Calculus 1

Dr. Yamar Hamwi

Al-Manara University

2024-2025

Calculus 1

Lecture 5

Derivatives

Chapter 3

Derivatives

The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composition $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g. We develop the rule in this section.

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The Chain Rule

EXAMPLE 3 Differentiate $\sin(x^2 + x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\text{inside}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside}\\ \text{left alone}}} \cdot \underbrace{(2x+1)}_{\substack{\text{derivative of}\\ \text{the inside}}}.$$

The Chain Rule

EXAMPLE 4 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$$g'(t) = \frac{d}{dt}\tan(5 - \sin 2t)$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t)$$ Derivative of $\tan u$ with $u = 5 - \sin 2t$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right)$$ Derivative of $5 - \sin u$ with $u = 2t$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t).$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

The Chain Rule with Powers of a Function

EXAMPLE 5 The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$ — Power Chain Rule with $u = 5x^3 - x^4, n = 7$

$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$

(b) $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$

$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$ — Power Chain Rule with $u = 3x - 2, n = -1$

$= -1(3x-2)^{-2}(3)$

$= -\frac{3}{(3x-2)^2}$

In part (b) we could also find the derivative with the Quotient Rule.

(c) $\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$ — Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n, n \neq -1$.

$= 5\sin^4 x \cos x$ ■

The Chain Rule with Powers of a Function

جَامعة المَنارة
MANARA UNIVERSITY

EXAMPLE 6

$$\frac{d}{dx}(|x|) = \frac{d}{dx}\sqrt{x^2}$$

$$= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2)$$ Power Chain Rule with $u = x^2, n = 1/2, x \neq 0$

$$= \frac{1}{2|x|} \cdot 2x$$ $\sqrt{x^2} = |x|$

$$= \frac{x}{|x|}, \quad x \neq 0.$$

Implicit Differentiation

encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $dy\backslash dx$ by *implicit differentiation*. This section describes the technique

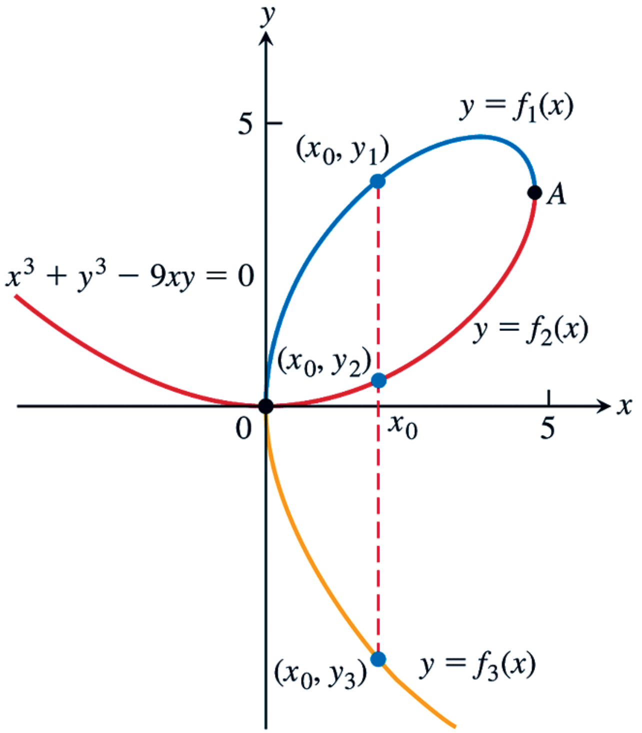


FIGURE 3.29 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Implicit Differentiation

EXAMPLE 1 Find dy/dx if $y^2 = x$.

$$y^2 = x$$

The Chain Rule gives

$$\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}.$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Implicit Differentiation

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

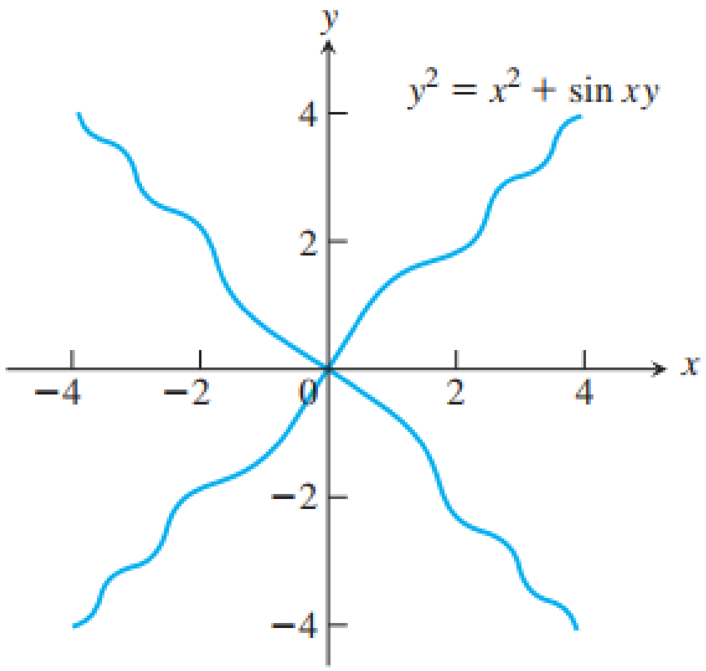
$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$



Derivatives of Higher Order

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

Tangent Lines

EXAMPLE 5 Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.32).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Differentiate both sides with respect to x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

Treat xy as a product and y as a function of x.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for dy/dx.

