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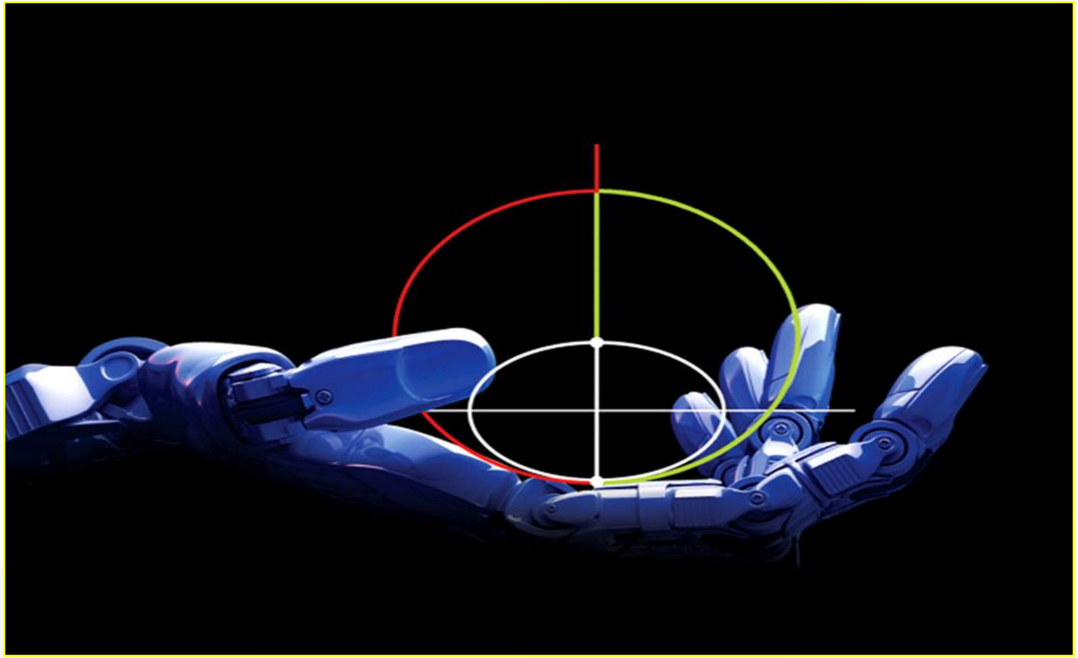
Time Domain Analysis



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Time Domain Analysis of First-Order System

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots.. + a_{n-1} y' + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \ldots.. + b_{m-1} x' + b_m x$$

- Where x is the input of the system and y is the output of the system.

Laplace Transformation

$$L\left| f(t) \right| = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$\text{Transfer function} = G(s) = \left. \frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]} \right|_{\text{zero initial conditions}}$$

$$G(s) = \frac{L[output]}{L[input]} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots.. + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1} s + a_n}$$

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$$a\frac{dx_o}{dt}+bx_o=cx_i(t)$$

$$(as+b)X_0(s)=cX_i(s)$$

$$G(s)=\frac{X_o}{X_i}(s)=\frac{c}{as+b}$$

$$G(s)=\frac{\frac{c}{b}}{1+\frac{a}{b}s}$$

$$G(s)=\frac{K}{1+Ts}$$

K: steady-state gain constant

T: time constant (seconds)

Steady-state gain

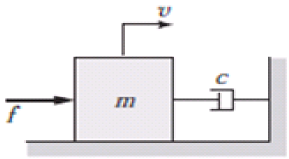
The steady-state of a **TF** can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input, ***u***, and an output, ***y***. Then we can calculate the steady-state gain, ***K***, from:

$$K = \frac{y_2 - y_1}{u_2 - u_1}$$

Time constant

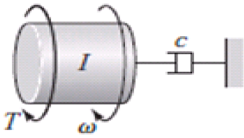
In brief, the time constant relates to the analytical solution for the unit step response of a first order differential equation, and is the time taken for the output to reach **63%** of the steady-state value

