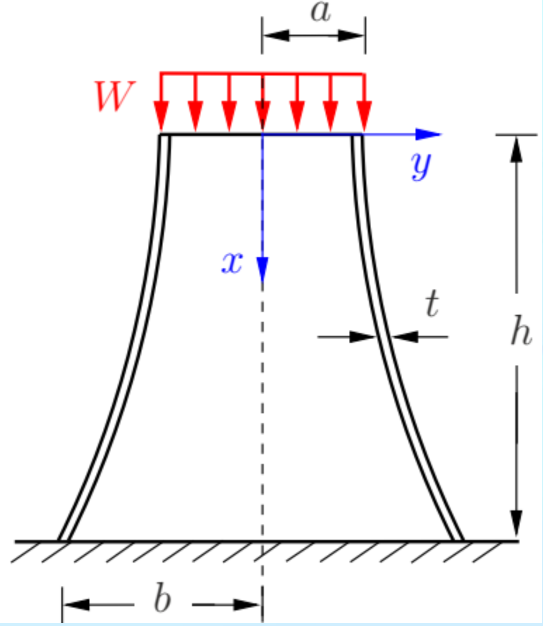
Note:

- For $\rho g = 0$ ("weightless bar") no minimum exists. The largest stress occurs at $x = a$.

- The minimum will be located within the bar, only if $a < x^* < h$ or $\rho g A_0 a^2/(2h) < F < \rho g A_0(h^2 + a^2)/(2h)$ holds.

Example 4 The contour of a light-house with circular thin-walled cross section follows a hyperbolic equation

$$y^2 - \frac{b^2 - a^2}{h^2}x^2 = a^2$$



Determine the stress distribution as a consequence of weight W of the lighthouse head (the weight of the structure can be neglected). Given: $b = 2a, t \ll a$

Solution As the weight W is the only acting external load, the normal force N is constant (compression): $N = -W$

The cross section area A is changing. It is approximated by (thin-walled structure with $t \ll y$)

$$A(x) = 2\pi yt = 2\pi t\sqrt{a^2 + \frac{b^2 - a^2}{h^2}x^2} = 2\pi t\sqrt{a^2 + \frac{3a^2}{h^2}x^2} = 2\pi at\sqrt{1 + \frac{3x^2}{h^2}}$$

The stress follows now as $$\sigma = \frac{N}{A} = -W\Big/\left(2\pi at\sqrt{1 + \frac{3x^2}{h^2}}\right)$$

Especially at the top & bottom position we get: $\sigma(x = 0) = -W/2\pi at, \sigma(x = h) = -W/4\pi at$

Note: The stress at the top is twice as large as the stress at the bottom, which is a inefficient use of material. This situation changes if the weight of the thin-walled structure is included in the analysis.