Calculus 1

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2024-2025

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Calculus 1

Lecture 6

Applications of Derivatives

Chapter 4
Applications of Derivatives

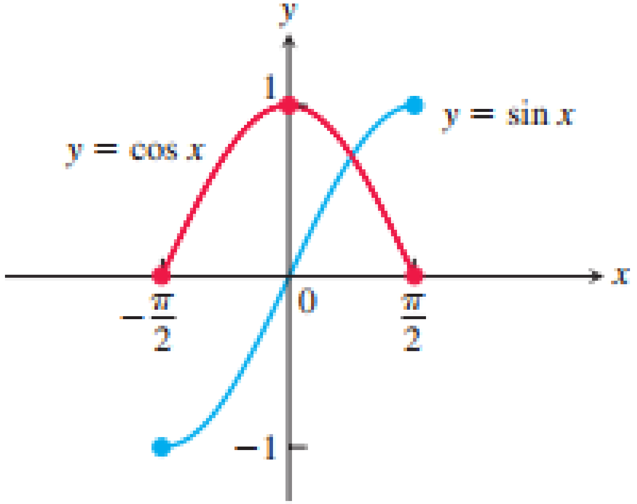
Extreme Values of Functions on Closed Intervals

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

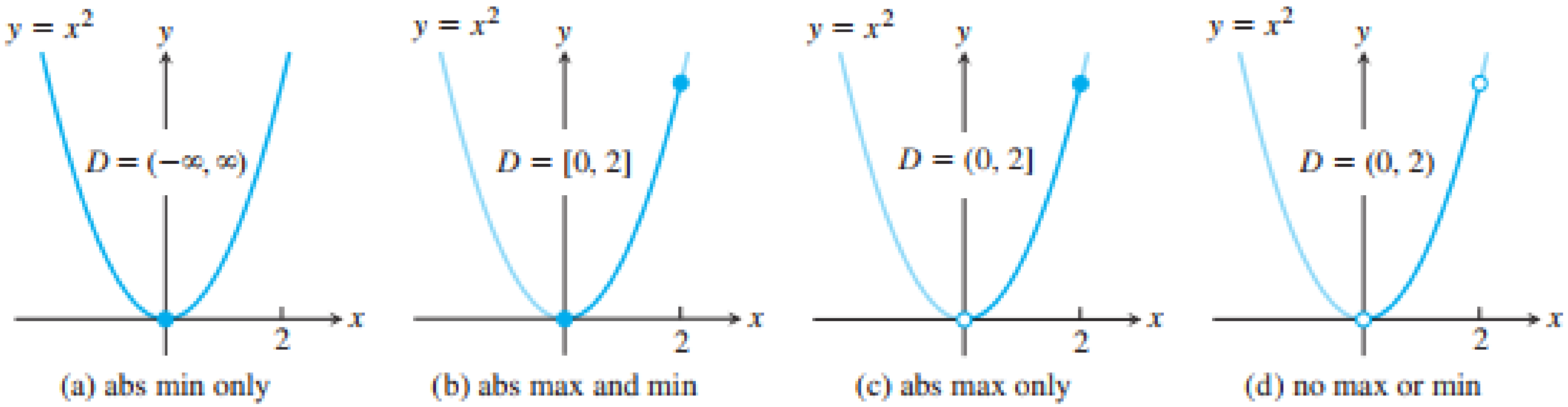
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$



Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1

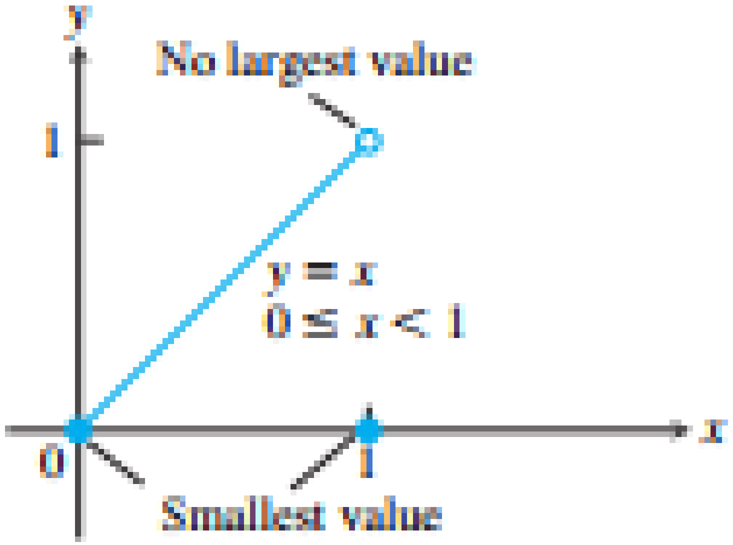
Extreme Values of Functions on Closed Intervals



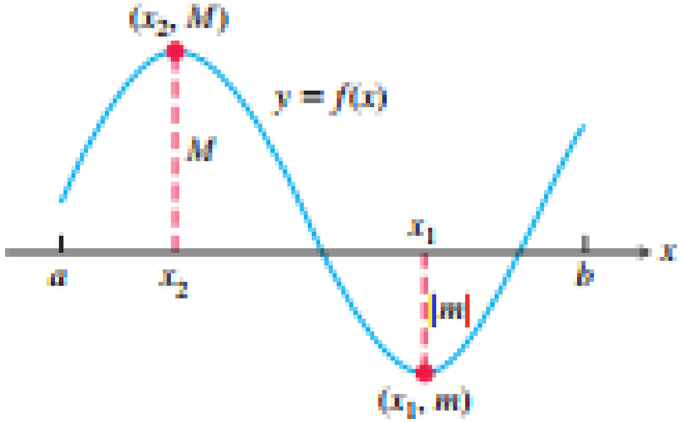
Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

Extreme Values of Functions on Closed Intervals

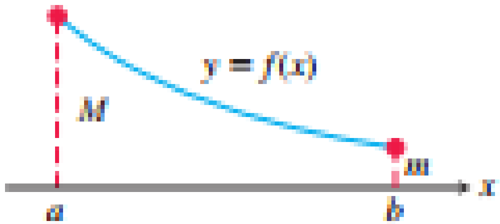
THEOREM 1—The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.