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at $(2, 4)$ is the line through $(2, 4)$ with slope $4/5$:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

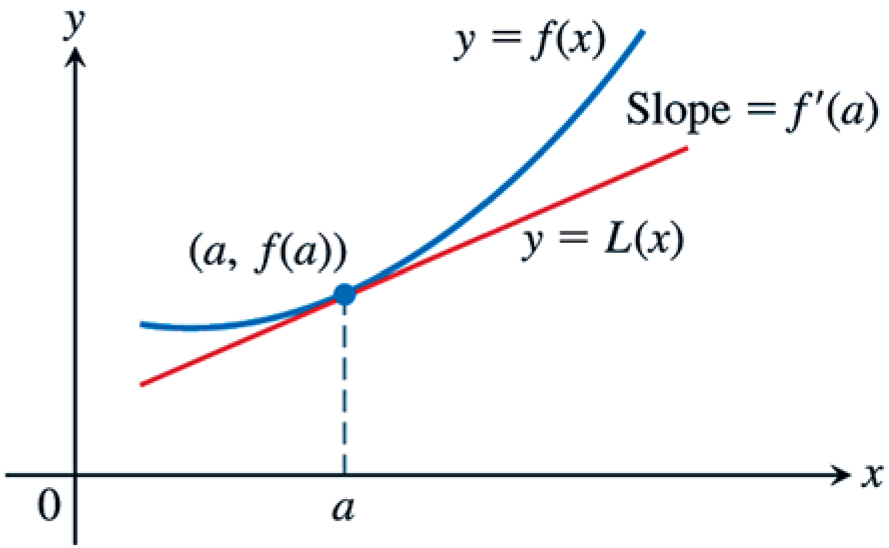
The normal to the curve at $(2, 4)$ is the line perpendicular to the tangent there, the line through $(2, 4)$ with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

■

Linearization



DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Linearization

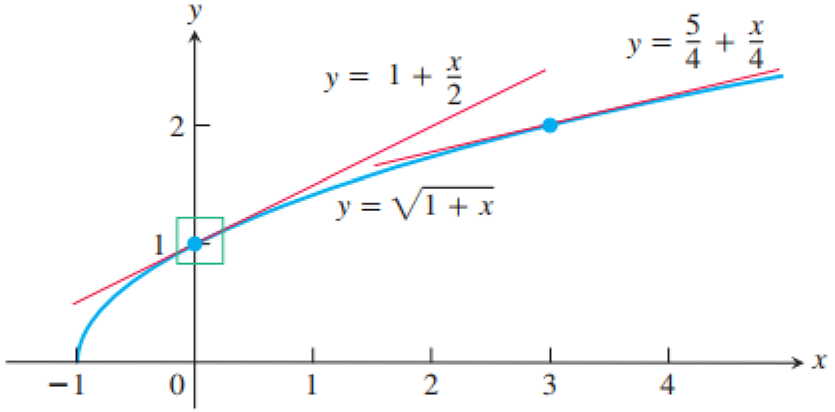
EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$

Solution Since

$$f'(x) = \frac{1}{2}(1+x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{1}{2}(x-0) = 1 + \frac{x}{2}.$$



$$\sqrt{1+x} \approx 1 + (x/2)$$

Linearization

Approximation	True value	$\lvert$True value $-$ approximation$\rvert$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

Linearization

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

Differentials

The geometric meaning of differentials

Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned} \Delta L &= L(a + dx) - L(a) \\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)} \\ &= f'(a)\,dx. \end{aligned}$$