Examples of First-Order Systems



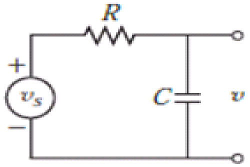
$$m\frac{dv}{dt} + cv = f$$

$$T = \frac{m}{c}$$



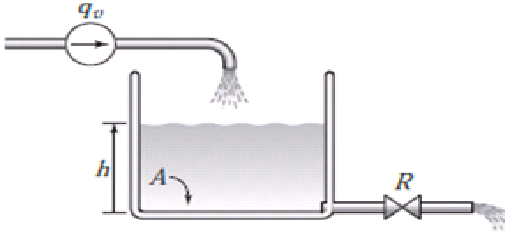
$$I\frac{dw}{dt} + cw = T$$

$$T = \frac{I}{c}$$



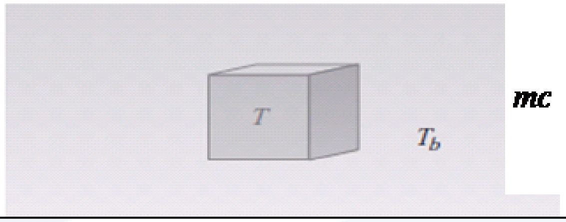
$$RC\frac{dv}{dt} + v = v_s$$

$$T = RC$$



$$AR\frac{dh}{dt} + \rho gh = Rq_v$$

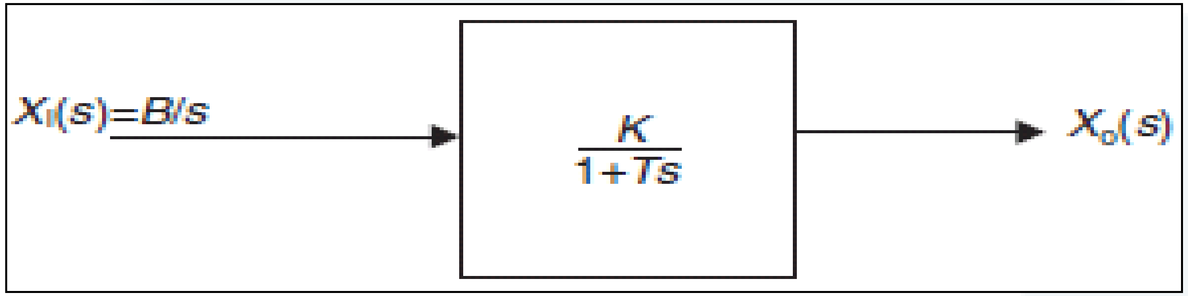
$$T = \frac{AR}{\rho g}$$



$$mc_p R\frac{dT}{dt} + T = T_b$$

$$T = mc_p R$$

Step Response of First-Order System



$$X_o(s) = \frac{BK}{s(1+Ts)} = BK\frac{\frac{1}{T}}{s(s+\frac{1}{T})}$$

$$x_o(t) = BK(1-e^{-\frac{t}{T}})$$

Time function $f(t)$	Laplace transform $\mathcal{L}[f(t)] = F(s)$
$1-e^{-at}$	$\frac{a}{s(s+a)}$

B=1(unit step)

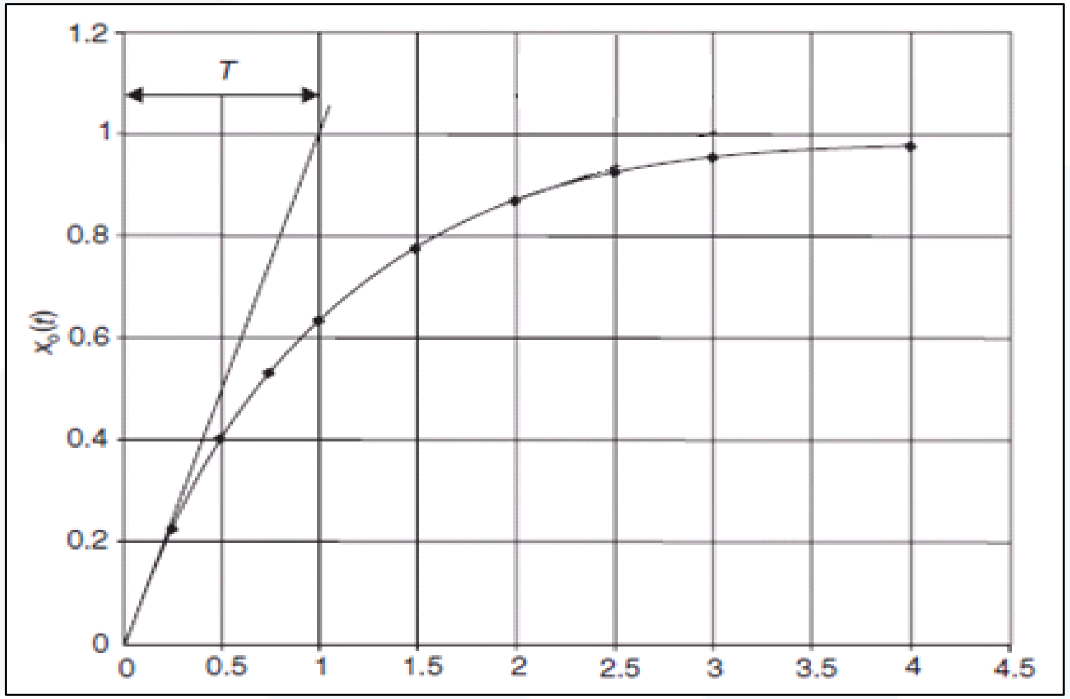
K=1(unity gain)

