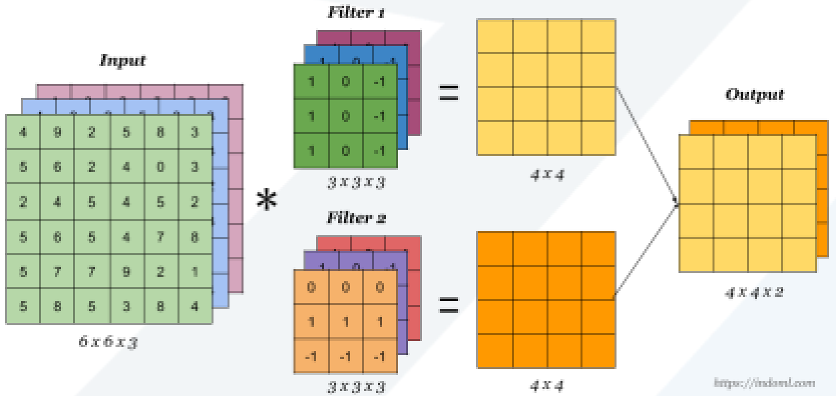
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 8: Eigenvalues and Eigenvectors



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Informatics

Manara University

Chapter 7

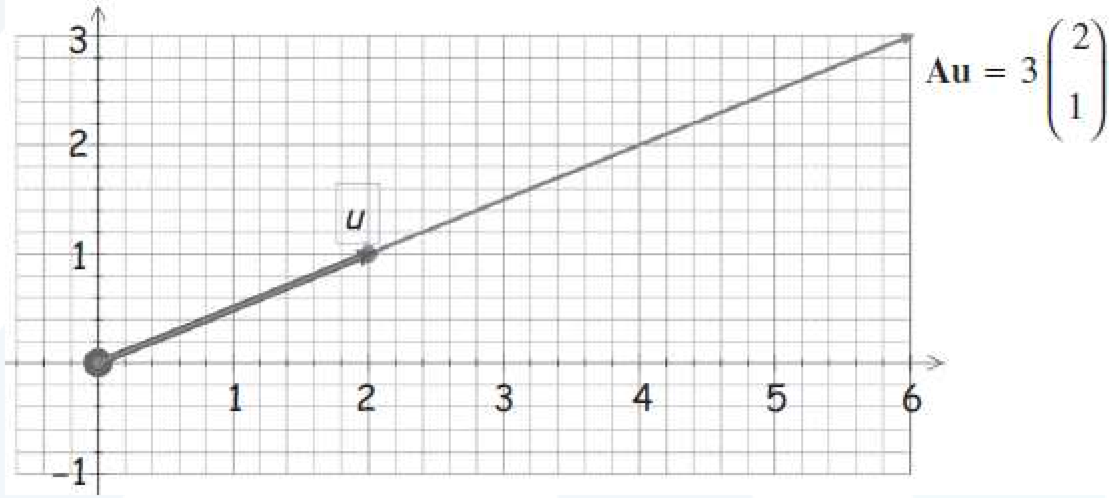
Eigenvalues and Eigenvectors

1. Eigenvalues and Eigenvectors

- By knowing the eigenvalues and eigenvectors of a matrix we can easily find its determinant, decide whether the matrix has an inverse and determine the powers of the matrix.
- For an example of linear algebra at work: Google's search engine, which relies upon eigenvalues and eigenvectors to rank pages with respect to relevance.
- Example 1:

Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A\boldsymbol{u} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3\boldsymbol{u}$$



- Matrix A transforms the vector $\boldsymbol{u}$ by scalar multiplying it $\Rightarrow$ the transformation only changes the length of the vector $\boldsymbol{u}$ unless $\lambda = \pm 1$ (length unchanged).
- Note: the relation $A\boldsymbol{u} = \lambda\boldsymbol{u}$ says that the matrix A applied to $\boldsymbol{u}$ gives a vector in the same or opposite (negative λ) direction of $\boldsymbol{u}$.
- Definition: For a non-zero vector $\boldsymbol{u}$ in R^n the scalar λ is called an eigenvalue of the $n\times n$ matrix A and the vector $\boldsymbol{u}$ is called an eigenvector corresponding to λ, which satisfies $A\boldsymbol{u} = \lambda\boldsymbol{u}$.