Differentials

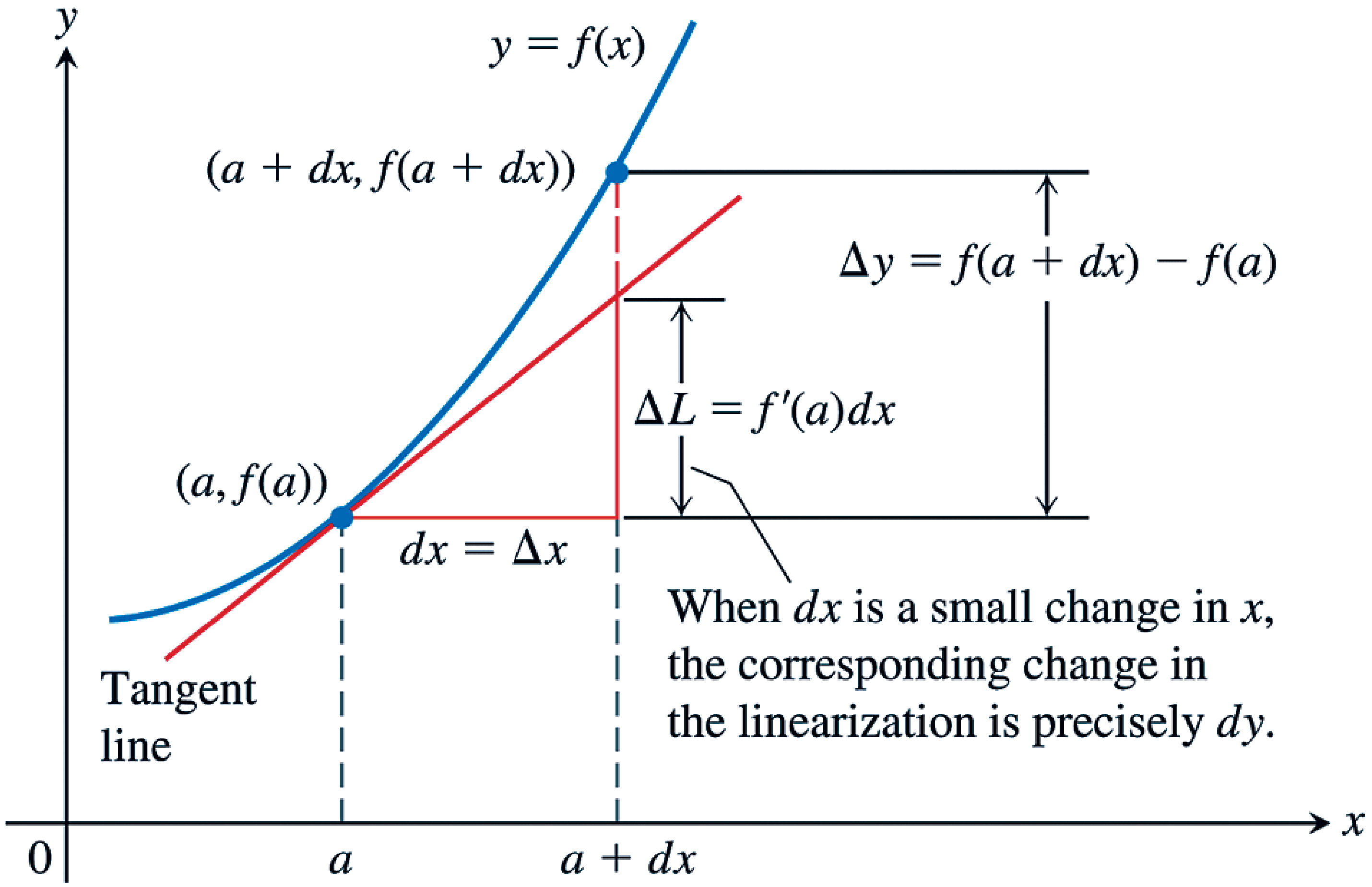


FIGURE 3.56 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Differentials

dy **represents the amount the tangent line rises or falls when *x* changes by an amount *dx* = Δ*x*.**

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f**. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Differentials

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u \, du.$$

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ■

Leibnitz Theorem Formula

The first derivative could be written as;

$$(uv)' = u'v+uv'$$

Now if we differentiate the above expression again, we get the second derivative;

$$(uv)'' = [(uv)']'$$

$$= (u'v+uv')'$$

$$= (u'v)'+(uv')'$$

$$= u''v + u'v' + u'\ v' + uv''$$

$$= u''v + 2u'v' + uv''$$

Leibnitz Theorem

$$(uv)^n = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^i$$

Where $\binom{n}{i}$ represents the number of i-combinations on n elements.

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g + \binom{n}{1} f^{(n-1)} \cdot g^{(1)} + \cdots$$

$$\ldots + \binom{n}{k} f^{(n-k)} \cdot g^{(k)} + \cdots + f \quad \cdot g^{(n)}$$

Tangent Lines and the Derivative at a Point

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$$

Example : Find the second derivative of the product of the functions x^2, and Tanx, using lebiniz rule.

Solution:

The given functions are $f(x) = x^2$, and $g(x) = \text{Tanx}$.

The leibniz rule for the product of two functions is $(f(x).g(x))'' = f''(x).g(x) + 2f'(x).g'(x) + f(x).g'(x)$.

$$\frac{d^2}{dx^2}.x^2.\text{Tanx} = \text{Tanx}.\frac{d^2}{dx^2}.x^2 + 2\frac{d}{dx}.x^2.\frac{d}{dx}.\text{Tanx} + x^2.\frac{d^2}{dx^2}$$

$$\frac{d^2}{dx^2}.x^2.\text{Tanx} = \text{Tanx}.2 + 2.2x.\text{Sec}^2x + x^2.2\text{Secx}.\text{Secx}.\text{Tan}$$

$$\frac{d^2}{dx^2}.x^2.\text{Tanx} = 2\text{Tanx} + 4x.\text{Sec}^2x + 2x^2\text{Sec}^2x.\text{Tanx}$$

Therefore the derivative of the product of two functions using leibniz rule is $2\text{Tanx} + 4x.\text{Sec}^2x + 2x^2\text{Sec}^2x.\text{Tanx}$.

Exercises

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = e^{-x}$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \frac{1}{x} + 7x$

1. $f(u)=6u-9 \Rightarrow f'(u)=6 \Rightarrow f'(g(x))=6; g(x)=\frac{1}{2}x^4 \Rightarrow g'(x)=2x^3;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=6\cdot 2x^3=12x^3$

2. $f(u)=2u^3 \Rightarrow f'(u)=6u^2 \Rightarrow f'(g(x))=6(8x-1)^2; g(x)=8x-1 \Rightarrow g'(x)=8;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=6(8x-1)^2\cdot 8=48(8x-1)^2$

3. $f(u)=\sin u \Rightarrow f'(u)=\cos u \Rightarrow f'(g(x))=\cos(3x+1); g(x)=3x+1 \Rightarrow g'(x)=3;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=(\cos(3x+1))(3)=3\cos(3x+1)$

4. $f(u)=\cos u \Rightarrow f'(u)=-\sin u \Rightarrow f'(g(x))=-\sin(e^{-x});\ \ g(x)=e^{-x} \Rightarrow g'(x)=-e^{-x};$ therefore,
$\frac{dy}{dx}=f'(g(x))g'(x)=-\sin(e^{-x})(-e^{-x})=e^{-x}\sin(e^{-x})$