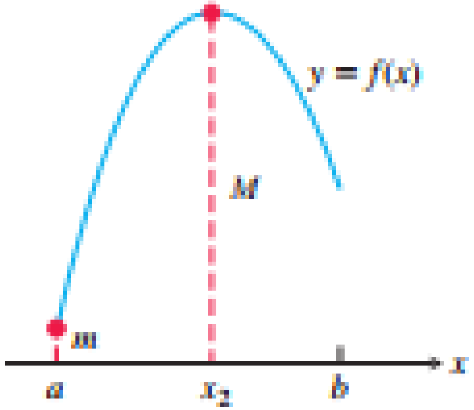
Extreme Values of Functions on Closed Intervals



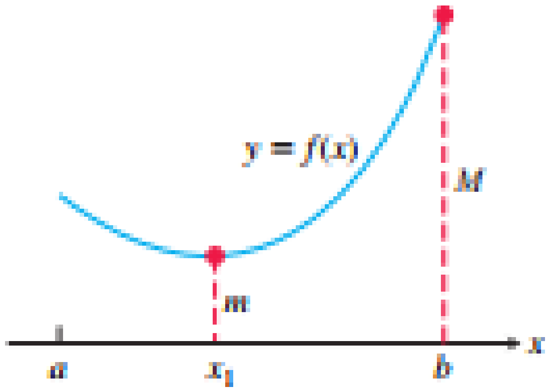
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint

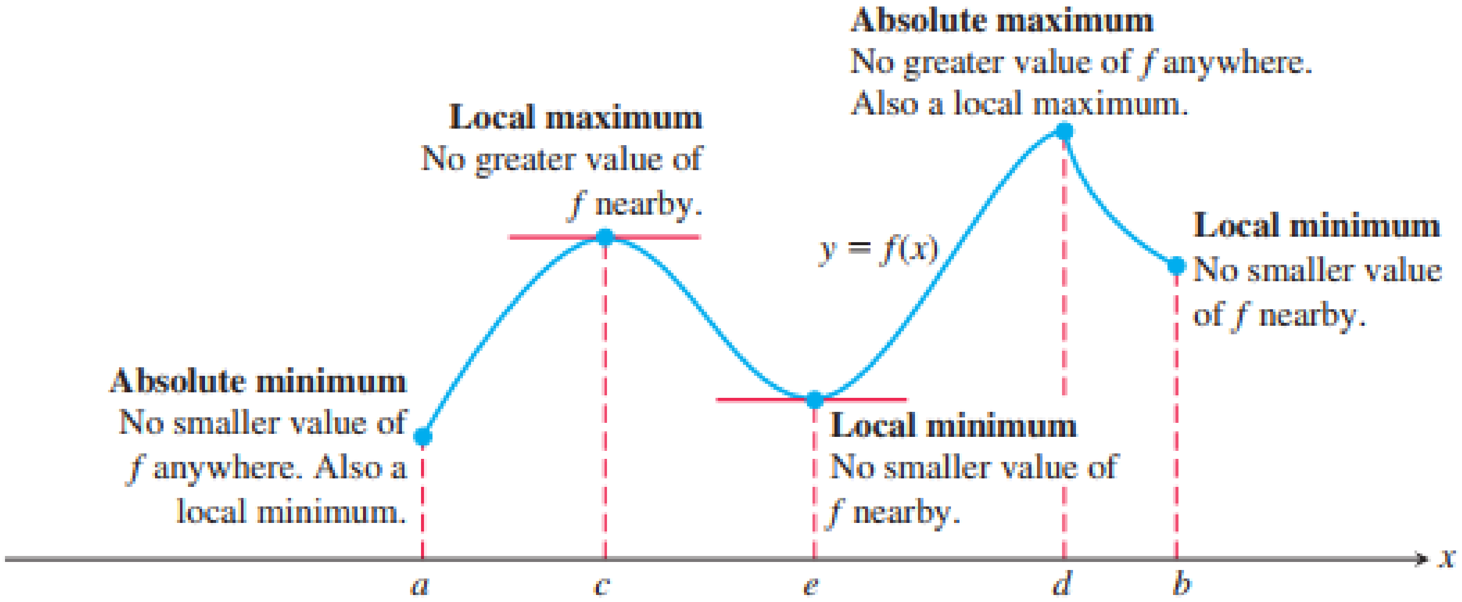


Minimum at interior point, maximum at endpoint

Local (Relative) Extreme Values

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.



Finding Extrema

THEOREM 2—The First Derivative Theorem for Local Extreme Values
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Finding Extrema

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Finding Extrema

Solution The function is differentiable over its entire domain, so the only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1.$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Finding Extrema

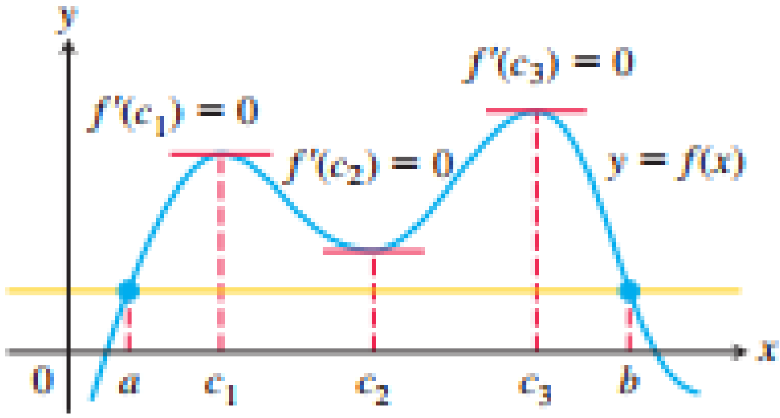
Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

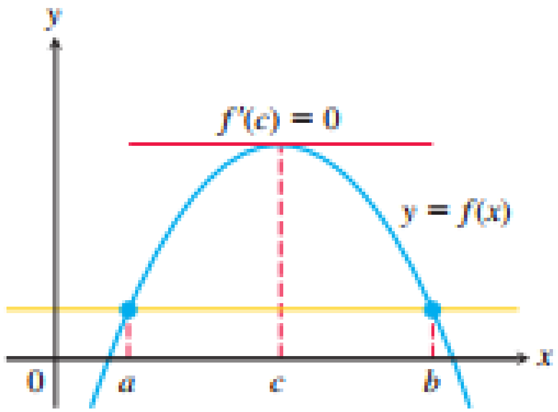
$$8 - 4t^3 = 0 \qquad \text{or} \qquad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum).

Rolle's Theorem

THEOREM 3—Rolle's Theorem
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.





(a)

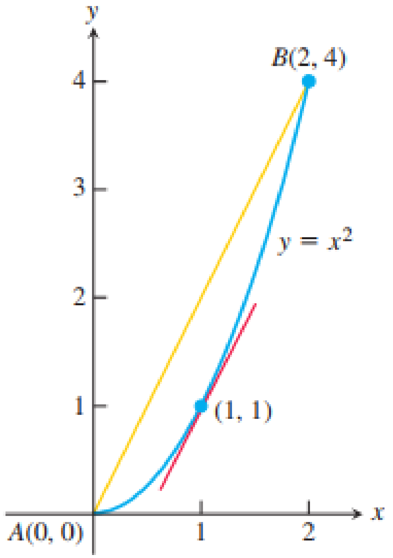
The Mean Value Theorem

THEOREM 4—The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

EXAMPLE 2 The function $f(x) = x^2$ (Figure 4.17) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■



The Mean Value Theorem

Find all values of c in the open interval (1, 4) such that $f'(c) = \dfrac{f(4) - f(1)}{4 - 1}$

$$f'(x) = \frac{4}{x^2} = \frac{f(4) - f(1)}{4 - 1} = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 2$$

