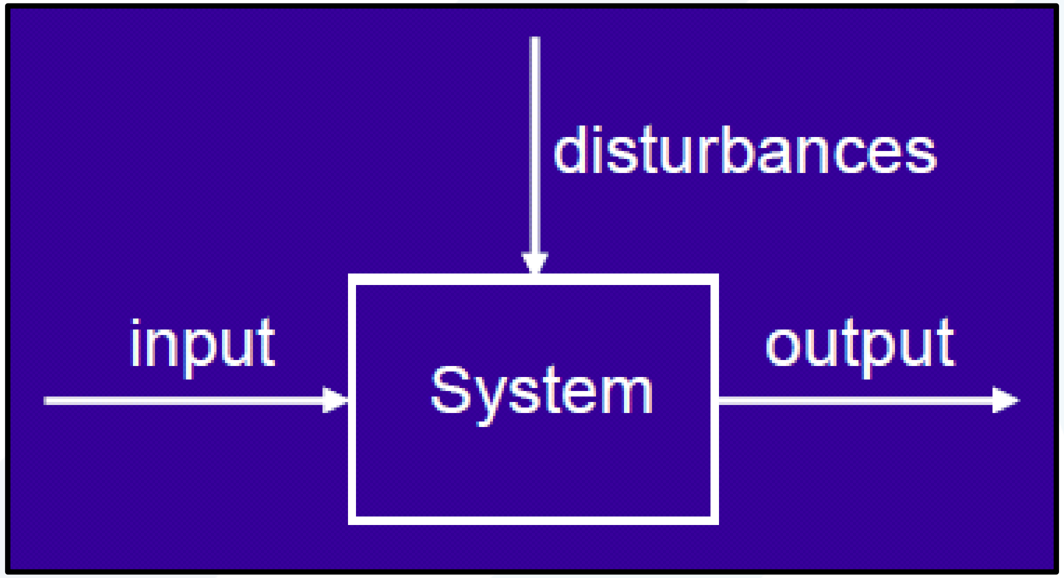
جامعة المنارة
كلية الهندسة
قسم الروبوتيك و الأنظمة الذكية
مقرر النمذجة و المطابقة

Design of Full Order State Observer



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العام الدراسي 2024-2025

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Principle of Duality. We shall now discuss the relationship between controllability and observability. We shall introduce the principle of duality, due to Kalman, to clarify apparent analogies between controllability and observability.

Consider the system S_1 described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where
$\mathbf{x}$ = state vector (n-vector)
$\mathbf{u}$ = control vector (r-vector)
$\mathbf{y}$ = output vector (m-vector)
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{B}$ = $n \times r$ matrix
$\mathbf{C}$ = $m \times n$ matrix

and the dual system S_2 defined by

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v}$$

$$\mathbf{n} = \mathbf{B}^*\mathbf{z}$$

where $\mathbf{z}$ = state vector (n-vector)

$\mathbf{v}$ = control vector (m-vector)

$\mathbf{n}$ = output vector (r-vector)

$\mathbf{A}^*$ = conjugate transpose of $\mathbf{A}$

$\mathbf{B}^*$ = conjugate transpose of $\mathbf{B}$

$\mathbf{C}^*$ = conjugate transpose of $\mathbf{C}$

The principle of duality states that the system S_1 is completely state controllable (observable) if and only if system S_2 is completely observable (state controllable).

To verify this principle, let us write down the necessary and sufficient conditions for complete state controllability and complete observability for systems S_1 and S_2.

For system S_1:

1. A necessary and sufficient condition for complete state controllability is that the rank of the matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$$

be n.

2. A necessary and sufficient condition for complete observability is that the rank of the matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be n.

For system S_2:

1. A necessary and sufficient condition for complete state controllability is that the rank of the matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be n.

2. A necessary and sufficient condition for complete observability is that the rank of the matrix

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

be n.

By comparing these conditions, the truth of this principle is apparent. By use of this principle, the observability of a given system can be checked by testing the state controllability of its dual.

Dual Problem. The problem of designing a full-order observer becomes that of determining the observer gain matrix $\mathbf{K}_e$ such that the error dynamics defined by Equation are asymptotically stable with sufficient speed of response. (The asymptotic stability and the speed of response of the error dynamics are determined by the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.) Hence, the design of the full-order observer becomes that of determining an appropriate $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has desired eigenvalues. Thus, the problem here becomes the same as the pole-placement problem we discussed In fact, the two problems are mathematically the same. This property is called duality.

