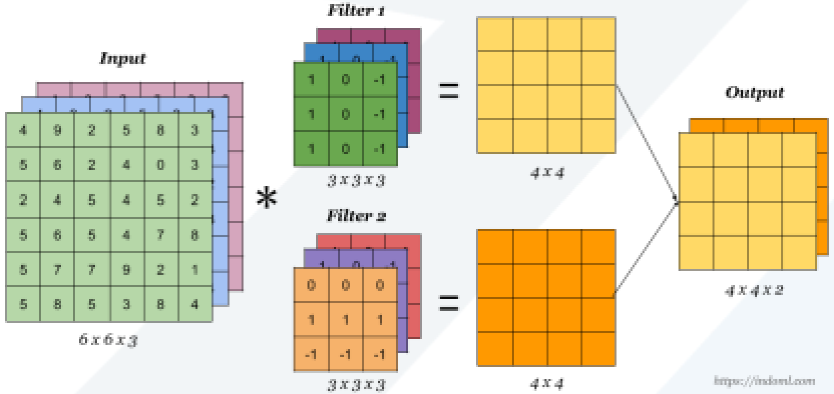
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 5: General Vector Spaces



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Informatics

Manara University

Chapter 4

General Vector Spaces

1. Real Vector Spaces

- **Definition:** Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and every scalar c and d, then V is called a vector space.

Addition:

(1) $\boldsymbol{u}+\boldsymbol{v}$ is in V — Closure under addition

(2) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ — Commutative property

(3) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$ — Associative property

(4) V has a zero vector $\mathbf{0}$: for every $\boldsymbol{u}$ in V, $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$ — Additive identity

(5) For every $\boldsymbol{u}$ in V, there is a vector in V denoted by $-\boldsymbol{u}$: $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$

Scalar identity

Scalar multiplication:

(6) $c\boldsymbol{u}$ is a vector in V — Closure under scalar multiplication

(7) $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ — Distributive property

(8) $(c+d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ — Distributive property

(9) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ — Associative property

(10) $1(\boldsymbol{u}) = \boldsymbol{u}$ — Scalar identity

- Notes:

(1) A vector space $(V, +, .)$ consists of four entities:
a nonempty set V of vectors, a set of scalars, and two operations $(+, .)$

(2) $V = \{\mathbf{0}\}$ zero vector space

- Examples of vector spaces:

(1) Euclidean vector space: $V = R^n$

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$ vector addition

$$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{mxn}$ (the set of all $m \times n$ matrices with real values)

Example: $(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$ vector addition

$$k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) n-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$kp(x) = ka_0 + ka_1x + \cdots + ka_nx^n$$

(4) Function space: $V = c(-\infty, \infty)$ (the set of all real functions)

$$(f + g)(x) = f(x) + g(x)$$

$$(kf)(x) = kf(x)$$

- **Theorem 1: (Properties of scalar multiplication)**

  Let v any element of a vector space V, and let c be any scalars. Then the following properties are true:

  (1) $0\mathbf{v} = \mathbf{0}$ (2) $c\mathbf{0} = \mathbf{0}$ (3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$ (4) $(-1)\mathbf{v} = -\mathbf{v}$

- Note: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Example 1: $V = R^2 =$ the set of all ordered pairs of real numbers
  vector addition: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$
  scalar multiplication: $c(u_1, u_2) = (cu_1, 0)$ Verify that V is not a vector space
  $1(1, 1) = (1, 0) \neq (1, 1) \Rightarrow V$ with the given operations is not a vector space.

- Example 2: Set of all real polynomials of degree n Is Not a vector space. Why?

2. Subspaces of Vector Spaces

- Definition: A non-empty subset W of a vector space V is called a subspace of V if it is also a vector space with respect to the same vector addition and scalar multiplication as V.

- Trivial subspace: Every vector space V has at least two subspaces:

  (1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

  (2) V is a subspace of V.

- Theorem 2: (Test for a subspace)

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold:

  (1) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are in W, then $\boldsymbol{u}+\boldsymbol{v}$ is in W.