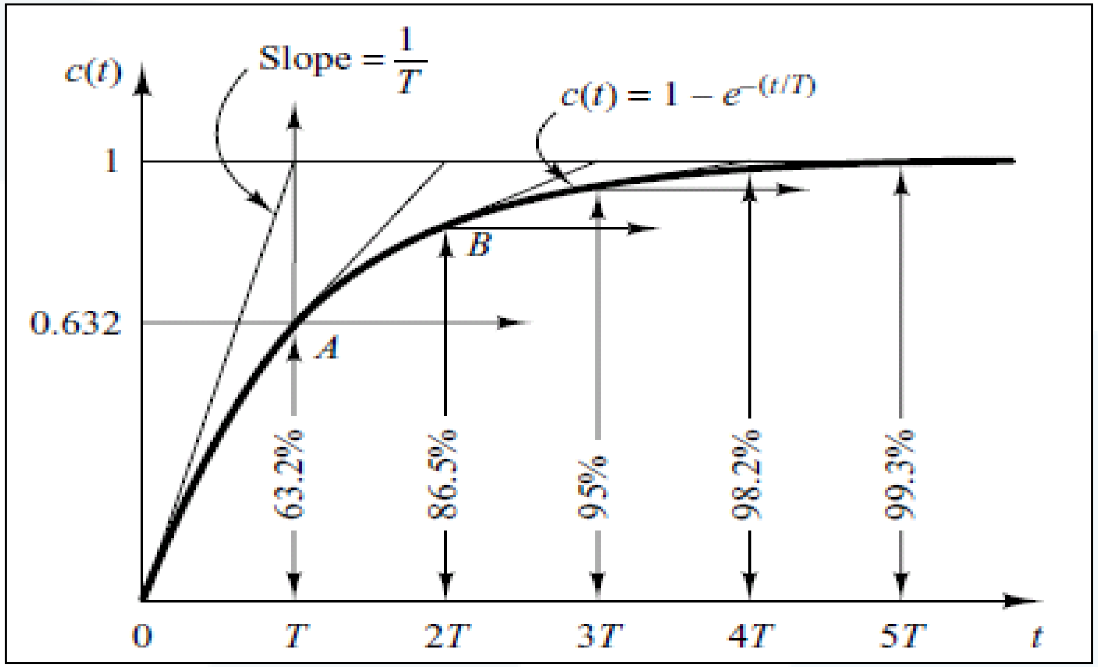
$$x_o(t) = 1-e^{-\frac{t}{T}}$$

The system time constant is the intersection of the slope at t=0 with the final value line

$$\frac{dx_o}{dt} = 0 - \left(-\frac{1}{T}\right)e^{-\frac{t}{T}} = \frac{1}{T}e^{-\frac{t}{T}}$$

$$\left.\frac{dx_o}{dt}\right|_{t=0} = \frac{1}{T}$$





In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t = 3T, 4T$, and $5T$, the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value.