Eigenvalue

$$A\boldsymbol{u} = \lambda\boldsymbol{u}$$

Eigenvector

- Example 2: (Verifying eigenvalues and eigenvectors)

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \overset{\text{Eigenvalue}}{2}\underset{\text{Eigenvector}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = 2\boldsymbol{u}$$

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \overset{\text{Eigenvalue}}{3}\underset{\text{Eigenvector}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 3\boldsymbol{v}$$

- Example 3: (Verifying eigenvalues and eigenvectors)

Let $A = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$A\boldsymbol{u} = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \overset{\text{Eigenvalue}}{2}\underset{\text{Eigenvector}}{\boldsymbol{u}}$$

- Theorem 1: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

  (1) An eigenvalue of A is a scalar λ such that that $\det(\lambda I - A) = 0$.

  (2) The eigenvectors of A corresponding to λ are the nonzero solutions of the homogeneous system $(\lambda I - A)\boldsymbol{u} = \mathbf{0}$.

- Note: $A\boldsymbol{u} = \lambda\boldsymbol{u} \Rightarrow (\lambda I - A)\boldsymbol{u} = \mathbf{0}$ has nonzero solutions iff $\det(\lambda I - A) = 0$.

- The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of A.

- The polynomial $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$ is called the characteristic polynomial of A.

- Example 4: (Finding eigenvalues and eigenvectors)

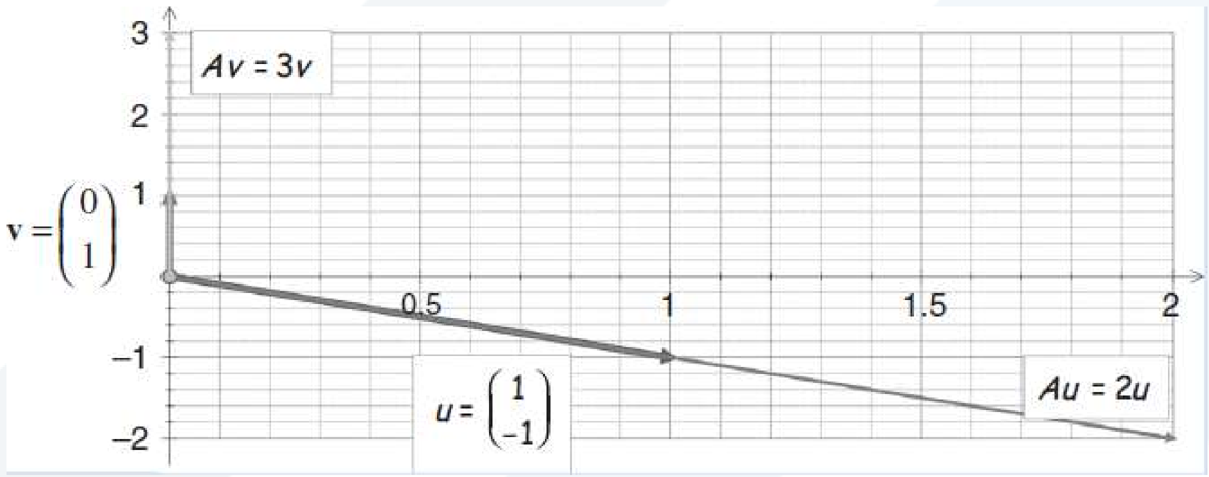
  Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

Characteristic equation: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3) = 0$

Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = 3$

$$(1)\ \lambda_1 = 2:\ (\lambda_1 I - A)\boldsymbol{u} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix},\ t \neq 0$$

$$(2)\ \lambda_1 = 3:\ (\lambda_1 I - A)\boldsymbol{u} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ t \neq 0$$