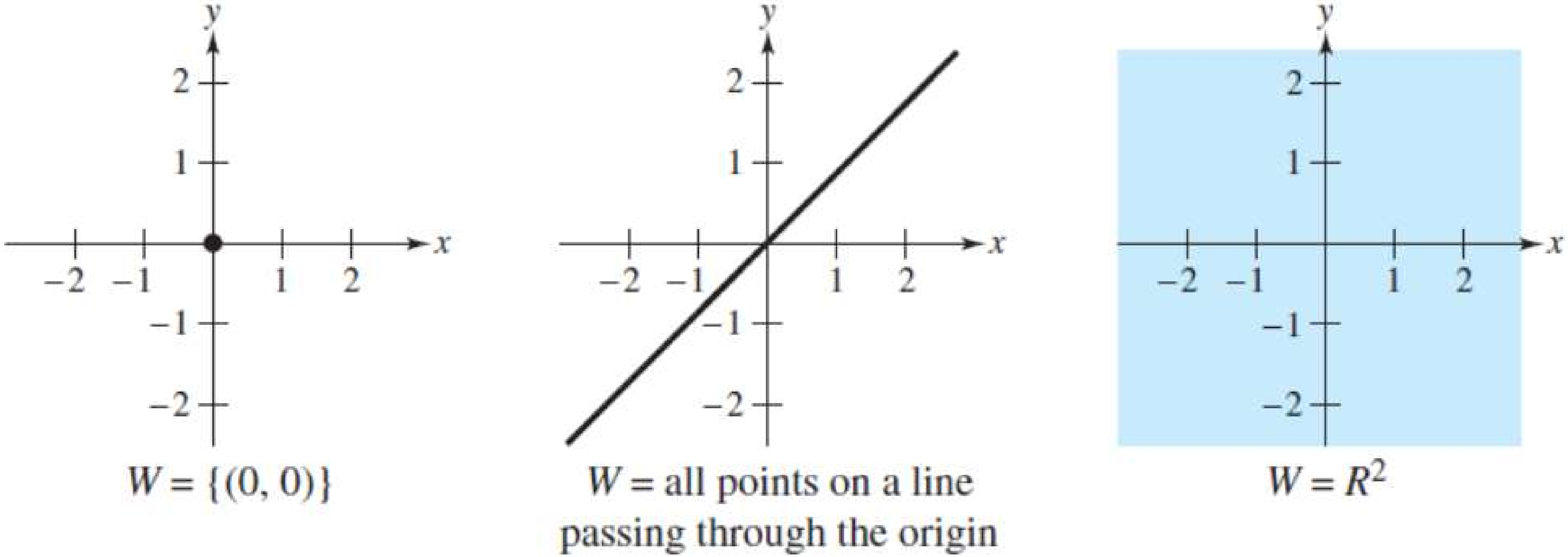
  (2) If $\boldsymbol{u}$ is in W and c is any scalar, then $c\boldsymbol{u}$ is in W.

- Notes:

  (1) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are in W, c and d are any scalars, then $c\boldsymbol{u} + d\boldsymbol{v}$ is in $W. \Rightarrow W$ is a subspace of V.

(2) If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V.

- Example 3: Subspaces of R^2



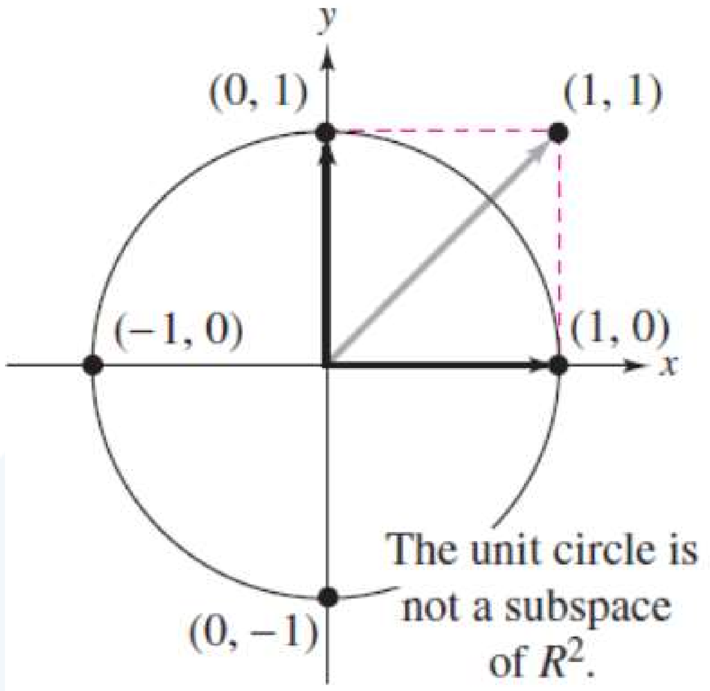
(1) $\{\mathbf{0}\}$ (2) Lines through the origin (3) R^2

- Example 4: (A Subset of R^2 That Is Not a Subspace)

  Show that the subset of R^2 consisting of all points on $x^2 + y^2 = 1$ is not a subspace.

  points (1, 0) and (0, 1) are in the subset, but their sum (1, 0) + (0, 1) = (1, 1) is not.