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

In designing the full-order state observer, we may solve the dual problem, that is, solve the pole-placement problem for the dual system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

$$n = \mathbf{B}^*\mathbf{z}$$

assuming the control signal v to be

$$v = -\mathbf{K}\mathbf{z}$$

If the dual system is completely state controllable, then the state feedback gain matrix $\mathbf{K}$ can be determined such that matrix $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ will yield a set of the desired eigenvalues.

If $\mu_1, \mu_2, \ldots, \mu_n$ are the desired eigenvalues of the state observer matrix, then by taking the same μ_i's as the desired eigenvalues of the state-feedback gain matrix of the dual system, we obtain

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$

Noting that the eigenvalues of $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ and those of $\mathbf{A} - \mathbf{K}^*\mathbf{C}$ are the same, we have

$$|s\mathbf{I} - (\mathbf{A}^* - \mathbf{C}^*\mathbf{K})| = |s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$$

Comparing the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}^*\mathbf{C})|$ and the characteristic polynomial $|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})|$ for the observer system , we find that $\mathbf{K}_e$ and $\mathbf{K}^*$ are related by

$$\mathbf{K}_e = \mathbf{K}^*$$

Thus, using the matrix $\mathbf{K}$ determined by the pole-placement approach in the dual system, the observer gain matrix $\mathbf{K}_e$ for the original system can be determined by using the relationship $\mathbf{K}_e = \mathbf{K}^*$.

Necessary and Sufficient Condition for State Observation. As discussed, a necessary and sufficient condition for the determination of the observer gain matrix $\mathbf{K}_e$ for the desired eigenvalues of $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is that the dual of the original system

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*v$$

be completely state controllable. The complete state controllability condition for this dual system is that the rank of

$$\left[\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be n. This is the condition for complete observability of the original system .

Once we select the desired eigenvalues (or desired characteristic equation), the full-order state observer can be designed, provided the plant is completely observable. The desired eigenvalues of the characteristic equation should be chosen so that the state observer responds at least two to five times faster than the closed-loop system considered. As stated earlier, the equation for the full-order state observer is

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

It is noted that thus far we have assumed the matrices **A**, **B**, and **C** in the observer to be exactly the same as those of the physical plant. If there are discrepancies in **A**, **B**, and **C** in the observer and in the physical plant, the dynamics of the observer error are no longer governed by Equation $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$

This means that the error may not approach zero as expected. Therefore, we need to choose $\mathbf{K}_e$ so that the observer is stable and the error remains acceptably small in the presence of small modeling errors.

Direct-Substitution Approach to Obtain State Observer Gain Matrix $\mathbf{K}_e$. Similar to the case of pole placement, if the system is of low order, then direct substitution of matrix $\mathbf{K}_e$ into the desired characteristic polynomial may be simpler. For example, if $\mathbf{x}$ is a 3-vector, then write the observer gain matrix $\mathbf{K}_e$ as

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

Substitute this $\mathbf{K}_e$ matrix into the desired characteristic polynomial:

$$|s\mathbf{I} - (\mathbf{A} - \mathbf{K}_e\mathbf{C})| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

By equating the coefficients of the like powers of s on both sides of this last equation, we can determine the values of k_{e1}, k_{e2}, and k_{e3}. This approach is convenient if $n = 1$, 2, or 3, where n is the dimension of the state vector $\mathbf{x}$. (Although this approach can be used when $n = 4, 5, 6, \ldots$, the computations involved may become very tedious.)

Determining Observer Gain Matrix K_e with MATLAB

The closed-loop poles of the observer are the eigenvalues of matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.

Referring to the duality problem between the pole-placement problem and observer-design problem, we can determine $\mathbf{K}_e$ by considering the pole-placement problem for the dual system. That is, we determine $\mathbf{K}_e$ by placing the eigenvalues of $\mathbf{A}^* - \mathbf{C}^*\mathbf{K}$ at the desired place. Since $\mathbf{K}_e = \mathbf{K}^*$, for the full-order observer we use the command

$$K_e = \text{acker}(A',C',L)'$$

where L is the vector of the desired eigenvalues for the observer.