- Example 5: (Finding eigenvalues and eigenvectors)

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 & -3 \\ -1 & \lambda - 5 & -1 \\ -3 & -1 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 6)(\lambda + 2) = 0$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 6$ and $\lambda_3 = -2$

$$\lambda_1 = 3: \ (\lambda I - A)\boldsymbol{u} = \begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\lambda_2 = 6:\ (\lambda I - A)\boldsymbol{u} = \begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda_3 = -2:\ (\lambda I - A)\boldsymbol{u} = \begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ t \neq 0$$

- Definition: Let A be an $n \times n$ matrix and have the eigenvalues λ_1, λ_2, ..., λ_n. If λ occurs only once then we say λ is a simple eigenvalue. If λ occurs m times where $m > 1$ then we say λ is an eigenvalue with multiplicity of m.
- In the equation $(\lambda - 1)^3(\lambda - 2) = 0$ we have $\lambda_{1,2,3} = 1$ is an eigenvalue of multiplicity 3 and $\lambda_1 = 2$ is a simple eigenvalue.
- Example 6: (Finding eigenvalues and eigenvectors)

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -3 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0 \quad \text{Eigenvalue: } \lambda_{1,2,3} = 2$$

$$\lambda_{1,2,3} = 2: (\lambda I - A)\boldsymbol{u} = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

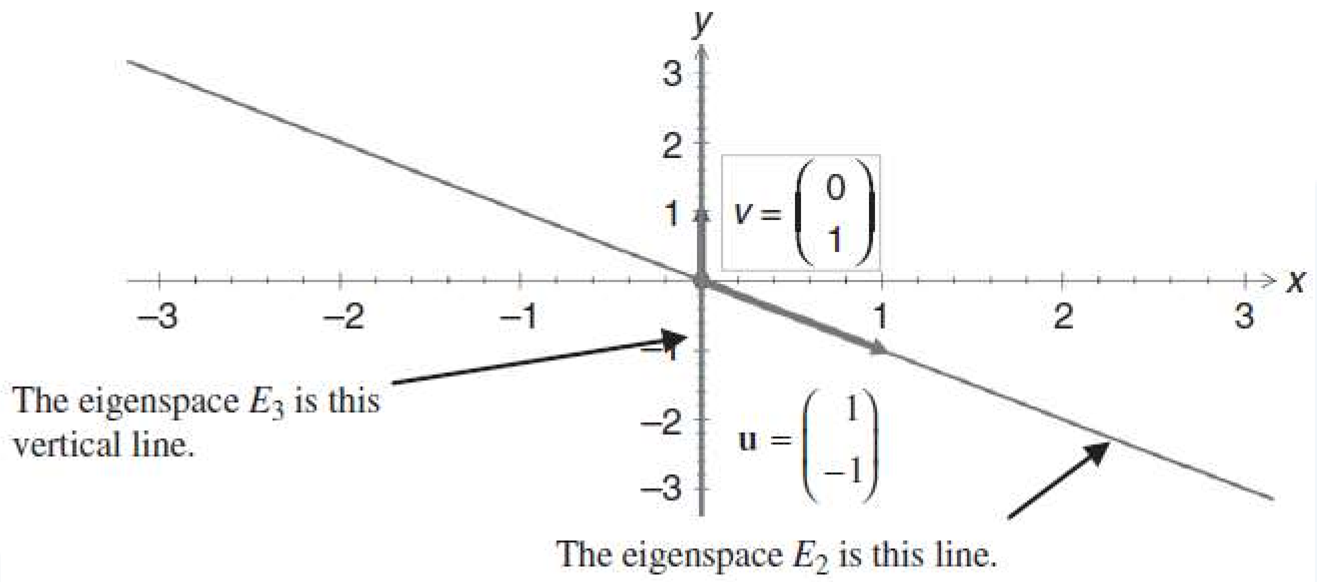
$$\begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -3t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