  (not closed under addition)



- Example 5: Subspaces of R^3

  (1) {**0**} $\quad$ **0** = (0, 0, 0)

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

- Example 6: (Determining subspaces of R^2)

  Which of the following two subsets is a subspace of R^2?

  (a) The set of points on the line given by $x+2y=0$. Yes

  (b) The set of points on the line given by $x+2y=1$. No

- Example 7: (A subspace of $M_{2\times2}$)

  Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication.

- Example 8: (The set of singular matrices is not a subspace of $M_{2\times2}$)

  Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations.

- Theorem 3: (The intersection of two subspaces is a subspace)
  If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

3. Spanning Sets and Linear Independence

- Definition: A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V if $\mathbf{v}$ can be written in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ where $c_1, c_2, \ldots, c_k$ are scalars.

- Example 9: (Finding a Linear Combination)
  Write the vector $\mathbf{v} = 1 + x + x^2$ in P_2 as a linear combination of vectors in the set $S = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = 1 - x, \mathbf{v}_3 = 1 - x^2\}$.

$$\mathbf{v} = 1 + x + x^2 = 3\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3.$$

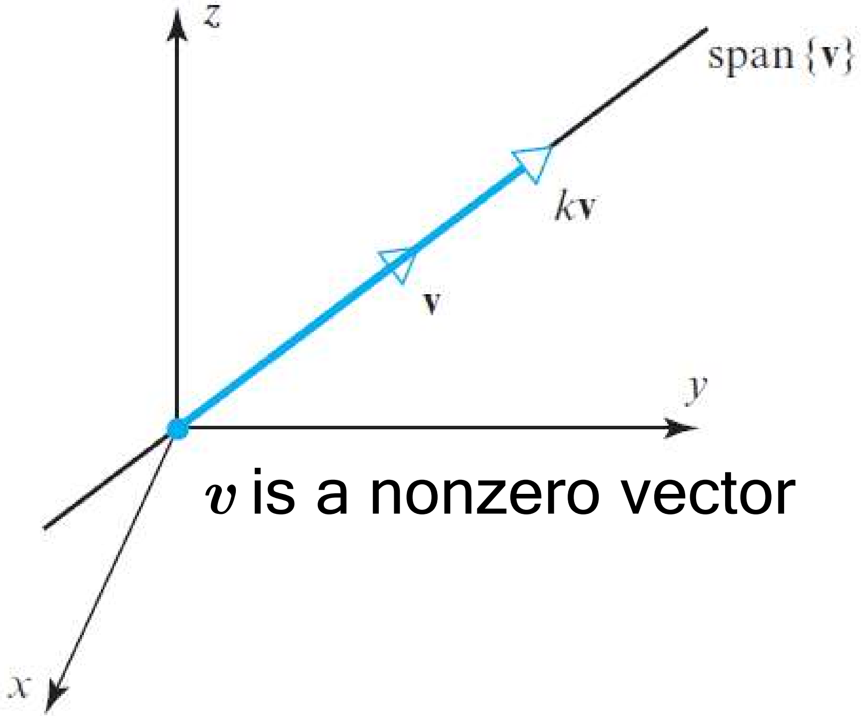
- Definition: Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of a vector space V. The set S is a spanning set of V if every vector in V can be written as a linear combination of vectors in S. In such cases it is said that S spans V.
- The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial $p(x) = a + bx + cx^2$ in P_2 can be written as: $p(x) = a(1) + b(x) + c(x^2)$.
- Definition: If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a set of a vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S.

$$\text{span}(S) = \left\{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k \,\middle|\, \forall c_i \in R\right\}$$

The span of S is denoted by: span(S) or span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