5. $f(u)=\sqrt{u} \Rightarrow f'(u)=\frac{1}{2\sqrt{u}} \Rightarrow f'(g(x))=\frac{1}{2\sqrt{\sin x}};\ \ g(x)=\sin x \Rightarrow g'(x)=\cos x;$ therefore,
$\frac{dy}{dx}=f'(g(x))g'(x)=\frac{\cos x}{2\sqrt{\sin x}}$

6. $f(u)=\sin u \Rightarrow f'(u)=\cos u \Rightarrow f'(g(x))=\cos(x-\cos x); g(x)=x-\cos x \Rightarrow g'(x)=1+\sin x;$
therefore $\frac{dy}{dx}=f'g(x))g'(x)=(\cos(x-\cos x))(1+\sin x)$

7. $f(u)=\tan u \Rightarrow f'(u)=\sec^2 u \Rightarrow f'(g(x))=\sec^2(\pi x^2);\ g(x)=\pi x^2 \Rightarrow g'(x)=2\pi x;$

therefore $\frac{dy}{dx}=f'(g(x))g'(x)=\sec^2(\pi x^2)(2\pi x)=2\pi x\sec^2(\pi x^2)$

8. $f(u)=-\sec u \Rightarrow f'(u)=-\sec u\tan u \Rightarrow f'(g(x))=-\sec\left(\frac{1}{x}+7x\right)\tan(\frac{1}{x}+7x);\ g(x)=\frac{1}{x}+7x \Rightarrow$

$g'(x)=-\frac{1}{x^2}+7;$ therefore, $\frac{dy}{dx}=f'(g(x))g'(x)=\left(\frac{1}{x^2}-7\right)\sec(\frac{1}{x}+7x)\tan(\frac{1}{x}+7x)$

Exercises

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5\cos^{-4} x$

9. With $u=(2x+1)$, $y=u^5$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=5u^4\cdot 2=10(2x+1)^4$

10. With $u=(4-3x)$, $y=u^9$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=9u^8\cdot(-3)=-27(4-3x)^8$

11. With $u=\left(1-\frac{x}{7}\right)$, $y=u^{-7}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-7u^{-8}\cdot\left(-\frac{1}{7}\right)=\left(1-\frac{x}{7}\right)^{-8}$

12. With $u=\frac{\sqrt{x}}{2}-1$, $y=u^{-10}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-10u^{-11}\cdot\left(\frac{1}{4\sqrt{x}}\right)=-\frac{1}{4\sqrt{x}}\left(\frac{\sqrt{x}}{2}-1\right)^{-11}$

13. With $u=\left(\frac{x^2}{8}+x-\frac{1}{x}\right)$, $y=u^4$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=4u^3\cdot\left(\frac{x}{4}+1+\frac{1}{x^2}\right)=4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3\left(\frac{x}{4}+1+\frac{1}{x^2}\right)$

14. With $u=3x^2-4x+6$, $y=u^{1/2}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2}u^{-1/2}\cdot(6x-4)=\frac{3x-2}{\sqrt{3x^2-4x+6}}$

15. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x)\tan(\tan x))\sec^2 x$

16. With $u = \pi - \frac{1}{x}$, $y = \cot u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\csc^2 u)\left(\frac{1}{x^2}\right) = -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)$

17. With $u = \tan x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \sec^2 x = 3\tan^2 x \sec^2 x$

18. With $u = \cos x$, $y = 5u^{-4}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-20u^{-5})(-\sin x) = 20(\cos^{-5} x)(\sin x)$

Exercises

Find y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$

72. $y = (1 - \sqrt{x})^{-1}$

73. $y = \frac{1}{9}\cot(3x - 1)$

74. $y = 9\tan\left(\frac{x}{3}\right)$

75. $y = x(2x + 1)^4$

76. $y = x^2(x^3 - 1)^5$

71. $y=\left(1+\frac{1}{x}\right)^3 \Rightarrow y'=3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)=-\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y''=\left(-\frac{3}{x^2}\right)\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)^2-\left(1+\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{3}{x^2}\right)$

$=\left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right)+\left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2=\frac{6}{x^4}\left(1+\frac{1}{x}\right)+\frac{6}{x^3}\left(1+\frac{1}{x}\right)^2=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

72. $y=\left(1-\sqrt{x}\right)^{-1} \Rightarrow y'=-\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-1/2}\right)=\frac{1}{2}\left(1-\sqrt{x}\right)^{-2}x^{-1/2}$

$\Rightarrow y''=\frac{1}{2}\left[\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-3/2}\right)+x^{-1/2}(-2)\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$

$=\frac{1}{2}\left[\frac{-1}{2}x^{-3/2}\left(1-\sqrt{x}\right)^{-2}+x^{-1}\left(1-\sqrt{x}\right)^{-3}\right]=\frac{1}{2}x^{-1}\left(1-\sqrt{x}\right)^{-3}\left[-\frac{1}{2}x^{-1/2}\left(1-\sqrt{x}\right)+1\right]$

$=\frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right)=\frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

73. $y=\frac{1}{9}\cot(3x-1) \Rightarrow y'=-\frac{1}{9}\csc^2(3x-1)(3)=-\frac{1}{3}\csc^2(3x-1) \Rightarrow y''=\left(-\frac{2}{3}\right)(\csc(3x-1)\cdot\frac{d}{dx}\csc(3x-1))$