Time Domain Analysis of Second-Order System

$$a\frac{d^2x_o}{dt^2}+b\frac{dx_o}{dt}+cx_o=ex_i(t)$$

$$(as^2+bs+c)X_0(s)=eX_i(s)$$

$$G(s)=\frac{X_o}{X_i}(s)=\frac{e}{as^2+bs+c}$$

$$G(s)=\frac{\frac{e}{c}}{\frac{a}{c}s^2+\frac{b}{c}s+1}$$

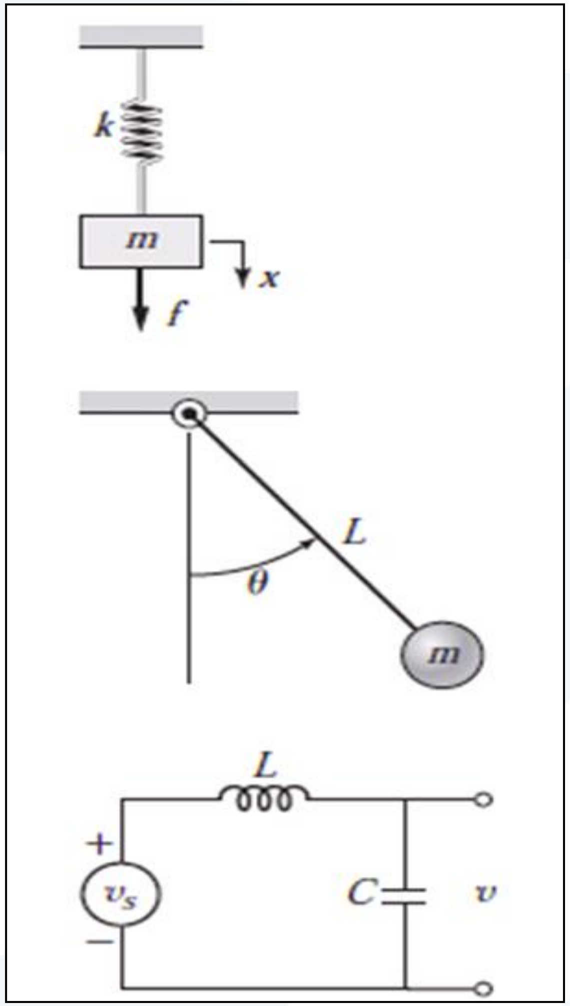
$$G(s)=\frac{K}{\frac{1}{w_n{}^2}s^2+\frac{2\zeta}{w_n}s+1}$$

$$G(s)=\frac{Kw_n{}^2}{s^2+2\zeta w_n s+w_n{}^2}$$

K: steady-state gain constant

w_n: undamped natural frequency(rad/s)

ζ : damping ratio



$$m\frac{d^2x}{dt^2} + kx = f(t)$$

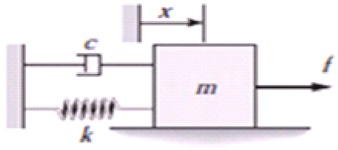
$$\omega_n = \sqrt{\frac{k}{m}}$$

$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

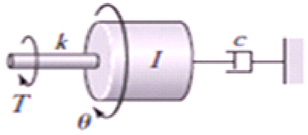
$$\omega_n = \sqrt{\frac{g}{L}}$$

$$LC\frac{d^2v}{dt^2} + v = v_s$$

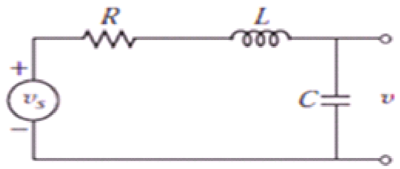
$$\omega_n = \frac{1}{\sqrt{LC}}$$



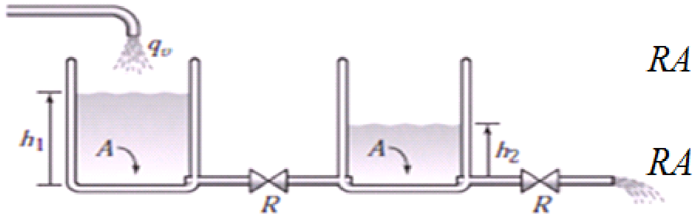
$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f$$



$$I\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+k\theta=T$$

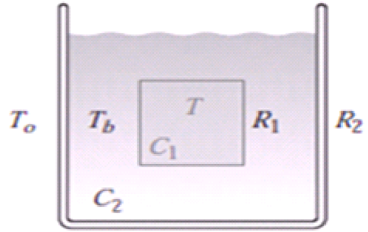


$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v=v_s$$



$$RA\frac{dh_1}{dt}+\rho g(h_1-h_2)=Rq_v$$

$$RA\frac{dh_2}{dt}+\rho g(h_2-h_1)+\rho g h_2=0$$



$$R_1C_1\frac{dT}{dt}+T=T_b$$

$$R_1R_2C_2\frac{dT_b}{dt}+(R_1+R_2)T_b=R_2T+R_1T_o$$

Roots of the Characteristic Equation

The time response of any system has two components:

(a) **Transient response**

(b) **Steady-state response**

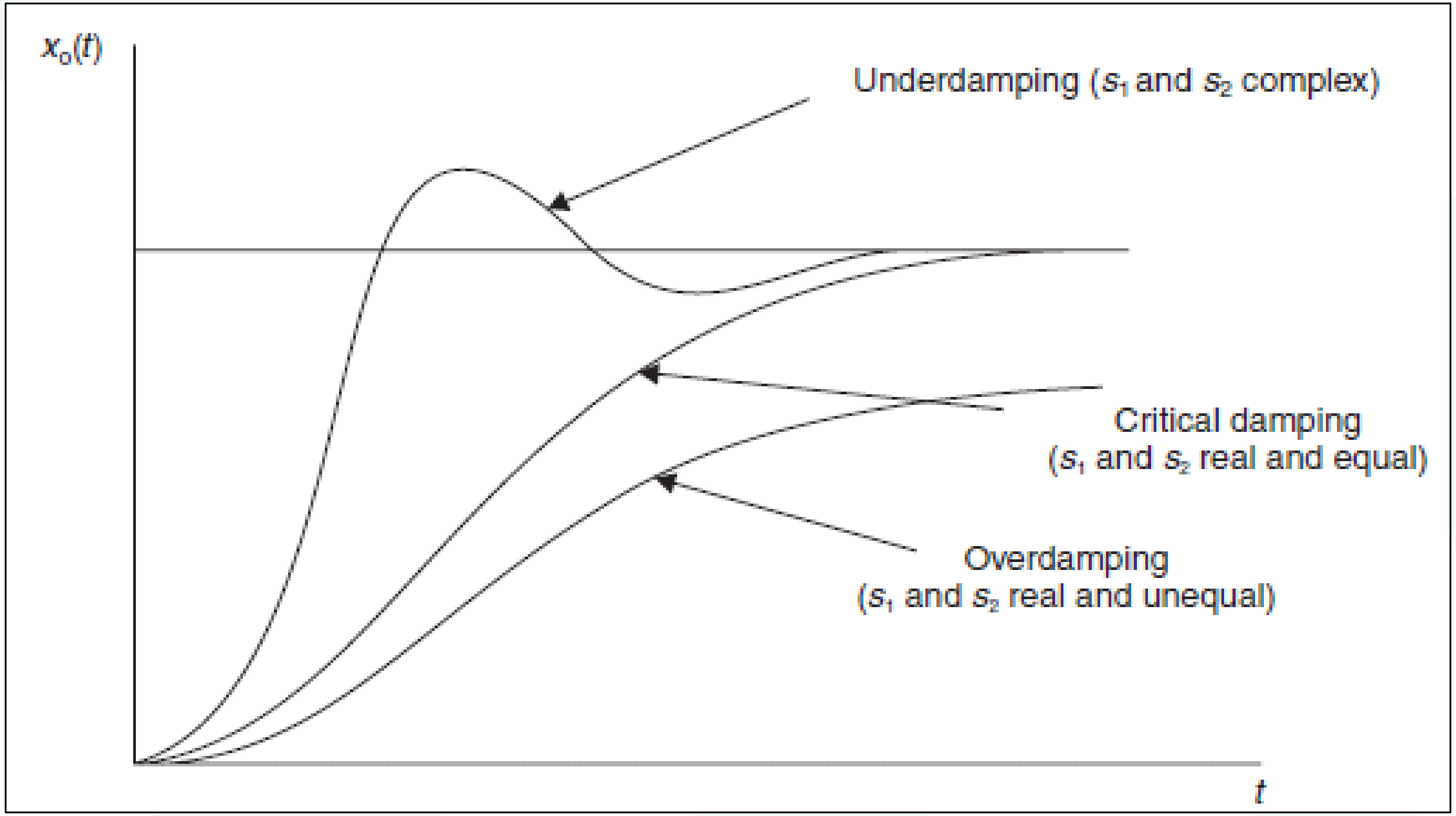
$$(as^2 + bs + c)X_0(s) = 0$$

$$(as^2 + bs + c) = 0$$

Characteristic Equation

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant	*Roots*	*Transient response type*
$b^2 > 4ac$	s_1 and s_2 real and unequal	Overdamped Transient Response
$b^2 = 4ac$	s_1 and s_2 real and equal	Critically Damped Transient Response
$b^2 < 4ac$	s_1 and s_2 complex conjugate of the form: $s_1, s_2 = -\sigma \pm j\omega$	Underdamped Transient Response


$x_o(t)$
Underdamping (s_1 and s_2 complex)
Critical damping
(s_1 and s_2 real and equal)
Overdamping
(s_1 and s_2 real and unequal)
t

Critical Damping and Damping Ratio

Critical damping

When the damping coefficient C of a second-order system has its critical value C_c, the system, when disturbed, will reach its steady-state value in the minimum time without overshoot. This is when the roots of the Characteristic Equation have equal negative real roots.