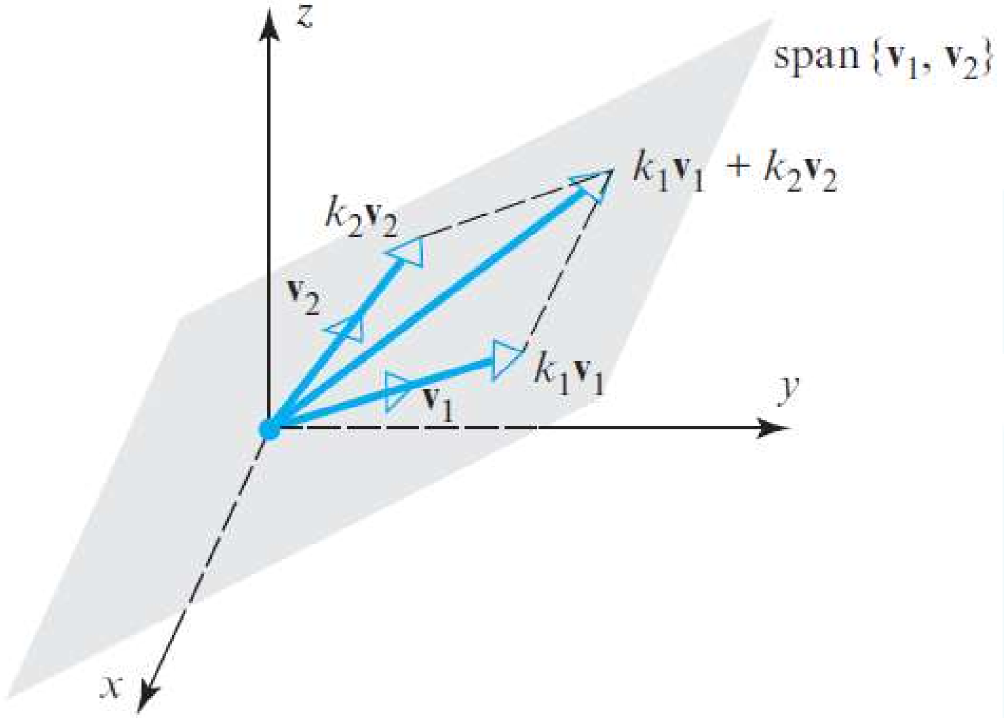
When span(S) $= V$, it is said that V is spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$, or that S spans V.

- Example 10: (A Geometric View of Spanning in R^3)



v is a nonzero vector

span$\{v\}$ is the line through the origin determined by v



span$\{v_1, v_2\}$ is the plane through the origin determined by v_1 and v_2

- **Theorem 4: (Span(S) is a subspace of V)**

  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a set of vectors in a vector space V, then

  (a) span(S) is a subspace of V.

  (b) span(S) is the smallest subspace of V that contains S.

- **Example 11: (Finding subspace spanned by a set of vectors)**

  Find the vector subspace spanned by the vectors $\{\boldsymbol{v}_1 = (1, 1, 1),\ \boldsymbol{v}_2 = (1, 2, 3)\}$

$$\boldsymbol{x} = (x, y, z) \in \text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2) \Rightarrow \boldsymbol{x} = \alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 = \alpha(1, 1, 1) + \beta(1, 2, 3)$$

$$\begin{aligned} x &= \alpha + \beta \\ y &= \alpha + 2\beta \\ z &= \alpha + 3\beta \end{aligned} \Rightarrow \begin{aligned} \alpha &= x - \beta \\ y &= x + \beta \\ z &= x + 2\beta \end{aligned} \Rightarrow 2y - z = x$$

$$\Rightarrow \text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{(x, y, z) \in R^3 | x - 2y + z = 0\}$$

- Definition: A set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ in a vector space V linearly independent (LI) when the vector equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_k\boldsymbol{v}_k = \mathbf{0}$ has only the trivial solution $c_1 = c_2 = \ldots\ c_k = 0$.
  If there are also nontrivial solutions, then S is linearly dependent (LD).

- Example 12: (Testing for linearly independent)
  Determine whether $S = \{\boldsymbol{v}_1 = 1 + x - 2x^2, \boldsymbol{v}_2 = 2 + 5x - x^2, \boldsymbol{v}_3 = x + x^2\}$ in P_2 is LI or LD

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0} \Rightarrow \begin{array}{rcrcrl} c_1 & + & 2c_2 & & & = 0 \\ c_1 & + & 5c_2 & + & c_3 & = 0 \\ -2c_1 & - & c_2 & + & c_3 & = 0 \end{array}$$

$$\Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss Elimination}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