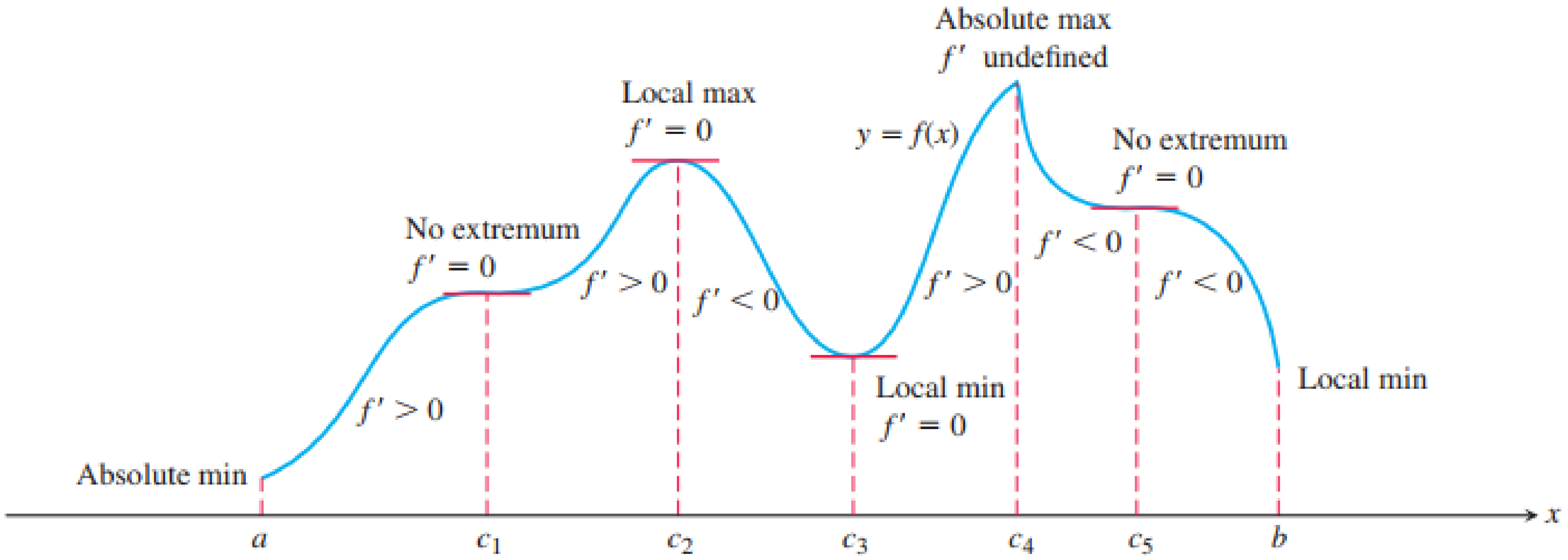
So, in the interval (1, 4), c = 2

First Derivative Test for Local Extrema

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.



First Derivative Test for Local Extrema

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

First Derivative Test for Local Extrema

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

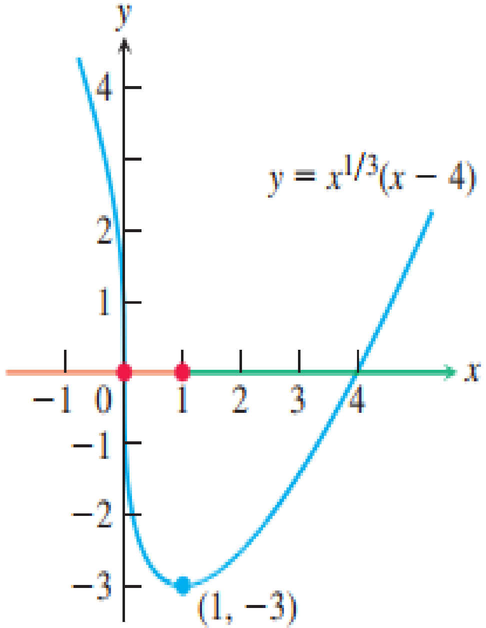
Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}(x^{4/3} - 4x^{1/3}) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

First Derivative Test for Local Extrema

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing



The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$.

First Derivative Test for Local Extrema

$$f(x) = (x-2)^{2/3} + 1$$

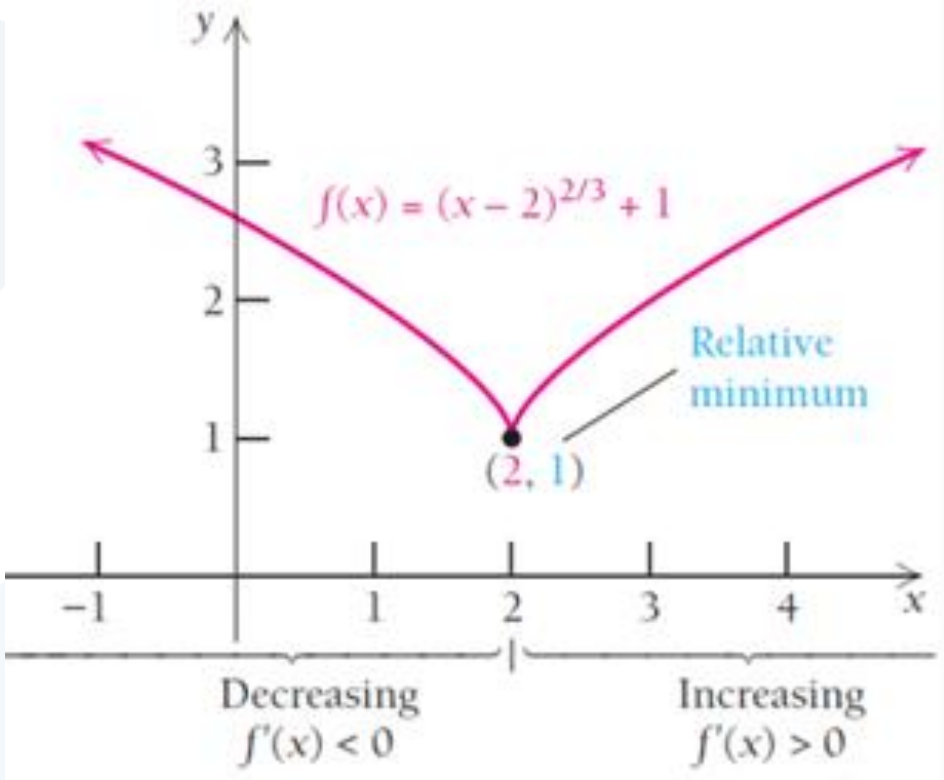
$$f'(x) = \frac{2}{3}(x-2)^{-1/3} = \frac{2}{3\sqrt[3]{x-2}}$$

$f'(x)$ does not exist at $x = 2$. Thus, 2 is a critical value

Interval	$-\infty < x < 2$	$2 < x < \infty$
Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$f'(0) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing

Therefore, by the First-Derivative Test, f has a relative maximum at $x = 2$

$$f(2) = 1$$



Concavity

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

A function whose graph is concave up is also often called **convex**.