Eigenspace

- If λ is an eigenvalue of a square matrix A with an eigenvector $\boldsymbol{u}$ then every non-zero scalar multiplication of $\boldsymbol{u}$, is also an eigenvector corresponding to λ.
- If A is an $n \times n$ matrix with an eigenvalue of λ, then the set S of all eigenvectors of A corresponding to λ together with the zero vector, 0, is a subspace of R^n:

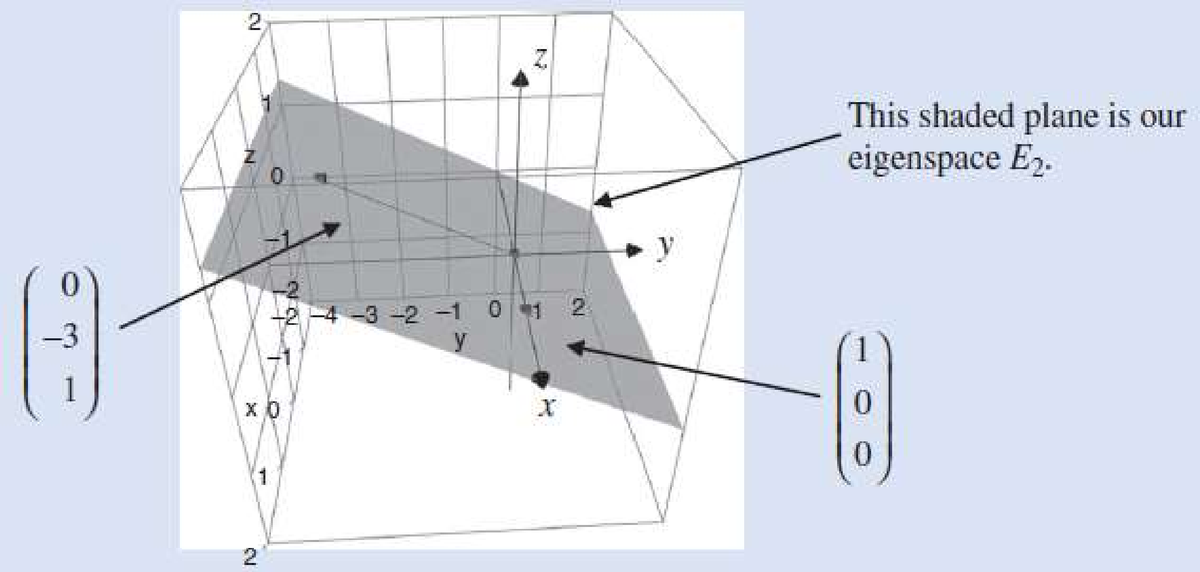
$$S = \{0\} \cup \{\boldsymbol{u} | \boldsymbol{u} \text{ is an eigenvector corresponding to } \lambda\}$$

- This subspace S is called an eigenspace of λ and is denoted by E_λ, that is $E_\lambda = S$.
- For example, the eigenspace associated with Example 4 for the eigenvalue $\lambda_1 = 2$ is the eigenvector $\boldsymbol{u} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for $\lambda_2 = 3$ the eigenvector $\boldsymbol{v} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



- The eigenspace associated with Example 6 for the eigenvalue $\lambda_{1,2,3} = 2$ is:

$$E_2 = \left\{ s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\-3\\1\end{bmatrix} \right\}$$



- Note: A set of basis vectors B of the eigenspace E_2 are given by:

$$B = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\1\end{bmatrix} \right\}$$

- Theorem 2: (Eigenvalues of triangular matrices)

  If A is an $n \times n$ diagonal or triangular matrix, then its eigenvalues are the entries on its main diagonal.

- **Example 7: (Finding eigenvalues)**

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$, $\quad |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3) = 0$

$\lambda_1 = 2,\ \lambda_2 = 1$ and $\lambda_3 = -3$

- **Theorem 3: (Eigenvalues of triangular matrices)**

If A is an $n\text{x}n$ matrix with eigenvector $\boldsymbol{u}$ corresponding to eigenvalue λ.