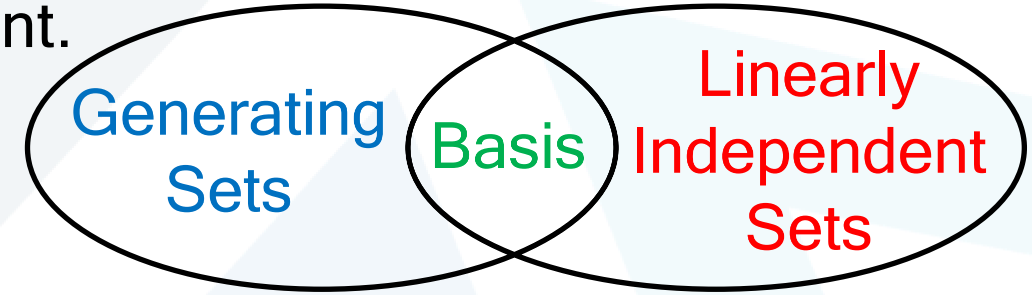
$\Rightarrow$ Infinitely many solutions

$\Rightarrow$ S is linearly dependent

4. Basis and Dimension

- **Definition:** A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is a basis for V when the conditions below are true:

  1. S spans V. 2. S is linearly independent.



- The standard basis for R^n:

  $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1 = (1,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,\ldots,0)$, $\mathbf{e}_n = (0,0,\ldots,1)$

  Example: R^4 $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

- The standard basis for $M_{m\times n}$ matrix space: $\{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$

  Example: M_{2x2} $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

- Theorem 5: (Uniqueness of basis representation)
  If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.
- Theorem 6: (Bases and linear dependence)
  If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.
- Theorem 7: (Number of vectors in a basis)
  If a vector space V has one basis with n vectors, then every basis for V has n vectors.
- Definition: A vector space V is called finite dimensional, if it has a basis consisting of a finite number of elements.

- Definition: The dimension of a finite dimensional vector space V is defined to be the number of vectors in a basis for V.

  V: a vector space, S: a basis for $V \Rightarrow \dim(V) = \#(S)$ (the number of vectors in S)

- Notes:

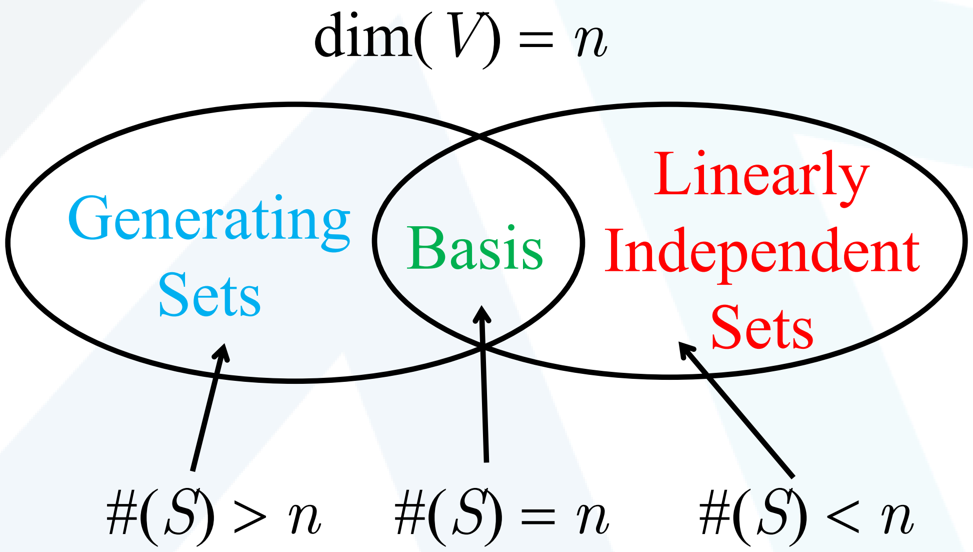
  (1) $\dim(\{\mathbf{0}\}) = 0$

  (2) $\dim(V) = n,\ S \subseteq V$

  S: a LI set $\Rightarrow \#(S) \le n$

  S: a generating set $\Rightarrow \#(S) \ge n$

  S: a basis $\Rightarrow \#(S) = n$



5. Rank and Nullity of a Matrix

The Three Fundamental Spaces of a Matrix If A is an $m\text{x}n$ matrix, then

- **Definition:** The subspace of R^n spanned by the row vectors of A is denoted by row$(A) = RS(A)$ and is called the row space of A.
- **Definition:** The subspace of R^m spanned by the column vectors of A is denoted by col$(A) = CS(A)$ and is called the column space of A.
- **Definition:** The solution space of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$, which is a subspace of R^n, is denoted by null$(A) = NS(A)$ and is called the null space of A.
- **Theorem 8: (Row and column space have equal dimensions)**
  If A is an $m\text{x}n$ matrix, then the row space and the column space of A have the same dimension $\dim(RS\ (A)) = \dim(CS\ (A))$.