The Second Derivative Test for Concavity

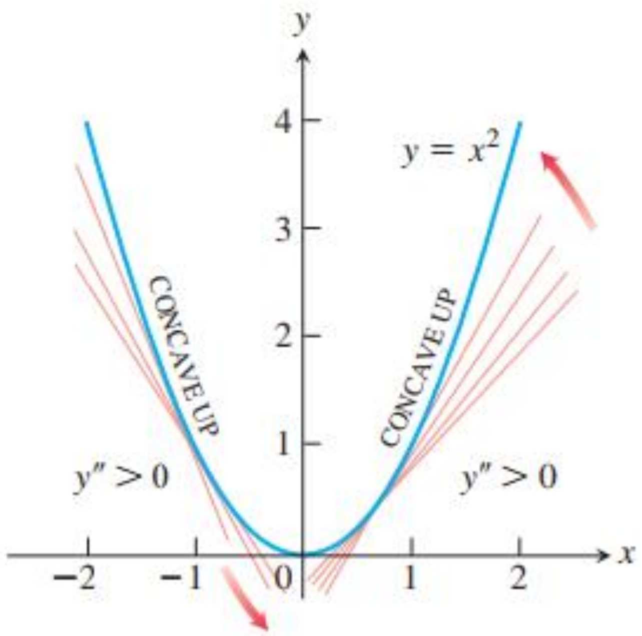
Let $y = f(x)$ be twice-differentiable on an interval I.

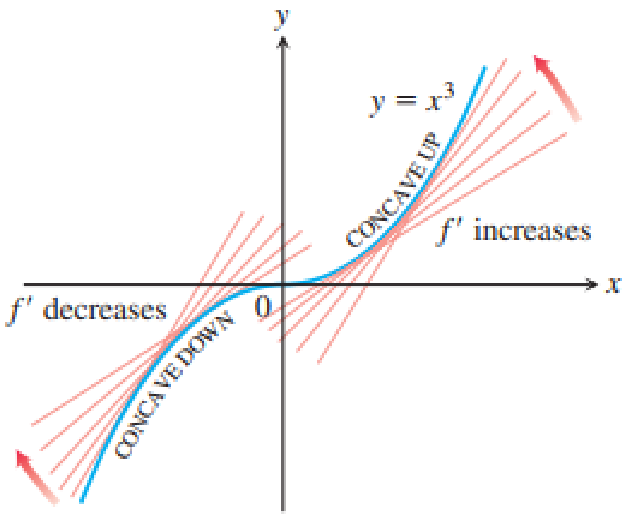
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$, where $y'' = 6x < 0$, and concave up on $(0, \infty)$, where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ∎

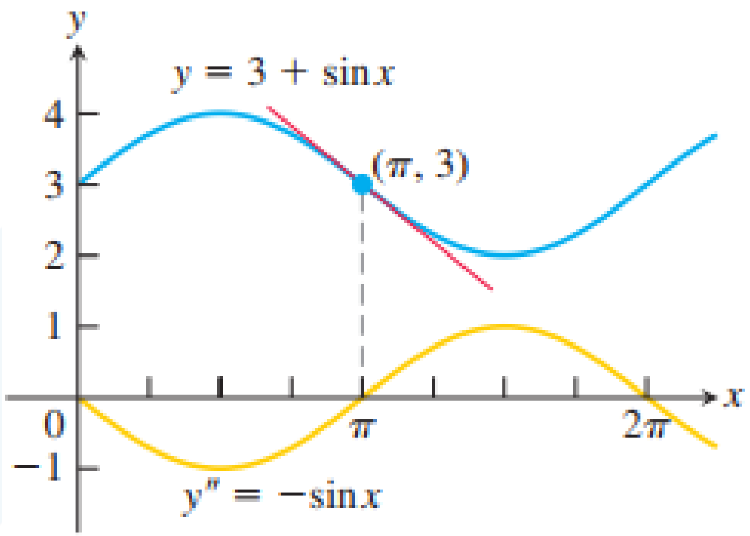




Concavity

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■



Points of Inflection

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Points of Inflection

EXAMPLE 3 Determine the concavity and find the inflection points of the function

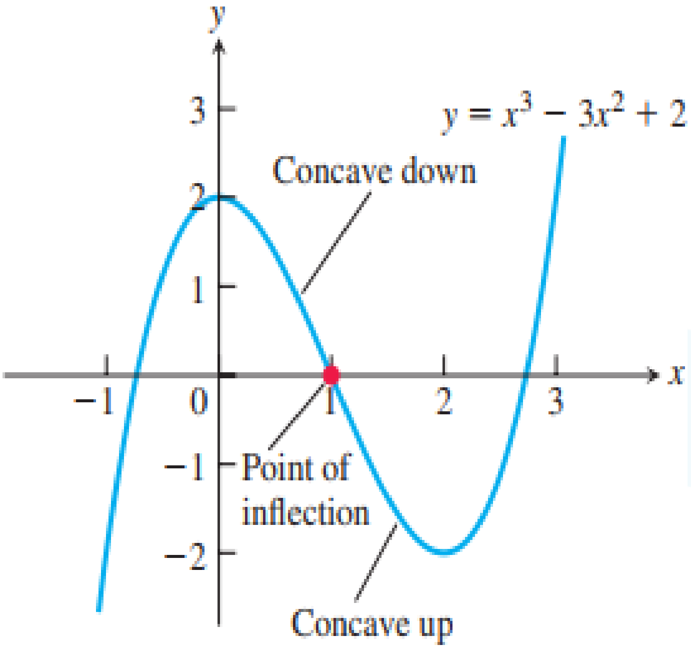
$$f(x) = x^3 - 3x^2 + 2.$$

Solution We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \qquad f''(x) = 6x - 6.$$

To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point $(1, 0)$ where the concavity changes.

The graph of f is shown in Figure 4.27. Notice that we did not need to know the shape of this graph ahead of time in order to determine its concavity. ■



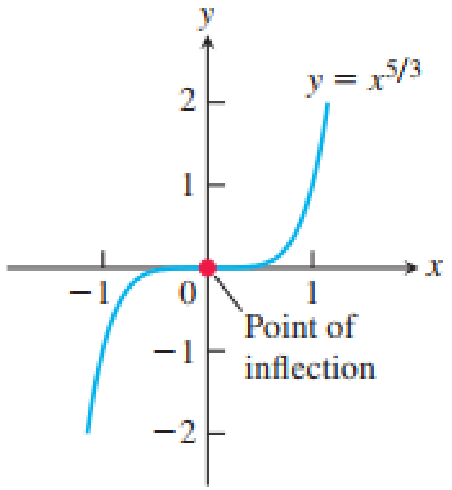
Points of Inflection

The next example illustrates that a function can have a point of inflection where the first derivative exists but the second derivative fails to exist.