Example

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

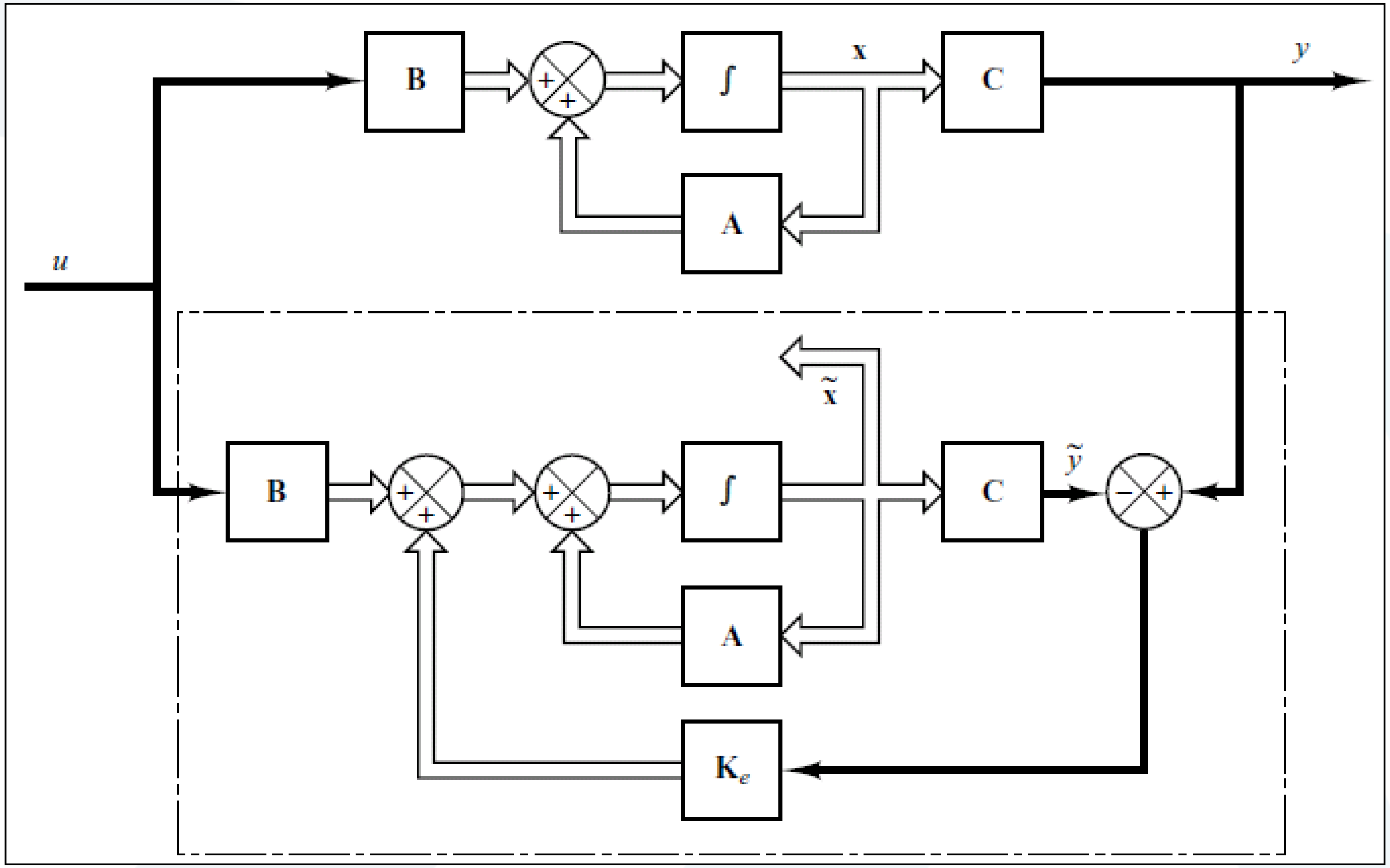
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Design a full-order state observer, assuming that the system configuration is identical to that shown in Figure . Assume that the desired eigenvalues of the observer matrix are

$$\mu_1 = -10, \qquad \mu_2 = -10$$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Solution

The design of the state observer reduces to the determination of an appropriate observer gain matrix $\mathbf{K}_e$.

Let us examine the observability matrix. The rank of

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is 2. Hence, the system is completely observable and the determination of the desired observer gain matrix is possible.