$=-\frac{2}{3}\csc(3x-1)(-\csc(3x-1)\cot(3x-1)\cdot\frac{d}{dx}(3x-1))=2\csc^2(3x-1)\cot(3x-1)$

74. $y=9\tan\left(\frac{x}{3}\right) \Rightarrow y'=9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right)=3\sec^2\left(\frac{x}{3}\right) \Rightarrow y''=3\cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right)=2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

75. $y=x(2x+1)^4 \Rightarrow y'=x\cdot 4(2x+1)^3(2)+1\cdot(2x+1)^4=(2x+1)^3(8x+(2x+1))=(2x+1)^3(10x+1)$

$\Rightarrow y''=(2x+1)^3(10)+3(2x+1)^2(2)(10x+1)=2(2x+1)^2(5(2x+1)+3(10x+1))=2(2x+1)^2(40x+8)$

$=16(2x+1)^2(5x+1)$

76. $y=x^2(x^3-1)^5 \Rightarrow y'=x^2\cdot 5(x^3-1)^4(3x^2)+2x(x^3-1)^5=x(x^3-1)^4[15x^3+2(x^3-1)]=(x^3-1)^4(17x^4-2x)$

$\Rightarrow y''=(x^3-1)^4(68x^3-2)+4(x^3-1)^3(3x^2)(17x^4-2x)=2(x^3-1)^3[(x^3-1)(34x^3-1)+6x^2(17x^4-2x)]$

$=2\left(x^3-1\right)^3\left(136x^6-47x^3+1\right)$

Exercises

In Exercises 79–84, find the value of $(f \circ g)'$ at the given value of x.

79. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

80. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

81. $f(u) = \cot \frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

82. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

83. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

84. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

79. $g(x)=\sqrt{x} \Rightarrow g'(x)=\frac{1}{2\sqrt{x}} \Rightarrow g(1)=1$ and $g'(1)=\frac{1}{2}$; $f(u)=u^5+1 \Rightarrow f'(u)=5u^4 \Rightarrow f'(g(1))=f'(1)=5$; therefore, $(f\circ g)'(1)=f'(g(1))\cdot g'(1)=5\cdot\frac{1}{2}=\frac{5}{2}$

80. $g(x)=(1-x)^{-1} \Rightarrow g'(x)=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2} \Rightarrow g(-1)=\frac{1}{2}$ and $g'(-1)=\frac{1}{4}$; $f(u)=1-\frac{1}{u} \Rightarrow f'(u)=\frac{1}{u^2}$ $\Rightarrow f'(g(-1))=f'\left(\frac{1}{2}\right)=4$; therefore, $(f\circ g)'(-1)=f'(g(-1))g'(-1)=4\cdot\frac{1}{4}=1$

81. $g(x)=5\sqrt{x} \Rightarrow g'(x)=\frac{5}{2\sqrt{x}} \Rightarrow g(1)=5$ and $g'(1)=\frac{5}{2}$; $f(u)=\cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u)=-\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right)=\frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right)$ $\Rightarrow f'(g(1))=f'(5)=-\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right)=-\frac{\pi}{10}$; therefore, $(f\circ g)'(1)=f'(g(1))g'(1)=-\frac{\pi}{10}\cdot\frac{5}{2}=-\frac{\pi}{4}$

82. $g(x)=\pi x \Rightarrow g'(x)=\pi \Rightarrow g\left(\frac{1}{4}\right)=\frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right)=\pi$; $f(u)=u+\sec^2 u \Rightarrow f'(u)=1+2\sec u\cdot\sec u\tan u$ $=1+2\sec^2 u\tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right)=f'\left(\frac{\pi}{4}\right)=1+2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}=5$; therefore, $(f\circ g)'\left(\frac{1}{4}\right)=f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right)=5\pi$

83. $g(x)=10x^2+x+1 \Rightarrow g'(x)=20x+1 \Rightarrow g(0)=1$ and $g'(0)=1$; $f(u)=\frac{2u}{u^2+1} \Rightarrow f'(u)=\frac{(u^2+1)(2)-(2u)(2u)}{(u^2+1)^2}$ $=\frac{-2u^2+2}{(u^2+1)^2} \Rightarrow f'(g(0))=f'(1)=0$; therefore, $(f\circ g)'(0)=f'(g(0))g'(0)=0\cdot 1=0$

84. $g(x)=\frac{1}{x^2}-1 \Rightarrow g'(x)=-\frac{2}{x^3} \Rightarrow g(-1)=0$ and $g'(-1)=2$; $f(u)=\left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u)=2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$ $=2\left(\frac{u-1}{u+1}\right)\cdot\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2}=\frac{2(u-1)(2)}{(u+1)^3}=\frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1))=f'(0)=-4$; therefore, $(f\circ g)'(-1)=f'(g(-1))g'(-1)=(-4)(2)=-8$

Exercises

85. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

86. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

85. $y = f(g(x)),\ f'(3) = -1,\ g'(2) = 5,\ g(2) = 3 \Rightarrow y' = f'(g(x))g'(x) \Rightarrow y'\big|_{x=2} = f'(g(2))g'(2) = f'(3)\cdot 5$
$= (-1)\cdot 5 = -5$

86. $r = \sin(f(t)),\ f(0) = \frac{\pi}{3},\ \ f'(0) = 4 \Rightarrow \frac{dr}{dt} = \cos(f(t))\cdot f'(t) \Rightarrow \frac{dr}{dt}\Big|_{t=0} = \cos(f(0))\cdot f'(0) = \cos\left(\frac{\pi}{3}\right)\cdot 4 = \left(\frac{1}{2}\right)\cdot 4 = 2$