EXAMPLE 4 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

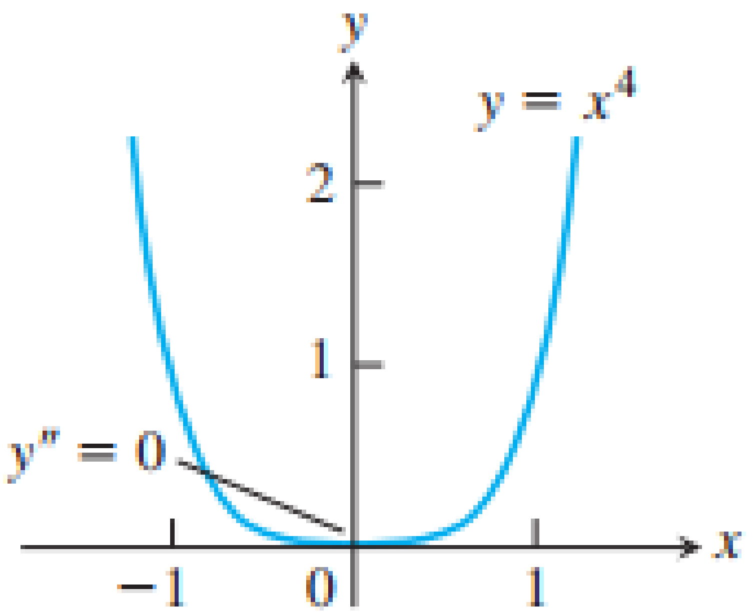
fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.28. ■



Points of Inflection

The following example shows that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE 5 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.29). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. The curve is concave up everywhere. ■

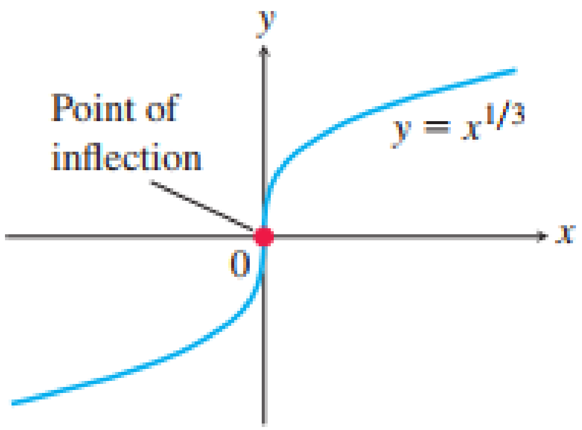


Points of Inflection

EXAMPLE 6 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

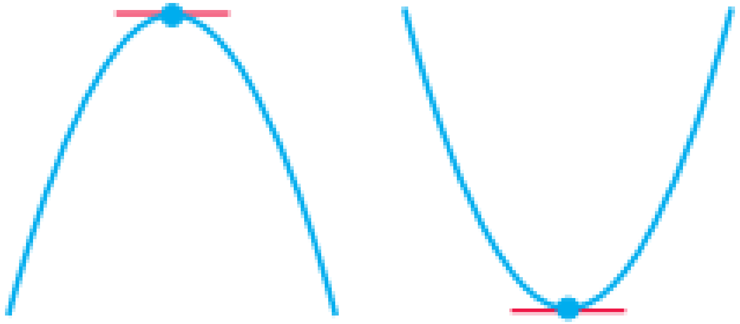
However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.30. ■



Second Derivative Test for Local Extrema

THEOREM 5—Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Procedure for Graphing

EXAMPLE 8 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Procedure for Graphing

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Procedure for Graphing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

Procedure for Graphing

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(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

The general shape of the curve is shown in the accompanying figure.

Procedure for Graphing

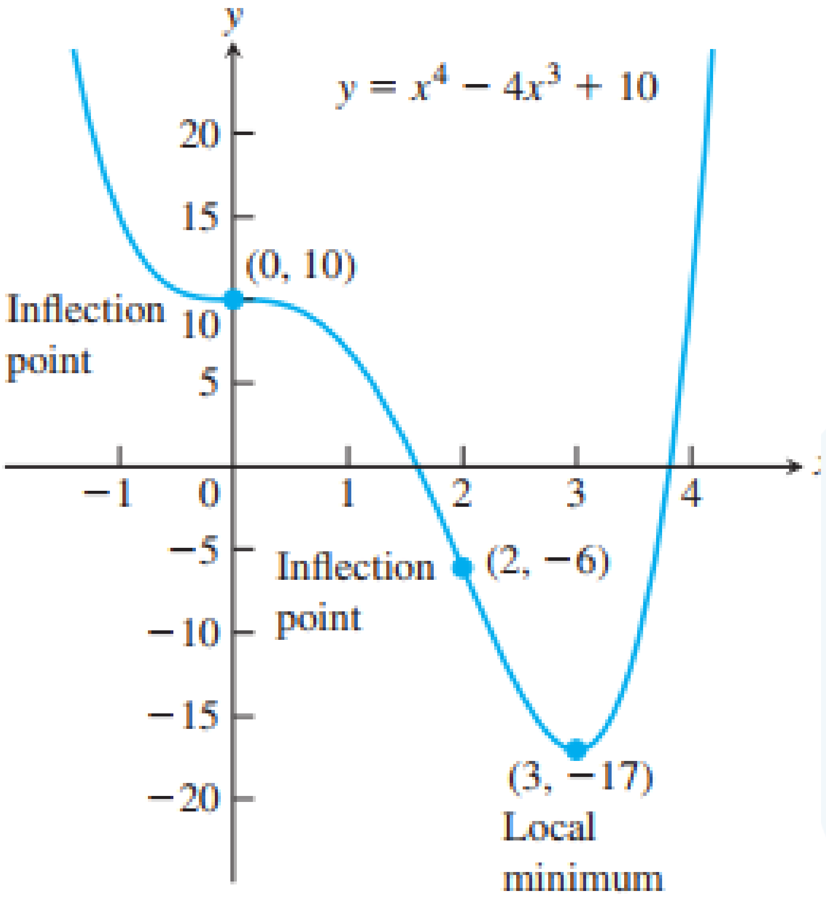


FIGURE 4.31 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 8).

Procedure for Graphing

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Procedure for Graphing

Sketch the graph of $f(x)$

$$f(x) = \frac{8}{x^2 - 4}$$

1. Intercepts:

   no x-intercept y-intercept $(0, -2)$

2. Asymptotes:

   $x = -2$ and $x = 2$ are vertical asymptotes $y = 0$ is horizontal asymptote

3. Asymptotes:

   Domain = $R \backslash \{-2, 2\}$ $f'(x) = \frac{-16x}{(x^2-4)^2}$, $f''(x) = \frac{16(3x^2+4)}{(x^2-4)^3}$

4. Critical values:

   $f'(x) = 0 \Rightarrow x = 0$,

   $f'(x)$ does not exist at -2 and 2, but neither value is in the domain of f

Procedure for Graphing

5. Increasing and/or decreasing; relative extrema:

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$
Sign of $f'(x)$	$f'(-3) > 0$	$f'(-1) > 0$	$f'(1) = -8 < 0$
Conclusion	Increasing	Increasing	Decreasing