(1) If m is a positive integer, then λ^m is an eigenvalue of the matrix A^m with the same eigenvector $\boldsymbol{u}$.

(2) If the matrix A is invertible, then the eigenvalue of the inverse matrix A^{-1} is $1/\lambda$ with the same eigenvector $\boldsymbol{u}$.

- **Example 8: (Finding eigenvalues)**

Find the eigenvalues of A^3 and A^{-1} of Example 5

The eigenvalues of A are: $\lambda_1 = 3$, $\lambda_2 = 6$ and $\lambda_3 = -2$

The eigenvalues of A^3 are: $\lambda_1 = 3^3 = 27$, $\lambda_2 = 6^3 = 216$ and $\lambda_3 = (-2)^3 = -8$

The eigenvalues of A^{-1} are: $\lambda_1 = 1/3 =$, $\lambda_2 = 1/6$ and $\lambda_3 = -1/2$

- **Theorem 4: (Determinant and Trace of a matrices)**

  If A is an $n x n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

  (1) The determinant of the matrix A is given by $\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$.

  (2) The trace of the matrix A is given by $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

- **Example 9:**

  Find the determinant and trace of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ given that the eigenvalues of A are 1, 4 and –1.

  $\det(A) = (1)(4)(-1) = -4$, $\operatorname{tr}(A) = 1 + 4 - 1 = 4$

- **Theorem 5: (Eigenvalues and Invertibility)**

  A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A

- **Theorem 6: (linearly independent eigenvectors)**

  Let A be an $n\text{x}n$ matrix with eigenvalues λ_1, λ_2, ..., λ_m. and corresponding eigenvectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, ..., $\boldsymbol{p}_m$ where $1 \le m \le n$. Then these eigenvectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, ... and $\boldsymbol{u}_m$ are linearly independent.

- **Theorem 7: (The Cayley Hamilton Theorem)**

  Let A be a square matrix of size $n\text{x}n$, and let $p(\lambda)$ be its characteristic polynomial, i.e. $p(\lambda) = \det(\lambda I - A)$, then:

$$p(A) = A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I = O$$

- Note: A consequence of the Cayley-Hamilton theorem is a new method for finding the inverse of a nonsingular matrix.

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = O \Rightarrow A\left[\frac{-1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1)\right] = I$$

$$\Rightarrow A^{-1} = \frac{-1}{c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1)$$

- Example 10: (Finding the inverse of a matrix using Cayley-Hamilton theorem)

Find the inverse of $A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$

$$p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$$

$$\Rightarrow A^{-1} = \tfrac{1}{4}(A^2 - 5A + 8I)$$

$$A^{-1} = \frac{1}{4}\left(\begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}^2 - 5\begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{4}\begin{bmatrix} 18 & 6 & 2 \\ 6 & 20 & 2 \\ 6 & -6 & 4 \end{bmatrix}$$

2. Diagonalization

- **Definition:** If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$.
- **Definition:** A square matrix A is said to be diagonalizable if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A.
- **Theorem 8: (Similar matrices have the same eigenvalues)**
  An $n\mathrm{x}n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.
- **Steps for diagonalizing an $n\times n$ square matrix:**
  Step 1: Find n linearly independent eigenvectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$

Step 2: Let $P = \left[\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \cdots \mid \boldsymbol{p}_n \right]$

Step 3: Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where $A\boldsymbol{p}_i = \lambda_i \boldsymbol{p}_i, \ \ i = 1, 2, \ldots, n$

- Example 11: (A diagonalizable matrix)

Show that the matrix A is diagonalizable $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then find a matrix P such that $P^{-1}AP$ is diagonal.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4, \lambda_{2,3} = -2$

$$\lambda_1 = 4,\ \boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_{2,3} = -2,\ \boldsymbol{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \boldsymbol{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} \boldsymbol{p}_2 & \boldsymbol{p}_1 & \boldsymbol{p}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Note: The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Example 12: (A matrix that is not diagonalizable)

Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$ Eigenvalue: $\lambda_{1,2} = 1$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ $\Rightarrow$ Eigenvector: $\boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

A does not have two $(n = 2)$ linearly independent eigenvectors, so A is not diagonalizable.