- Theorem 9: (Solution of a system of linear equations)

  The system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is in the column space of A.

- Definition: The dimension of the row (or column) space of a matrix A is called the rank of A and is denoted by $\operatorname{rank}(A)$: $\operatorname{rank}(A) = \dim(RS\,(A)) = \dim(CS\,(A))$.

- Definition: The dimension of the nullspace of A is called the nullity of A: $\operatorname{nullity}(A) = \dim(NS\,(A))$.

- Theorem 10: If A is any matrix, then $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

- Notes:

  (1) The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix.

(2) The number of leading 1's in the reduced row-echelon form of A is equal to the rank of A.

(3) The number of free variables in the reduced row-echelon form of A is equal to the nullity of A.

- Theorem 11: (Consistency of $A\boldsymbol{x} = \boldsymbol{b}$)

  If $\operatorname{rank}([A|\boldsymbol{b}]) = \operatorname{rank}(A)$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

- Notes:

  (1) If $\operatorname{rank}(A) = \operatorname{rank}(A|\boldsymbol{b}) = n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

  (2) If $\operatorname{rank}(A) = \operatorname{rank}(A|\boldsymbol{b}) < n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has ∞-many solutions.

  (3) If $\operatorname{rank}(A) < \operatorname{rank}(A|\boldsymbol{b})$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.

- Example 13: (Rank by Row Reduction)

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{bmatrix} \xrightarrow{\text{Gauss Elimination}} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) =$ 2 (2 non-zero rows) $\text{nullity}(A) =$ 2 (2 free variables)

- Example 14 : (Finding the solution set of a nonhomogeneous system)

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 \quad\quad + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A \vdots \mathbf{b}] = \left[\begin{array}{ccc:c} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 \quad\quad + x_3 &= 3 \\ x_2 - 2x_3 &= -4 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 3 - x_3 \\ x_2 &= -4 + 2x_3 \end{aligned}$$

letting $x_3 = t$, then the solutions are: $\{(3 - t, -4 + 2t, t) | t \in R\}$

So the system has infinitely many solutions (consistent)

- Note: $\text{rank}(A) = \text{rank}([A \vdots \mathbf{b}]) = 2$.
- Theorem 12: (Dimension Theorem for Matrices)

  If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.

- Example 15 : (Rank and nullity of a matrix)

Find the rank and nullity of $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 3$ (the number of nonzero rows in B)

$\text{nullity}(A) = n - \text{rank}(A) = 5 - 3 = 2$

- Summary of equivalent conditions for square matrices:

  If A is an $n\text{x}n$ matrix, then the following conditions are equivalent:

  (1) A is invertible

  (2) $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution for any $n\times 1$ matrix $\boldsymbol{b}$.

  (3) $A\boldsymbol{x}=\boldsymbol{0}$ has only the trivial solution.

  (4) A is row-equivalent to I_n.

  (5) $|A|\neq 0$.

  (6) $\text{rank}(A)=n$.

  (7) The n row vectors of A are linearly independent.

  (8) The n column vectors of A are linearly independent.

6. Coordinates and Change of Basis

- **Coordinate representation relative to a basis:** Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be an ordered basis for a vector space V and let $\boldsymbol{x}$ be a vector in V such that: $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n$.

  The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\boldsymbol{x}$ relative to the basis B. The coordinate matrix (or coordinate vector) of x relative to B is the column matrix in R^n whose components are the coordinates of $\boldsymbol{x}$.

$$[\boldsymbol{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- **Example 16 : (Coordinates and components in R^n)**

  Find the coordinate matrix of $\boldsymbol{x} = (-2, 1, 3)$ in R^3 relative to the standard basis S.

$$\boldsymbol{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$[\boldsymbol{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

- **Example 17 : (Finding a coordinate matrix relative to a nonstandard basis)**

  Find the coordinate matrix of $\boldsymbol{x} = (1, 2, -1)$ in R^3 relative to the (nonstandard) basis $B' = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$

$$\boldsymbol{x} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} \Rightarrow (1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

$$\Rightarrow \begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \text{i.e.} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{bmatrix} \xrightarrow{\text{G. J. Elimination}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Change of Basis In R^n

- Change of basis: Given the coordinates of a vector relative to a basis B, find the coordinates relative to another basis B'.