$f''(0) = -1 < 0 \Rightarrow (0, -2)$ relative maximum

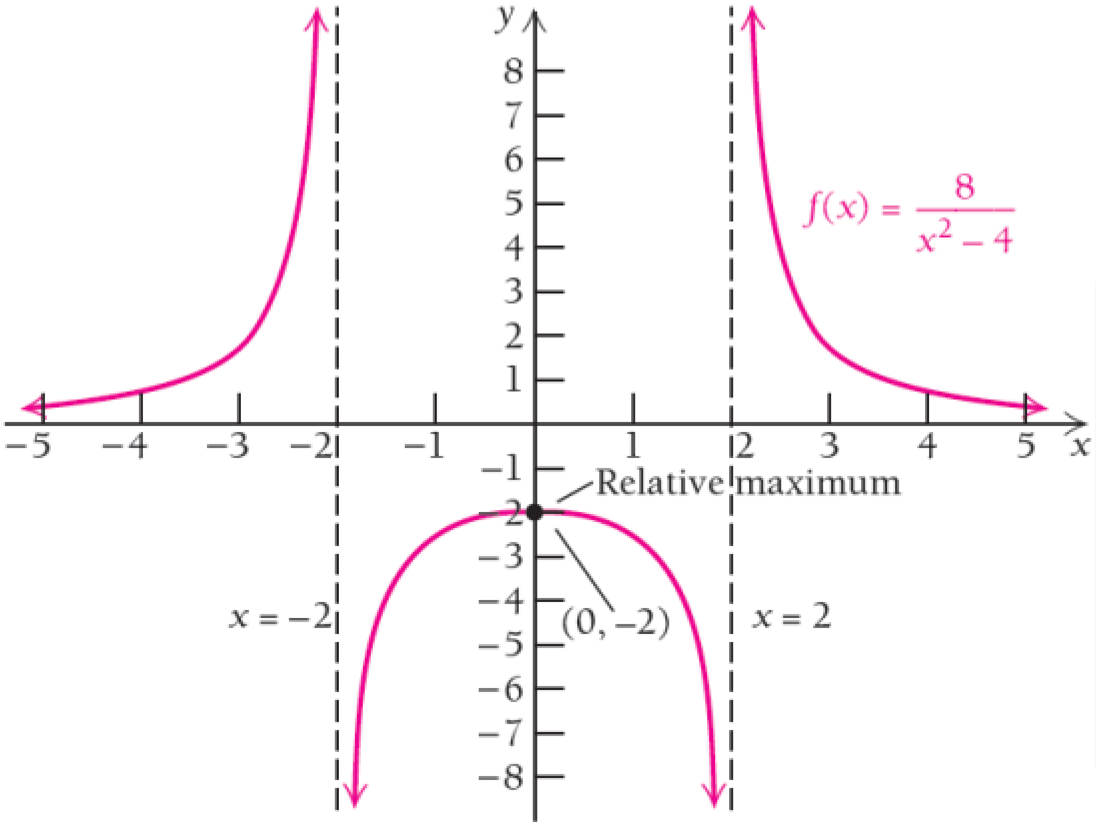
6. Concavity:

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) = -8 > 0$
Conclusion	Concave Up	Concave Down	Concave Up

7. Inflection points:

$f''(x) \neq 0$, $f''(x)$ does not exist at -2 and 2, but neither value is in the domain of f

no points of inflection



Procedure for Graphing

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$,
y-intercept at $y = 1$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

Procedure for Graphing

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing*. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.

6. *Asymptotes*. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Procedure for Graphing

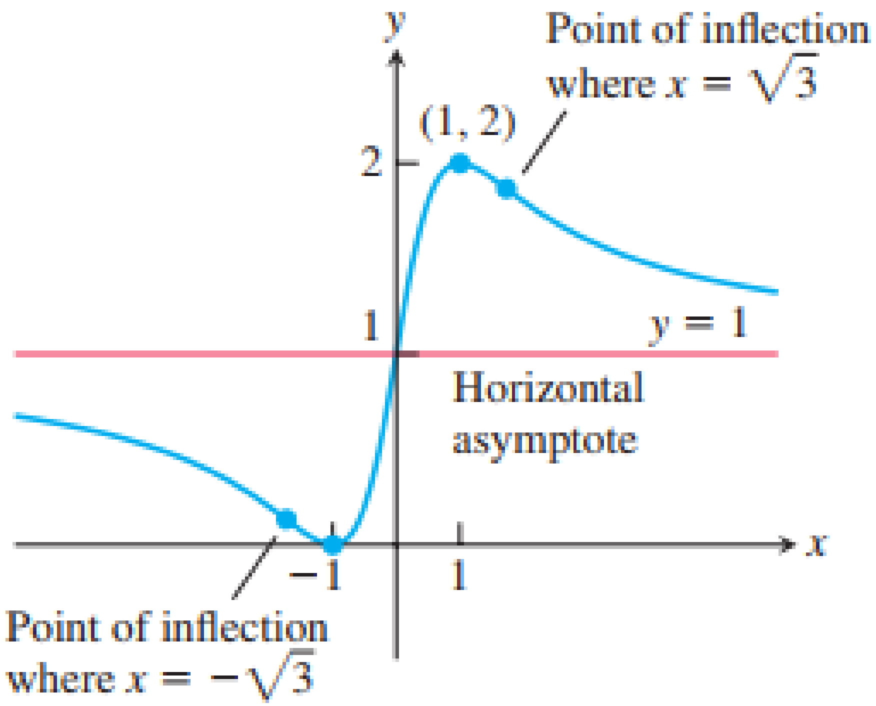
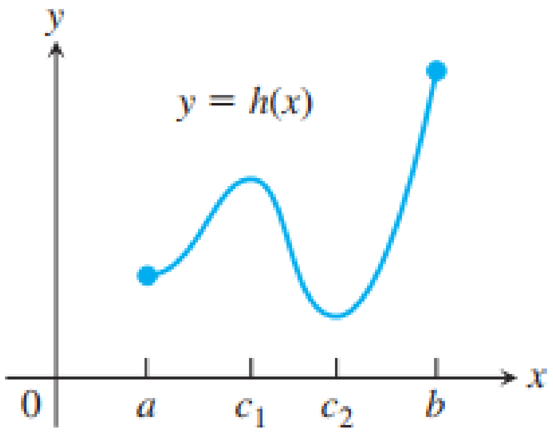


FIGURE 4.32 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 9).

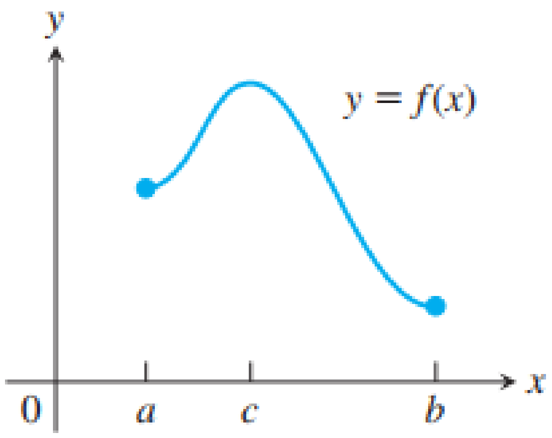
Exercices

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with .Theorem 1

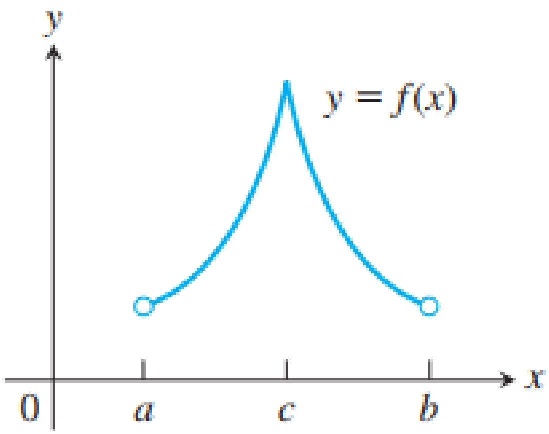
1.