Damping ratio

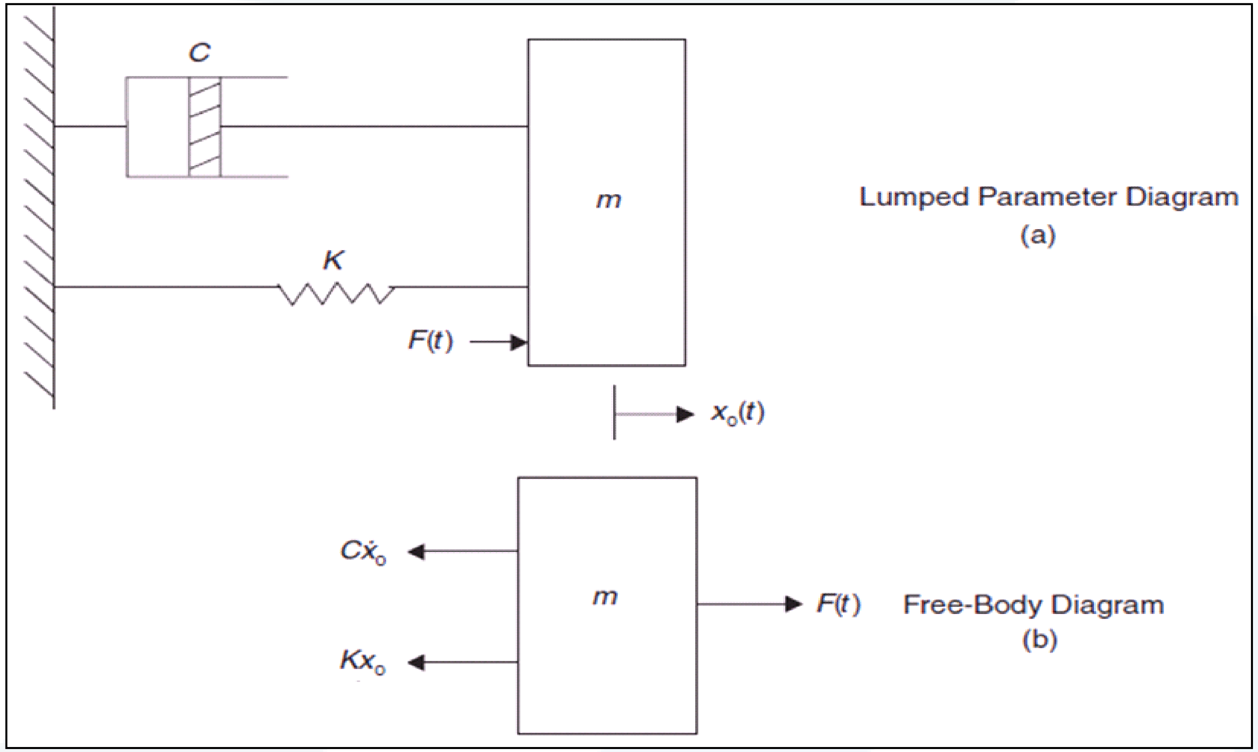
The ratio of the damping coefficient C in a second-order system compared with the value of the damping coefficient C_c required for critical damping is called the Damping Ratio ζ

$$\zeta = \frac{C}{C_c}$$

$\zeta = 0$	No damping
$\zeta < 1$	Underdamping
$\zeta = 1$	Critical damping
$\zeta > 1$	Overdamping

EXAMPLE

Find the value of the critical damping coefficient C_c in terms of K and m for the spring–mass–damper system shown in Figure



From Newton's second law

$$\sum Fx = m\ddot{x}_o$$

From the free-body diagram

$$F(t) - Kx_o(t) - C\dot{x}_o(t) = m\ddot{x}_o(t)$$

Taking Laplace transforms, zero initial conditions

$$F(s) - KX_o(s) - CsX_o(s) = ms^2X_o(s)$$

or

$$(ms^2 + Cs + K)X_o(s) = F(s)$$

Characteristic Equation is $ms^2 + Cs + K = 0$

$$\text{i.e.} \quad s^2 + \frac{C}{m} + \frac{K}{m} = 0$$

and the roots are

$$s_1, s_2 = \frac{1}{2}\left\{-\frac{C}{m} \pm \sqrt{\left(\frac{C}{m}\right)^2 - 4\frac{K}{m}}\right\}$$