- **Theorem 9: (Sufficient conditions for diagonalization)**

  If an $n\text{x}n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- **Example 13: (Determining whether a matrix is diagonalizable)**

  Determine whether the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ is diagonalizable

  A is a triangular matrix, its eigenvalues are the main diagonal entries $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -3$. These three values are distinct, so A is diagonalizable.

- **Theorem 10: (Powers of matrices)**

  If an $n\text{x}n$ matrix A is diagonalizable with $P^{-1}AP = D$ where D is a diagonal matrix then $A^k = PD^kP^{-1}$ where k is a positive integer, and

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- **Example 14: (Powers of matrices)**

Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Find A^6 given that $P = \begin{array}{c} \lambda = \;\; 2 \quad\; 2 \quad\; 1 \\ \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$ and $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$$A^6 = PD^6P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 2^6 & 0 \\ 0 & 0 & 1^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$

3. Symmetric Matrices and Orthogonal Diagonalization

- A square matrix A is symmetric if it is equal to its transpose: $A = A^T$.

- Example 15: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

(symmetric) (symmetric) (nonsymmetric)

- Theorem 11: (Eigenvalues of symmetric matrices)

  If A is an $n\text{x}n$ symmetric matrix, then the following properties are true:

  (1) A is diagonalizable.

  (2) All eigenvalues of A are real.

(3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors.

- A square matrix P is called orthogonal if it is invertible and $P^{-1} = P^T$.
- Example 16: (Orthogonal matrices)

(a) $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$

- Theorem 12: (Properties of orthogonal matrices)

An $n\mathrm{x}n$ matrix P is orthogonal iff its column vectors form an orthonormal set.

▪ Example 17: (Orthogonal matrices)

Show that the matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$ is orthogonal

If P is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\{\boldsymbol{p}_1, \quad \boldsymbol{p}_2, \quad \boldsymbol{p}_3\}$$

$$\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = \boldsymbol{p}_1 \cdot \boldsymbol{p}_3 = \boldsymbol{p}_2 \cdot \boldsymbol{p}_3 = 0, \quad \|\boldsymbol{p}_1\| = \|\boldsymbol{p}_2\| = \|\boldsymbol{p}_3\| = 1$$

- **Theorem 13: (Properties of symmetric matrices)**
  Let A be an $n\mathrm{x}n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

- **Example 18: (Eigenvectors of a symmetric matrix)**

  Show that any two eigenvectors of matrix $A=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ corresponding to distinct eigenvalues are orthogonal.

$$|\lambda I - A| = \begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) = 0$$

$\Rightarrow$ Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = 4$

$$(1)\ \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \Rightarrow \boldsymbol{u} = s\begin{bmatrix}-1\\ 1\end{bmatrix},\quad s \neq 0$$

$$(2)\ \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{u} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad t \neq 0$$

$\boldsymbol{u}.\boldsymbol{v} = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Orthogonal Diagonalization

- matrix A is orthogonally diagonalizable when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

- Theorem 14: (Fundamental theorem of symmetric matrices)
  Let A be an $n\text{x}n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric.

- Orthogonal diagonalization of a symmetric matrix:

  Let A be an $n \times n$ symmetric matrix.

  (1) Find all eigenvalues of A and determine the multiplicity of each.

  (2) For each eigenvalue of multiplicity 1, choose a unit eigenvector. (Find any eigenvector and then normalize it).