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$$

the characteristic equation for the observer becomes

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0$$

Define

$$\mathbf{K}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}$$

Then the characteristic equation becomes

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} [0 \quad 1] \right| = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix}$$

$$= s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

Since the desired characteristic equation is

$$s^2 + 20s + 100 = 0$$

we obtain

$$k_{e1} = 120.6, \qquad k_{e2} = 20$$

or

$$\mathbf{K}_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

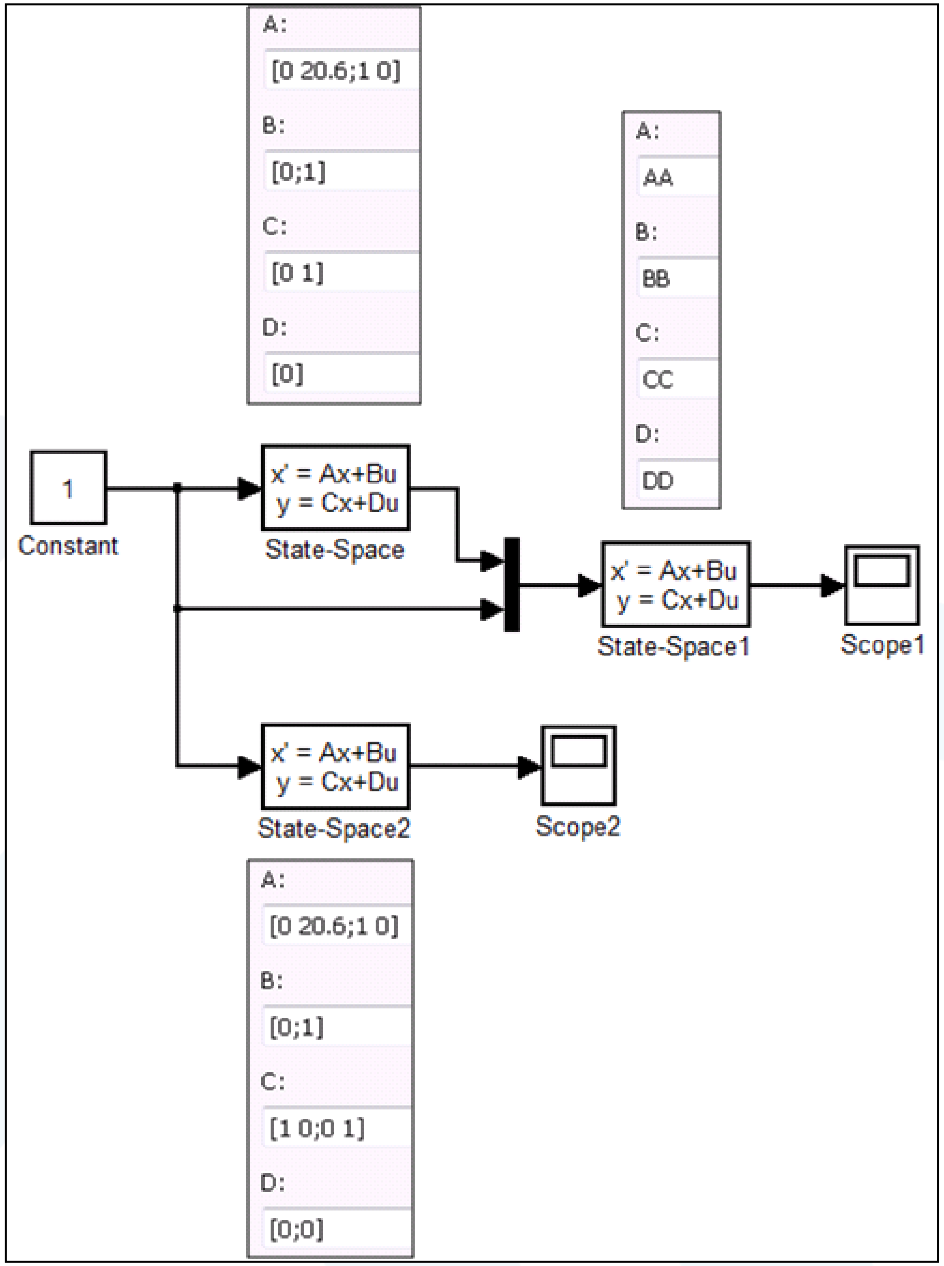
The equation for the full-order state observer is

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

or

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -100 \\ 1 & -20 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}y$$

```
A = [0 20.6;1 0];
B = [0;1];
C = [0 1];
JJ = [-10 -10];
KK= acker(A',C',JJ);
Ke=KK'
AA=(A-Ke*C);
BB=[Ke(1) 0;Ke(2) 1];
CC=[1 0;0 1];
DD=[0 0;0 0];
```