For critical damping, the discriminant is zero, hence the roots become

$$s_1 = s_2 = -\frac{C_c}{2m}$$

Also, for critical damping

$$\frac{C_c^2}{m^2} = \frac{4K}{m}$$

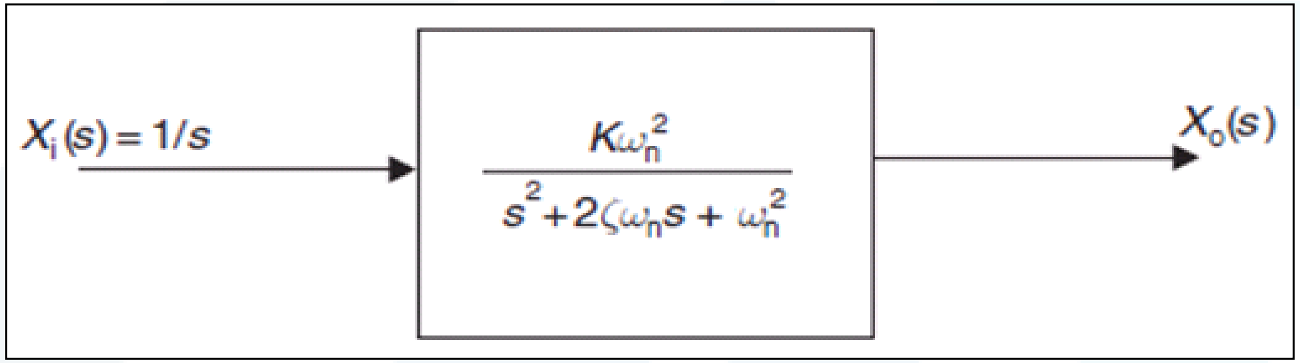
$$C_c^2 = \frac{4Km^2}{m}$$

giving

$$C_c = 2\sqrt{Km}$$

Generalized Second-Order System Response to a unit Step Input

Consider a second-order system whose steady-state gain is K, undamped natural frequency is ω_n and whose damping ratio is ζ, where $\zeta < 1$. For a unit step input, the block diagram is as shown in Figure



$$X_o(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$X_o(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

multiply by $s(s^2 + 2\zeta\omega_n s + \omega_n^2)$

$$K\omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs$$

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Equating coefficients

$$(s^2):\quad 0 = A + B$$

$$(s^1):\quad 0 = 2\zeta\omega_n A + C$$

$$(s^0):\quad K\omega_n^2 = \omega_n^2 A$$

$$A = K,\quad B = -K \quad \text{and} \quad C = -2\zeta\omega_n K$$

$$X_o(s) = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}\right]$$

Completing the square

$$X_o(s) = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2}\right\}\right] = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1 - \zeta^2}\right)^2}\right\}\right]$$

The terms in the brackets { } can be written in the standard forms

$$\text{Term (1)} = \frac{-s}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$\text{Term (2)} = -\left\{\frac{2\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right\}\left\{\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}\right\}$$

$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse transform

$$x_o(t) = K\left[1 - e^{-\zeta\omega_n t}\left\{\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

When $\zeta = 0$

$$x_o(t) = K[1 - e^0\{\cos\omega_n t + 0\}]$$
$$= K[1 - \cos\omega_n t]$$

From equation it can be seen that when there is no damping, a step input will cause the system to oscillate continuously at ω_n (rad/s).

Damped natural frequency ω_d

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

where ω_d is called the damped natural frequency.