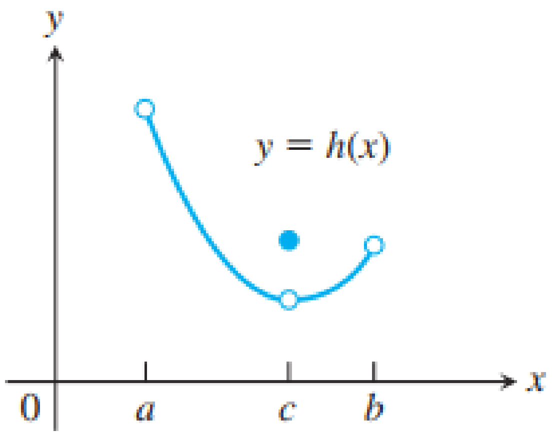
2.



3.



4.



1. An absolute minimum at $x = c_2$, an absolute maximum at $x = b$. Theorem 1 guarantees the existence of such extreme values because h is continuous on $[a, b]$.

2. An absolute minimum at $x = b$, an absolute maximum at $x = c$. Theorem 1 guarantees the existence of such extreme values because f is continuous on $[a, b]$.

3. No absolute minimum. An absolute maximum at $x = c$. Since the function's domain is an open interval, the function does not satisfy the hypotheses of Theorem 1 and need not have absolute extreme values.

4. No absolute extrema. The function is neither continuous nor defined on a closed interval, so it need not fulfill the conclusions of Theorem 1.

Exercices

find the absolute maximum and minimum values of each function on the given interval

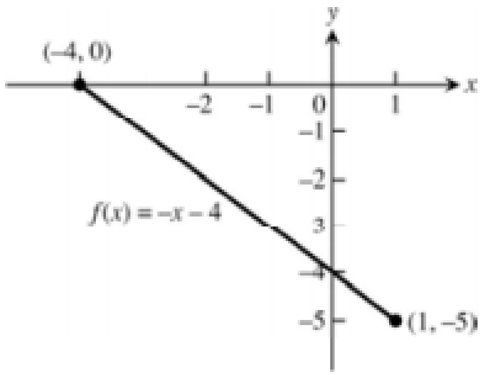
22. $f(x) = -x - 4, \quad -4 \leq x \leq 1$

23. $f(x) = x^2 - 1, \quad -1 \leq x \leq 2$

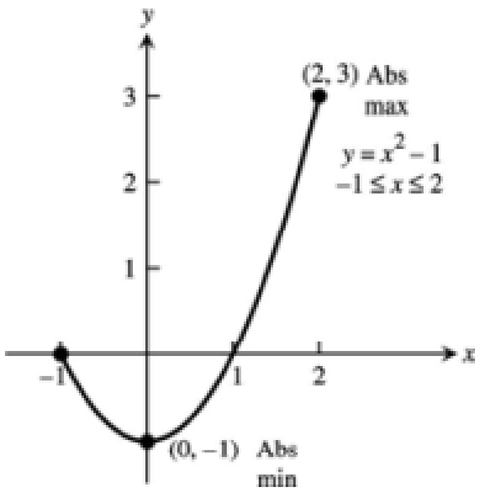
24. $f(x) = 4 - x^3, \quad -2 \leq x \leq 1$

25. $F(x) = -\frac{1}{x^2}, \quad 0.5 \leq x \leq 2$

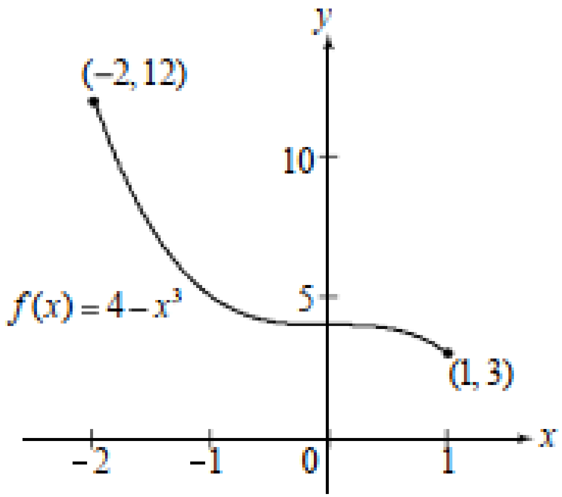
22. $f(x) = -x - 4 \Rightarrow f'(x) = -1 \Rightarrow$ no critical points; $f(-4) = 0,\ f(1) = -5 \Rightarrow$ the absolute maximum is 0 at $x = -4$ and the absolute minimum is -5 at $x = 1$



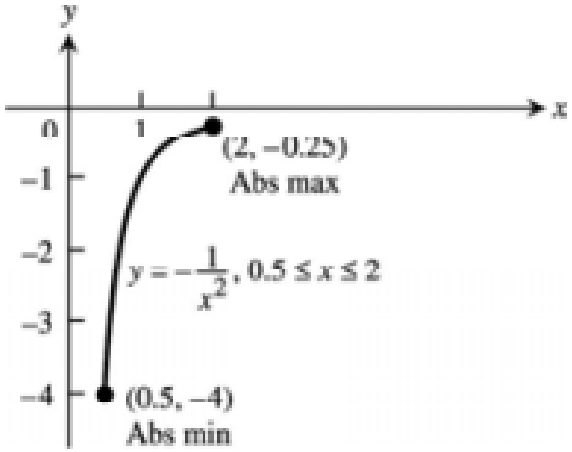
23. $f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow$ a critical point at $x = 0$; $f(-1) = 0,\ f(0) = -1,\ f(2) = 3 \Rightarrow$ the absolute maximum is 3 at $x = 2$ and the absolute minimum is -1 at $x = 0$



24. $f(x)=4-x^3 \Rightarrow f'(x)=-3x^2 \Rightarrow$ a critical point at $x=0$; $f(-2)=12$, $f(0)=4$, $f(1)=3 \Rightarrow$ the absolute maximum is 12 at $x=-2$ and the absolute minimum is 3 at $x=1$



25. $F(x)=-\frac{1}{x^2}=-x^{-2} \Rightarrow F'(x)=2x^{-3}=\frac{2}{x^3}$, however $x=0$ is not a critical point since 0 is not in the domain; $F(0.5)=-4$, $F(2)=-0.25 \Rightarrow$ the absolute maximum is -0.25 at $x=2$ and the absolute minimum is -4 at $x=0.5$