Exercises

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \dfrac{x - 1}{x + 1}$
8. $x^3 = \dfrac{2x - y}{x + 3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$

1. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$

Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

2. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 18y + 18x\frac{dy}{dx} \Rightarrow (3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$

3. $2xy + y^2 = x + y$:

Step 1: $\left(2x\frac{dy}{dx} + 2y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Step 2: $2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$

Step 3: $\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$

Step 4: $\frac{dy}{dx} = \frac{1 - 2y}{2x + 2y - 1}$

4. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

5. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1-\frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y) + 2y\right] = 2x[1 - x(x-y) - (x-y)^2]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1 - x(x-y) - (x-y)^2\right]}{-2x^2(x-y)+2y} = \frac{x\left[1-x(x-y)-(x-y)^2\right]}{y-x^2(x-y)} = \frac{x\left(1-x^2+xy-x^2+2xy-y^2\right)}{x^2y-x^3+y} = \frac{x-2x^3+3x^2y-xy^2}{x^2y-x^3+y}$

6. $(3xy+7)^2 = 6y \Rightarrow 2(3xy+7)\cdot\left(3x\frac{dy}{dx} + 3y\right) = 6\frac{dy}{dx} \Rightarrow 2(3xy+7)(3x)\frac{dy}{dx} - 6\frac{dy}{dx} = -6y(3xy+7)$

$\Rightarrow \frac{dy}{dx}[6x(3xy+7) - 6] = -6y(3xy+7) \Rightarrow \frac{dy}{dx} = -\frac{y(3xy+7)}{x(3xy+7)-1} = \frac{3xy^2+7y}{1-3x^2y-7x}$

7. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

8. $x^3=\frac{2x-y}{x+3y}\Rightarrow x^4+3x^3y=2x-y\Rightarrow 4x^3+9x^2y+3x^3y'=2-y'\Rightarrow(3x^3+1)y'=2-4x^3-9x^2y$

$\Rightarrow y'=\frac{2-4x^3-9x^2y}{3x^3+1}$

9. $x=\tan y\Rightarrow 1=(\sec^2 y)\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{\sec^2 y}=\cos^2 y$

10. $xy=\cot(xy)\Rightarrow x\frac{dy}{dx}+y=-\csc^2(xy)\left(x\frac{dy}{dx}+y\right)\Rightarrow x\frac{dy}{dx}+x\csc^2(xy)\frac{dy}{dx}=-y\csc^2(xy)-y$

$\Rightarrow\frac{dy}{dx}\left[x+x\csc^2(xy)\right]=-y\left[\csc^2(xy)+1\right]\Rightarrow\frac{dy}{dx}=\frac{-y\left[\csc^2(xy)+1\right]}{x\left[1+\csc^2(xy)\right]}=-\frac{y}{x}$

11. $x+\tan(xy)=0\Rightarrow 1+\left[\sec^2(xy)\right]\left(y+x\frac{dy}{dx}\right)=0\Rightarrow x\sec^2(xy)\frac{dy}{dx}=-1-y\sec^2(xy)\Rightarrow\frac{dy}{dx}=\frac{-1-y\sec^2(xy)}{x\sec^2(xy)}$

$=\frac{-1}{x\sec^2(xy)}-\frac{y}{x}=\frac{-\cos^2(xy)}{x}-\frac{y}{x}=\frac{-\cos^2(xy)-y}{x}$

12. $x^4+\sin y=x^3y^2\Rightarrow 4x^3+(\cos y)\frac{dy}{dx}=3x^2y^2+x^3\cdot 2y\frac{dy}{dx}\Rightarrow(\cos y-2x^3y)\frac{dy}{dx}=3x^2y^2-4x^3\Rightarrow\frac{dy}{dx}=\frac{3x^2y^2-4x^3}{\cos y-2x^3y}$

13. $y\sin\left(\frac{1}{y}\right)=1-xy\Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right]+\sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx}=-x\frac{dy}{dx}-y\Rightarrow\frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right)+\sin\left(\frac{1}{y}\right)+x\right]=-y$

$\Rightarrow\frac{dy}{dx}=\frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right)+\sin\left(\frac{1}{y}\right)+x}=\frac{-y^2}{y\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)+xy}$

14. $x\cos(2x+3y) = y\sin x \Rightarrow -x\sin(2x+3y)(2+3y') + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow -2x\sin(2x+3y) - 3xy'\sin(2x+3y) + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow \cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = (\sin x + 3x\sin(2x+3y))y'$

$\Rightarrow y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3x\sin(2x+3y)}$

Exercises

In Exercises 21–26, use implicit differentiation to find dy/dx and then d^2y/dx^2.

21. $x^2 + y^2 = 1$

22. $x^{2/3} + y^{2/3} = 1$

23. $y^2 = e^{x^2} + 2x$

24. $y^2 - 2x = 1 - 2y$

25. $2\sqrt{y} = x - y$

26. $xy + y^2 = 1$

27. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point (2, 2).

28. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point (0, −1).