Example

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 100 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Evaluate the coefficients of the state feedback gain matrix such that the closed-loop poles have the values

$$s = -5 - j8 \qquad s = -5 + j8$$

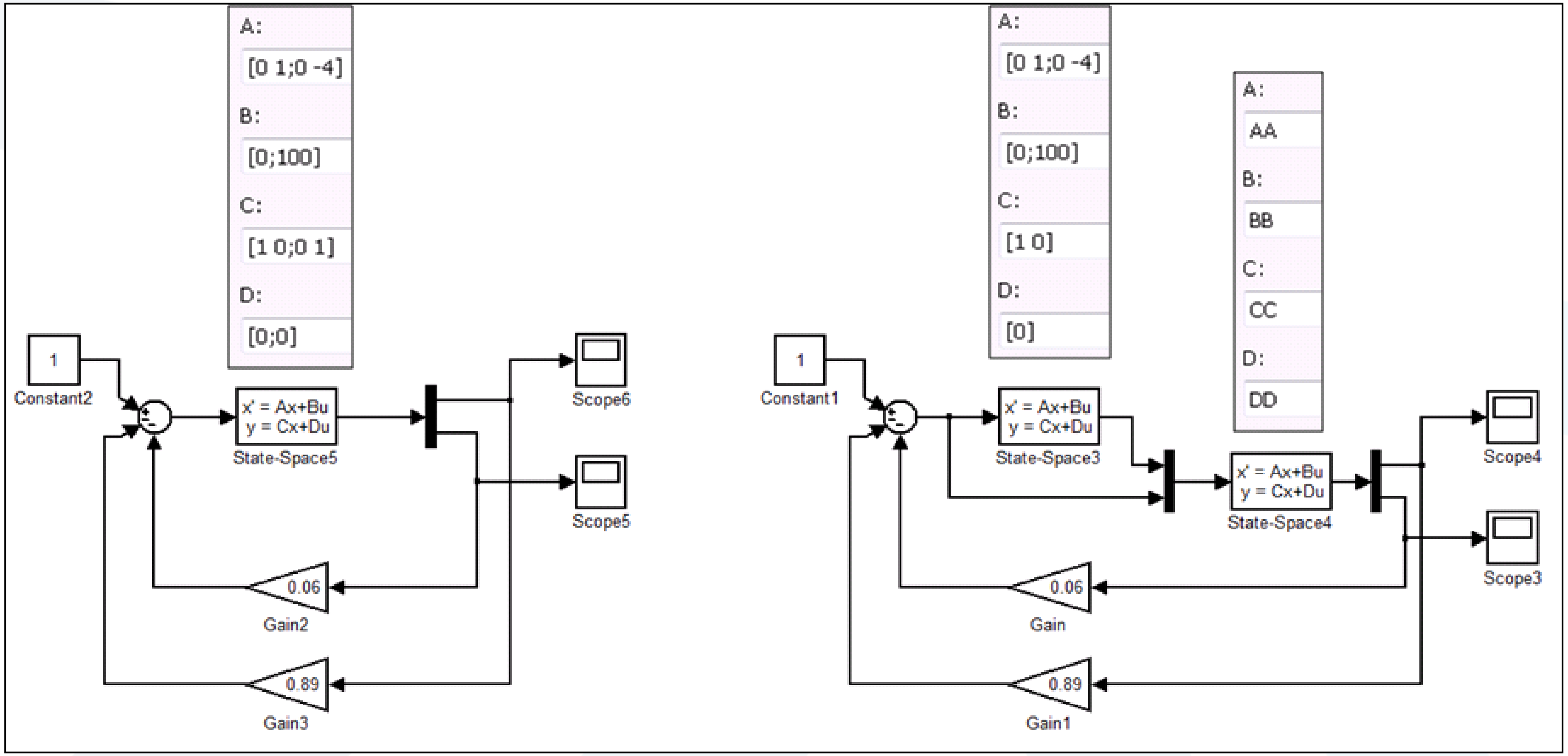
Solution

```
A = [0 1;0 -4];
B = [0;100];
J = [-5-8*j -5+8*j];
K = acker(A,B,J)

K =
    0.8900    0.0600
```

A = [0 1;0 -4];
C = [1 0];
JJ = [-10 -10];
KK= acker(A',C',JJ);
Ke=KK'
AA=(A-Ke*C);
BB=[Ke(1) 0;Ke(2) 100];
CC=[1 0;0 1];
DD=[0 0;0 0];

Ke =
16
36



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