  In Example 17, let B be the standard basis. Finding the coordinate matrix of $\boldsymbol{x} = (1, 2, -1)$ relative to the basis B' becomes solving for c_1, c_2, and c_3 in the matrix equation.

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\boldsymbol{x}]_{B'}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\boldsymbol{x}]_B}$$

P is the transition matrix from B' to B,

$P[\boldsymbol{x}]_{B'} = [\boldsymbol{x}]_B$ Change of basis from B' to B

$[\boldsymbol{x}]_{B'} = P^{-1}\ [\boldsymbol{x}]_B$ Change of basis from B to B'

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

P^{-1} $\quad [\boldsymbol{x}]_B$ $\quad [\boldsymbol{x}]_{B'}$

$$[\boldsymbol{x}]_{B'} = P^{-1} [\boldsymbol{x}]_B$$

Coordinate matrix of $\boldsymbol{x}$ relative to B'

Transition matrix from B to B'

Coordinate matrix of $\boldsymbol{x}$ relative to B

- **Theorem 13: (The inverse of a transition matrix)**

  If P is the transition matrix from a basis B' to a basis B in R^n, then

  (1) P is invertible.

  (2) The transition matrix from B to B' is P^{-1}.

- **Notes:**

$$B=\{\boldsymbol{u}_1,\ \boldsymbol{u}_2,\ \dots,\ \boldsymbol{u}_n\},\qquad B'=\{\boldsymbol{u}'_1,\ \boldsymbol{u}'_2,\ \dots,\ \boldsymbol{u}'_n\}$$

$$[\boldsymbol{v}]_B=[[\boldsymbol{u}'_1]_B,\ [\boldsymbol{u}'_2]_B,\ \dots,\ [\boldsymbol{u}'_n]_B]\quad [\boldsymbol{v}]_{B'}=P\ [\boldsymbol{v}]_{B'}$$

$$[\boldsymbol{v}]_{B'}=[[\boldsymbol{u}_1]_{B'},\ [\boldsymbol{u}_2]_{B'},\ \dots,\ [\boldsymbol{u}_n]_{B'}]\quad [\boldsymbol{v}]_B=P^{-1}\ [\boldsymbol{v}]_B$$

- **Theorem 14: (Transition matrix from B to B')**

Let $B=\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \dots,\ \boldsymbol{v}_n\}$ and $B'=\{\boldsymbol{u}_1,\ \boldsymbol{u}_2,\dots,\ \boldsymbol{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the nx$2n$ matrix $[B' \vdots B]$ as follows: $[B' \vdots B] \longrightarrow [I_n \vdots P^{-1}]$

- **Example 18: (Finding a transition matrix)**

$B=\{(-3,\ 2),\ (4,-2)\}$ and $B'=\{(-1,\ 2),\ (2,-2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B.

(b) Let $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\boldsymbol{v}]_B$

(c) Find the transition matrix from B to B'.

(a) $\underbrace{\begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}}_{B} \vdots \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}}_{B'}$ $\xrightarrow{\text{G. J. Elimination}}$ $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \vdots \underbrace{\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}}_{P}$ $\Rightarrow P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

(the transition matrix from B' to B)

(b) $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_B = P[\boldsymbol{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

- **Check:** $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{v} = (1)(-1, 2) + (2)(2, -2) = (3, -2)$

  $[\boldsymbol{v}]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = (-1)(3, -2) + (0)(4, -2) = (3, -2)$

(c) $\begin{bmatrix} -1 & 2 & \vdots & -3 & 4 \\ 2 & -2 & \vdots & 2 & -2 \end{bmatrix} \xrightarrow{\text{G. J. Elimination}} \begin{bmatrix} 1 & 0 & \vdots & -1 & 2 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

B' B I P^{-1}

(the transition matrix from B to B')

- **Check:** $PP^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

- **Example 19: (Finding a transition matrix)**

  Find the transition matrix from B to B' for The bases for R^3 below.

  $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{G. J. Elimination}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$

B' B I P^{-1}

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

the result is the same as that obtained in Example 17