21. $x^2+y^2=1 \Rightarrow 2x+2yy'=0 \Rightarrow 2yy'=-2x \Rightarrow \frac{dy}{dx}=y'=-\frac{x}{y}$; now to find $\frac{d^2y}{dx^2}$, $\frac{d}{dx}(y')=\frac{d}{dx}\left(-\frac{x}{y}\right)$

$\Rightarrow y''=\frac{y(-1)+xy'}{y^2}=\frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$ since $y'=-\frac{x}{y} \Rightarrow \frac{d^2y}{dx^2}=y''=\frac{-y^2-x^2}{y^3}=\frac{-y^2-(1-y^2)}{y^3}=\frac{-1}{y^3}$

22. $x^{2/3}+y^{2/3}=1 \Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right]=-\frac{2}{3}x^{-1/3} \Rightarrow y'=\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y''=\frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y'+y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}=\frac{x^{1/3}\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right)+y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{3}x^{-2/3}y^{-1/3}+\frac{1}{3}y^{1/3}x^{-4/3}=\frac{y^{1/3}}{3x^{4/3}}+\frac{1}{3y^{1/3}x^{2/3}}$

23. $y^2=e^{x^2}+2x \Rightarrow 2yy'=2x+2=2xe^{x^2}+2 \Rightarrow \frac{dy}{dx}=\frac{xe^{x^2}+1}{y} \Rightarrow \frac{d^2y}{dx^2}=\frac{y\left(2x^2e^{x^2}+e^{x^2}\right)-\left(xe^{x^2}+1\right)y'}{y^2}$

$=\frac{y\left(2x^2e^{x^2}\right)-\left(xe^{x^2}+1\right)\cdot\frac{xe^{x^2}+1}{y}}{y^2}=\frac{\left(2x^2y^2+y^2-2x\right)e^{x^2}-x^2e^{2x^2}-1}{y^3}$

24. $y^2-2x=1-2y \Rightarrow 2y\cdot y'-2=-2y' \Rightarrow y'(2y+2)=2 \Rightarrow y'=\frac{1}{y+1}=(y+1)^{-1}$; then $y''=-(y+1)^{-2}\cdot y'$

$=-(y+1)^{-2}(y+1)^{-1} \Rightarrow \frac{d^2y}{dx^2}=y''=\frac{-1}{(y+1)^3}$

25. $2\sqrt{y}=x-y \Rightarrow y^{-1/2}y'=1-y' \Rightarrow y'\left(y^{-1/2}+1\right)=1 \Rightarrow \frac{dy}{dx}=y'=\frac{1}{y^{-1/2}+1}=\frac{\sqrt{y}}{\sqrt{y}+1}$; we can differentiate the equation $y'\left(y^{-1/2}+1\right)=1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right)+\left(y^{-1/2}+1\right)y''=0 \Rightarrow \left(y^{-1/2}+1\right)y''=\frac{1}{2}[y']^2y^{-3/2}$

$$\Rightarrow \frac{d^2y}{dx^2}=y''=\frac{\frac{1}{2}\left(\frac{1}{y^{-1/2}+1}\right)^2y^{-3/2}}{(y^{-1/2}+1)}=\frac{1}{2y^{3/2}(y^{-1/2}+1)^3}=\frac{1}{2\left(1+\sqrt{y}\right)^3}$$

26. $xy+y^2=1 \Rightarrow xy'+y+2yy'=0 \Rightarrow xy'+2yy'=-y \Rightarrow y'(x+2y)=-y \Rightarrow y'=\frac{-y}{(x+2y)}$;

$$\frac{d^2y}{dx^2}=y''=\frac{-(x+2y)y'+y(1+2y')}{(x+2y)^2}=\frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right]+y\left[1+2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2}=\frac{\frac{1}{(x+2y)}[y(x+2y)+y(x+2y)-2y^2]}{(x+2y)^2}=\frac{2y(x+2y)-2y^2}{(x+2y)^3}$$

$$=\frac{2y^2+2xy}{(x+2y)^3}=\frac{2y(x+y)}{(x+2y)^3}$$

27. $x^3+y^3=16 \Rightarrow 3x^2+3y^2y'=0 \Rightarrow 3y^2y'=-3x^2 \Rightarrow y'=-\frac{x^2}{y^2}$; we differentiate $y^2y'=-x^2$ to find y'':

$$y^2y''+y'[2y\cdot y']=-2x \Rightarrow y^2y''=-2x-2y[y']^2 \Rightarrow y''=\frac{-2x-2y\left(-\frac{x^2}{y^2}\right)^2}{y^2}=\frac{-2x-\frac{2x^4}{y^3}}{y^2}=\frac{-2xy^3-2x^4}{y^5}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(2,2)}=\frac{-32-32}{32}=-2$$

28. $xy+y^2=1 \Rightarrow xy'+y+2yy'=0 \Rightarrow y'(x+2y)=-y \Rightarrow y'=\frac{-y}{(x+2y)} \Rightarrow y''=\frac{(x+2y)(-y')-(-y)(1+2y')}{(x+2y)^2}$; since $y'|_{(0,-1)}=-\frac{1}{2}$ we obtain $y''|_{(0,-1)}=\frac{(-2)\left(\frac{1}{2}\right)-(-1)(0)}{4}=-\frac{1}{4}$

Exercises

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \frac{1}{x}, \quad a = 1$
4. $f(x) = \sqrt[3]{x}, \quad a = -8$
5. $f(x) = \tan x, \quad a = \pi$
6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

   a. $\sin x$ **b.** $\cos x$ **c.** $\tan x$ **d.** e^x **e.** $\ln(1 + x)$

1. $f(x)=x^3-2x+3 \Rightarrow f'(x)=3x^2-2 \Rightarrow L(x)=f'(2)(x-2)+f(2)=10(x-2)+7 \Rightarrow L(x)=10x-13$ at $x=2$