  (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

  (4) Steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- **Example 19: (Orthogonal diagonalization)**

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

$$(1)\ \left|\lambda I - A\right| = (\lambda - 3)^2(\lambda + 6) = 0$$

Eigenvalues: $\lambda_1 = -6, \lambda_2 = 3$ (has a multiplicity of 2)

$$(2)\ \lambda_1 = -6,\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$(3)\ \lambda_2 = 3,\ \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

linearly Independent

Gram-Schmidt Process:

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{w}_3 = \boldsymbol{v}_3 - \frac{\boldsymbol{v}_3 . \boldsymbol{w}_2}{\boldsymbol{w}_2 . \boldsymbol{w}_2} \boldsymbol{w}_2 = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_3 = \frac{\boldsymbol{w}_3}{\|\boldsymbol{w}_3\|} = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$(4)\ P = \left[\boldsymbol{p}_1\ \boldsymbol{p}_2\ \boldsymbol{p}_3\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. Singular Value Decomposition

- The singular value decomposition (SVD) is one of the most important factorizations of a matrix. SVD factorization can be applied to any matrix; it does not need to be a square or symmetric matrix.

- Theorem 15:
  If A is an $m\text{x}n$ matrix, then:
  (a) A^TA is orthogonally diagonalizable.
  (b) The eigenvalues of A^TA are nonnegative.

- Definition: If A is an $m\times n$ matrix, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A^TA, then the numbers $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \cdots, \sigma_n = \sqrt{\lambda_n}$ are called the singular values of A.

- Example 20: (Singular Values)

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial of $A^T A$ is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

So the eigenvalues of $A^T A$ are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = 1$

- **Theorem 16: (Singular Value Decomposition)**

  If A is an $m\text{x}n$ matrix of rank k, then A can be expressed in the form $A = U\Sigma V^T$, where Σ has size $m{\times}n$ and can be expressed in partitioned form as:

$$\Sigma = \left[\begin{array}{c:c} D & 0_{k\times(n-k)} \\ \hdashline 0_{(m-k)\times k} & 0_{(m-k)\times(n-k)} \end{array}\right]$$

  in which D is a diagonal $k{\times}k$ matrix whose successive entries are the first k singular values of A in nonincreasing order ($\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$), U is an $m{\times}m$ orthogonal matrix, and V is an $n{\times}n$ orthogonal matrix.

$$\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i \qquad (i = 1, 2, \cdots, k)$$

$$D = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}$$

$V = [\boldsymbol{v}_1\ \boldsymbol{v}_2 \cdots \boldsymbol{v}_n]$ orthogonally diagonalizes A^TA

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_m\}$ is an extension of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ to an orthonormal basis for R^m.

- Example 21: (Singular Value Decomposition if A Is Not Square)

Find a singular value decomposition of the matrix of Example 20

The eigenvalues of A^TA are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = 1$$

The unit eigenvectors corresponding to λ_1 and λ_2 are

$$\boldsymbol{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

V orthogonally diagonalizes A^TA

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two of the three column vectors of U

To extend the orthonormal set $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to an orthonormal basis for R^3

The vector $\boldsymbol{u}_3 = [x\ y\ z]^T$ needs to be orthogonal to both vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$

$$\frac{\sqrt{6}}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

In matrix form, $\begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}} = \underset{U}{\begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}$$

$$A = U \quad \Sigma \quad V^T$$

Reduced Singular Value Decomposition

- The zero rows and columns of the matrix Σ are superfluous and can be eliminated

$$A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_k \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_k^T \end{bmatrix}$$

which is called a reduced singular value decomposition RSVD of A.

$$A = U_1\Sigma_1V_1^T = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^T + \cdots + \sigma_k\boldsymbol{u}_k\boldsymbol{v}_k^T$$

which is called a reduced singular value expansion of A.

- Example 22: (Reduced Singular Value Decomposition)

  Find a RSVD and a reduced singular value expansion of Example 21

Since A has rank 2, with $k=2$ the reduced SVD of A corresponding to is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T = \sqrt{3} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} + (1) \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \sqrt{3} \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$ the reduced singular value expansion