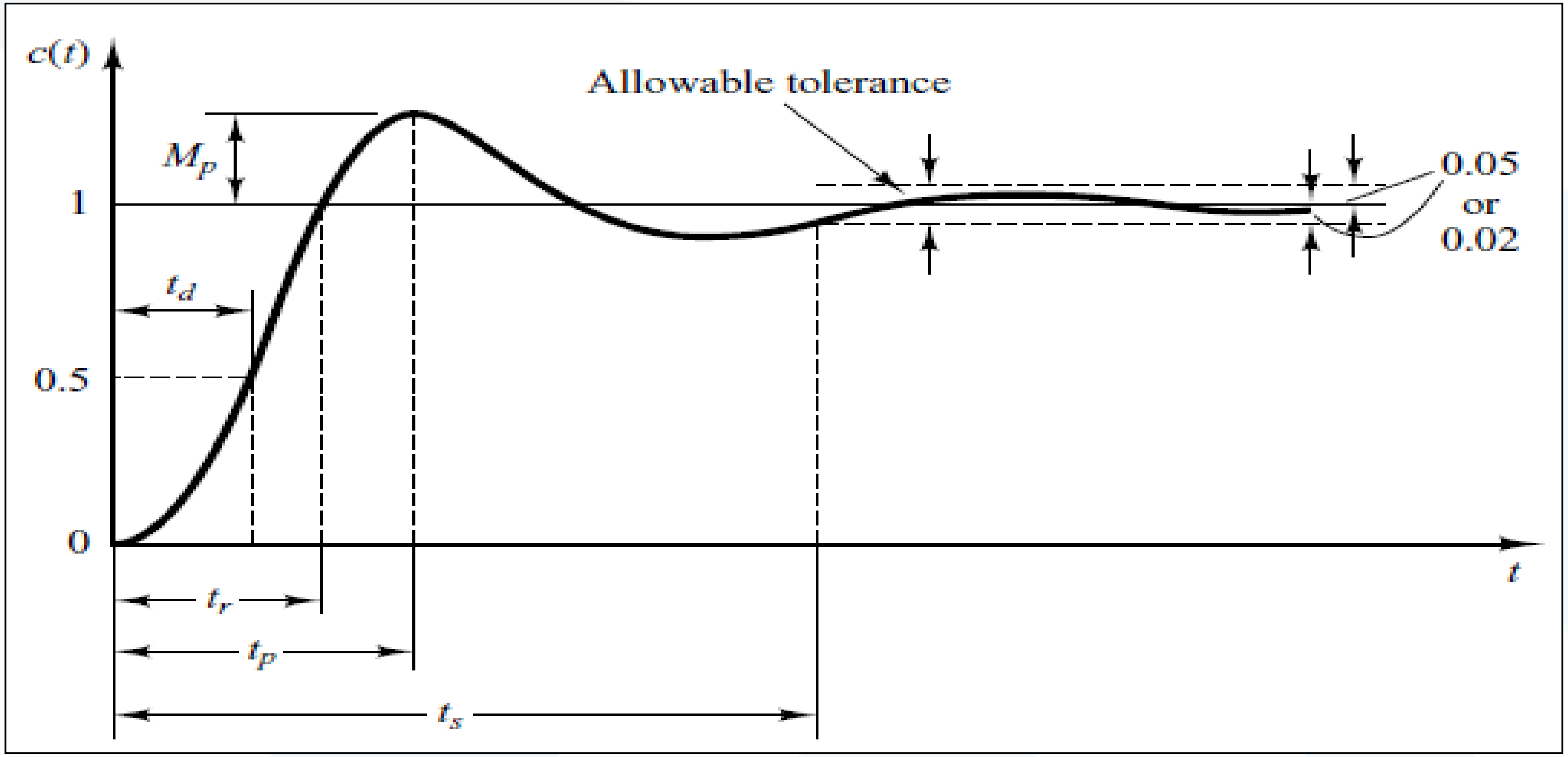
$$x_o(t) = K\left[1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\omega_d t\right\}\right]$$
$$= K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right] \qquad \tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Definitions of Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

These specifications are defined in what follows and are shown graphically in Figure



c(t)
Allowable tolerance
M_p
1
0.05
or
0.02
t_d
0.5
0
t
t_r
t_p
t_s

1. Delay time, t_d: The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time, t_r: The rise time is the time required for the response to rise from 10% to 90%, or 0% to 100% of its final value.
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

5. Settling time, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.

Second-Order System and Transient-Response Specifications

Peak time t_p: Referring to Equation , we may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$+ e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, dc/dt, evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin \omega_d t_p = 0$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot, $\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d}$$

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Maximum overshoot M_p: The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained from Equation as

$$M_p = c(t_p) - 1$$
$$= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$
$$= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$\sigma = \zeta\omega_n$

where σ is called the *attenuation*

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Settling time t_s: For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

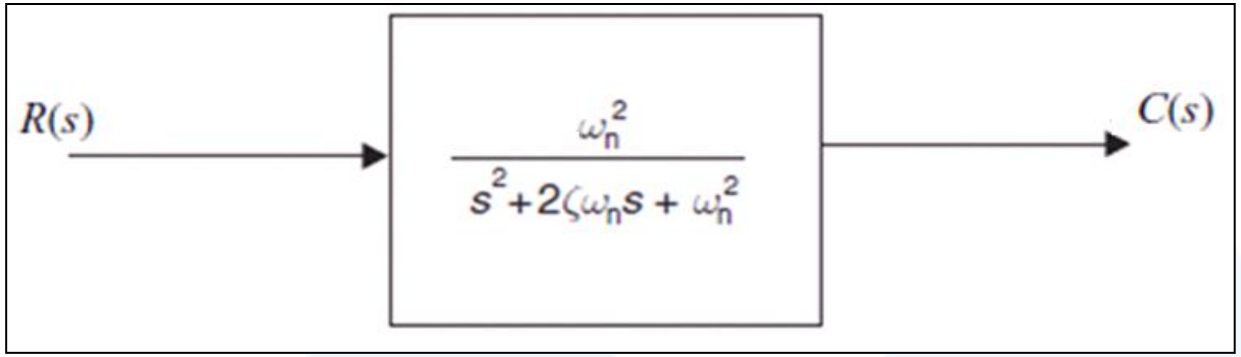
$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

or

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

EXAMPLE

Consider the system shown in Figure , where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.



From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1 - \zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Peak time t_p: The peak time is $$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s:

For the 2% criterion, the settling time is $t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33$ sec

For the 5% criterion, $t_s = \frac{3}{\sigma} = \frac{3}{3} = 1$ sec

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