2. $f(x)=\sqrt{x^2+9}=(x^2+9)^{1/2} \Rightarrow f'(x)=\left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+9}} \Rightarrow L(x)=f'(-4)(x+4)+f(-4)$
$=-\frac{4}{5}(x+4)+5 \Rightarrow L(x)=-\frac{4}{5}x+\frac{9}{5}$ at $x=-4$

3. $f(x)=x+\frac{1}{x} \Rightarrow f'(x)=1-x^{-2} \Rightarrow L(x)=f(1)+f'(1)(x-1)=2+0(x-1)=2$

4. $f(x)=x^{1/3} \Rightarrow f'(x)=\frac{1}{3x^{2/3}} \Rightarrow L(x)=f'(-8)(x-(-8))+f(-8)=\frac{1}{12}(x+8)-2 \Rightarrow L(x)=\frac{1}{12}x-\frac{4}{3}$

5. $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x \Rightarrow L(x)=f(\pi)+f'(\pi)(x-\pi)=0+1(x-\pi)=x-\pi$

6. (a) $f(x)=\sin x \Rightarrow f'(x)=\cos x \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$
(b) $f(x)=\cos x \Rightarrow f'(x)=-\sin x \Rightarrow L(x)=f(0)+f'(0)(x-0)=1 \Rightarrow L(x)=1$
(c) $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$
(d) $f(x)=e^x \Rightarrow f'(x)=e^x \Rightarrow L(x)=f(0)+f'(0)(x-0)=1+x \Rightarrow L(x)=1+x$
(e) $f(x)=\ln(1+x) \Rightarrow f'(x)=\frac{1}{1+x} \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$

Exercises

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

a. $f(x) = (1 - x)^6$ **b.** $f(x) = \dfrac{2}{1 - x}$

15. $f'(x) = k(1+x)^{k-1}$. We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

16. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$

(b) $f(x) = \frac{2}{1-x} = 2\left[1+(-x)\right]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$

Exercises

In Exercises 19–38, find dy.

19. $y = x^3 - 3\sqrt{x}$

20. $y = x\sqrt{1 - x^2}$

21. $y = \dfrac{2x}{1 + x^2}$

22. $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$

24. $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$

26. $y = \cos(x^2)$

27. $y = 4\tan(x^3/3)$

28. $y = \sec(x^2 - 1)$

29. $y = 3\csc\left(1 - 2\sqrt{x}\right)$

30. $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

19. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right)dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

20. $y = x\sqrt{1-x^2} = x(1-x^2)^{1/2} \Rightarrow dy = \left[(1)(1-x^2)^{1/2} + (x)\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\right]dx$

$= \left(1-x^2\right)^{-1/2}\left[(1-x^2) - x^2\right]dx = \frac{\left(1-2x^2\right)}{\sqrt{1-x^2}}dx$

21. $y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)(1+x^2)-(2x)(2x)}{(1+x^2)^2}\right)dx = \frac{2-2x^2}{(1+x^2)^2}\,dx$

22. $y = \frac{2\sqrt{x}}{3\left(1+\sqrt{x}\right)} = \frac{2x^{1/2}}{3\left(1+x^{1/2}\right)} \Rightarrow dy = \left(\frac{x^{-1/2}\left(3\left(1+x^{1/2}\right)\right) - 2x^{1/2}\left(\frac{3}{2}x^{-1/2}\right)}{9\left(1+x^{1/2}\right)^2}\right)dx = \frac{3x^{-1/2}+3-3}{9(1+x^{1/2})^2}\,dx \Rightarrow dy = \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}\,dx$

23. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}dy + y\,dx + x\,dy - dx = 0 \Rightarrow (3y^{1/2} + x)\,dy = (1-y)\,dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x}dx$

24. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2 dx + 2xy\,dy - 6x^{1/2}\,dx - dy = 0 \Rightarrow (2xy - 1)\,dy = (6x^{1/2} - y^2)dx \Rightarrow dy = \frac{6\sqrt{x} - y^2}{2xy - 1}dx$

25. $y = \sin(5\sqrt{x}) = \sin(5x^{1/2}) \Rightarrow dy = (\cos(5x^{1/2}))\left(\frac{5}{2}x^{-1/2}\right)dx \Rightarrow dy = \frac{5\cos(5\sqrt{x})}{2\sqrt{x}}dx$

26. $y = \cos(x^2) \Rightarrow dy = [-\sin(x^2)](2x)dx = -2x\sin(x^2)dx$

27. $y = 4\tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)(x^2)\,dx \Rightarrow dy = 4x^2\sec^2\left(\frac{x^3}{3}\right)dx$

28. $y = \sec\left(x^2 - 1\right) \Rightarrow dy = [\sec(x^2 - 1)\tan(x^2 - 1)](2x)\,dx = 2x[\sec(x^2 - 1)\tan(x^2 - 1)]dx$

29. $y = 3\csc(1 - 2\sqrt{x}) = 3\csc(1 - 2x^{1/2}) \Rightarrow dy = 3(-\csc(1 - 2x^{1/2}))\cot(1 - 2x^{1/2})\,(-x^{-1/2})\,dx$
$\Rightarrow dy = \frac{3}{\sqrt{x}}\csc(1 - 2\sqrt{x})\cot(1 - 2\sqrt{x})\,dx$

Thank you for your attention