Exercices

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x-1)$
2. $f'(x) = (x-1)(x+2)$
3. $f'(x) = (x-1)^2(x+2)$
4. $f'(x) = (x-1)^2(x+2)^2$
5. $f'(x) = (x-1)e^{-x}$
6. $f'(x) = (x-7)(x+1)(x+5)$
7. $f'(x) = \dfrac{x^2(x-1)}{x+2}, \quad x \neq -2$
8. $f'(x) = \dfrac{(x-2)(x+4)}{(x+1)(x-3)}, \quad x \neq -1, 3$
9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$
10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

1. (a) $f'(x) = x(x-1) \Rightarrow$ critical points at 0 and 1

   (b) $f' = +++ \underset{0}{|} --- \underset{1}{|} +++ \Rightarrow$ increasing on $(-\infty, 0)$ and $(1, \infty)$, decreasing on $(0, 1)$

   (c) Local maximum at $x = 0$ and a local minimum at $x = 1$

2. (a) $f'(x) = (x-1)(x+2) \Rightarrow$ critical points at -2 and 1

   (b) $f' = +++ \underset{-2}{|} --- \underset{1}{|} +++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(1, \infty)$, decreasing on $(-2, 1)$

   (c) Local maximum at $x = -2$ and a local minimum at $x = 1$

3. (a) $f'(x) = (x-1)^2(x+2) \Rightarrow$ critical points at -2 and 1

   (b) $f' = --- \underset{-2}{|} +++ \underset{1}{|} +++ \Rightarrow$ increasing on $(-2, 1)$ and $(1, \infty)$, decreasing on $(-\infty, -2)$

   (c) No local maximum and a local minimum at $x = -2$

4. (a) $f'(x) = (x-1)^2(x+2)^2 \Rightarrow$ critical points at -2 and 1

   (b) $f' = +++ \underset{-2}{|} +++ \underset{1}{|} +++ \Rightarrow$ increasing on $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$, never decreasing

   (c) No local extrema

5. (a) $f'(x) = (x-1)e^{-x} \Rightarrow$ critical point at $x = 1$

(b) $f' = ----\underset{1}{|}++++ \Rightarrow$ decreasing on $(-\infty, 1)$, increasing on $(1, \infty)$

(c) Local (and absolute) minimum at $x = 1$

6. (a) $f'(x) = (x-7)(x+1)(x+5) \Rightarrow$ critical points at -5, -1 and 7

(b) $f' = ---\underset{-5}{|}+++\underset{-1}{|}---\underset{7}{|}+++ \Rightarrow$ increasing on $(-5, -1)$ and $(7, \infty)$, decreasing on $(-\infty, -5)$ and $(-1, 7)$

(c) Local maximum at $x = -1$, local minima at $x = -5$ and $x = 7$

7. (a) $f'(x) = \dfrac{x^2(x-1)}{(x+2)} \Rightarrow$ critical points at $x = 0$, $x = 1$ and $x = -2$

(b) $f' = +++\underset{-2}{)(}---\underset{0}{|}---\underset{1}{|}+++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(1, \infty)$, decreasing on $(-2, 0)$ and $(0, 1)$

(c) Local minimum at $x = 1$

8. (a) $f'(x) = \frac{(x-2)(x+4)}{(x+1)(x-3)} \Rightarrow$ critical points at $x = 2$, $x = -4$, $x = -1$, and $x = 3$

   (b) $f' = +++\underset{-4}{|}---\underset{-1}{)(}+++\underset{2}{|}---\underset{3}{)(}+++ \Rightarrow$ increasing on $(-\infty, -4)$, $(-1, 2)$ and $(3, \infty)$, decreasing on $(-4, -1)$ and $(2, 3)$

   (c) Local maximum at $x = -4$ and $x = 2$

9. (a) $f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} \Rightarrow$ critical points at $x = -2$, $x = 2$ and $x = 0$.

   (b) $f' = +++\underset{-2}{|}---\underset{0}{)(}---\underset{2}{|}+++ \Rightarrow$ increasing on $(-\infty, -2)$ and $(2, \infty)$, decreasing on $(-2, 0)$ and $(0, 2)$

   (c) Local maximum at $x = -2$, local minimum at $x = 2$

10. (a) $f'(x) = 3 - \frac{6}{\sqrt{x}} = \frac{3\sqrt{x}-6}{\sqrt{x}} \Rightarrow$ critical points at $x = 4$ and $x = 0$

    (b) $f' = \underset{0}{(}---\underset{4}{|}+++ \Rightarrow$ increasing on $(4, \infty)$, decreasing on $(0, 4)$

    (c) Local minimum at $x = 4$

Exercices

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

7. $f(x) = x^3 - x^2, \quad [-1, 2]$
8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

1. When $f(x)=x^2+2x-1$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0}=f'(c) \Rightarrow 3=2c+2 \Rightarrow c=\frac{1}{2}$.

2. When $f(x)=x^{2/3}$ for $0 \le x \le 1$, then $\frac{f(1)-f(0)}{1-0}=f'(c) \Rightarrow 1=\left(\frac{2}{3}\right)c^{-1/3} \Rightarrow c=\frac{8}{27}$.

3. When $f(x)=x+\frac{1}{x}$ for $\frac{1}{2} \le x \le 2$, then $\frac{f(2)-f(1/2)}{2-1/2}=f'(c) \Rightarrow 0=1-\frac{1}{c^2} \Rightarrow c=1$.

4. When $f(x)=\sqrt{x-1}$ for $1 \le x \le 3$, then $\frac{f(3)-f(1)}{3-1}=f'(c) \Rightarrow \frac{\sqrt{2}}{2}=\frac{1}{2\sqrt{c-1}} \Rightarrow c=\frac{3}{2}$.

7. When $f(x)=x^3-x^2$ for $-1 \le x \le 2$, then $\frac{f(2)-f(-1)}{2-(-1)}=f'(c) \Rightarrow 2=3c^2-2c \Rightarrow c=\frac{1\pm\sqrt{7}}{3}$.
$\frac{1+\sqrt{7}}{3} \approx 1.22$ and $\frac{1-\sqrt{7}}{3} \approx -0.549$ are both in the interval $-1 \le x \le 2$.

8. When $g(x)=\begin{cases} x^3 & -2 \le x \le 0 \\ x^2 & 0 < x \le 2 \end{cases}$, then $\frac{g(2)-g(-2)}{2-(-2)}=g'(c) \Rightarrow 3=g'(c)$. If $-2 \le x < 0$, then $g'(x)=3x^2 \Rightarrow 3=g'(c)$
$\Rightarrow 3c^2=3 \Rightarrow c=\pm 1$. Only $c=-1$ is in the interval. If $0 < x \le 2$, then $g'(x)=2x \Rightarrow 3=g'(c) \Rightarrow 2c=3 \Rightarrow c=\frac{3}{2}$.

Thank you for your attention