جامعة المنارة
MANARA UNIVERSITY

التحليل الرياضي ١

المحاضرة 8+7

ميكاترونيكس ومعلوماتية

عملي

Prepared by
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التكامل

تكامل التوابع الكسرية

تمارين

1 احسب التكاملات الآتية:

- $\int \frac{x+4}{x^2+5x-6}\,dx$
- $\int_4^8 \frac{y\,dy}{y^2-2y-3}$
- $\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$

الحل

$$\int \frac{x+4}{x^2+5x-6}\,dx$$

$$\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6);\ x=1 \Rightarrow B=\frac{5}{7};\ x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7};$$

$$\int \frac{x+4}{x^2+5x-6}dx = \frac{2}{7}\int\frac{dx}{x+6} + \frac{5}{7}\int\frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}$$

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1)+B(y-3);\ y=-1 \Rightarrow B=\frac{-1}{-4}=\frac{1}{4};\ y=3 \Rightarrow A=\frac{3}{4};$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8\frac{dy}{y-3} + \frac{1}{4}\int_4^8\frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$$

$$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$$

تمارين

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy$$

$$\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1)+By; \quad y=0 \Rightarrow A=4;\ y=-1 \Rightarrow B=\frac{3}{-1}=-3;$$

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1}\frac{dy}{y} - 3\int_{1/2}^{1}\frac{dy}{y+1} = \Big[4\ln|y| - 3\ln|y+1|\Big]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\tfrac{1}{2} - 3\ln\tfrac{3}{2}\right)$$

$$= \ln\tfrac{1}{8} - \ln\tfrac{1}{16} + \ln\tfrac{27}{8} = \ln\left(\tfrac{27}{8}\cdot\tfrac{1}{8}\cdot 16\right) = \ln\tfrac{27}{4}$$

2 احسب التكاملات الآتية:

- $\int_{0}^{1} \frac{x^3\,dx}{x^2+2x+1}$
- $\int_{-1}^{0} \frac{x^3\,dx}{x^2-2x+1}$
- $\int \frac{dx}{(x^2-1)^2}$

الحل

$$\int_{0}^{1} \frac{x^3\,dx}{x^2+2x+1}$$

بالقسمة المطولة نحصل على

$$\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$$

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1)+B = Ax+(A+B)$$

$$\Rightarrow A=3,\ A+B=2 \Rightarrow A=3,\ B=-1;$$

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$$\int_0^1 \frac{x^3dx}{x^2+2x+1} = \int_0^1 (x-2)dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1 = \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$$

$$\int_{-1}^{0} \frac{x^3\,dx}{x^2 - 2x + 1}$$

بالقسمة المطولة نحصل على $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$

$$\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B = Ax + (-A+B) \Rightarrow A = 3, -A+B = -2 \Rightarrow A = 3, B = 1;$$

$$\int_{-1}^0 \frac{x^3dx}{x^2-2x+1} = \int_{-1}^0 (x+2)dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$$

$$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$$

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$$\int \frac{dx}{(x^2-1)^2}$$

$$\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$$

$$x = -1 \Rightarrow C = \tfrac{1}{4}; \quad x = 1 \Rightarrow D = \tfrac{1}{4};$$

$$\text{coefficient of } x^3 = A + B \Rightarrow A + B = 0;$$

$$\text{constant} = A - B + C + D \Rightarrow A - B + C + D = 1$$

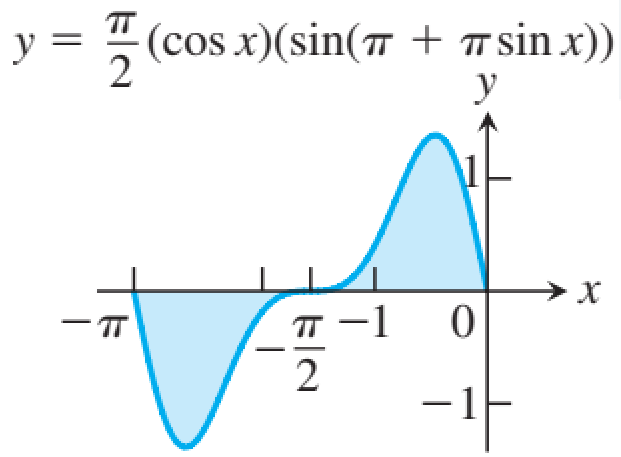
$$\Rightarrow A - B = \tfrac{1}{2} \Rightarrow A = \tfrac{1}{4} \Rightarrow B = -\tfrac{1}{4};$$

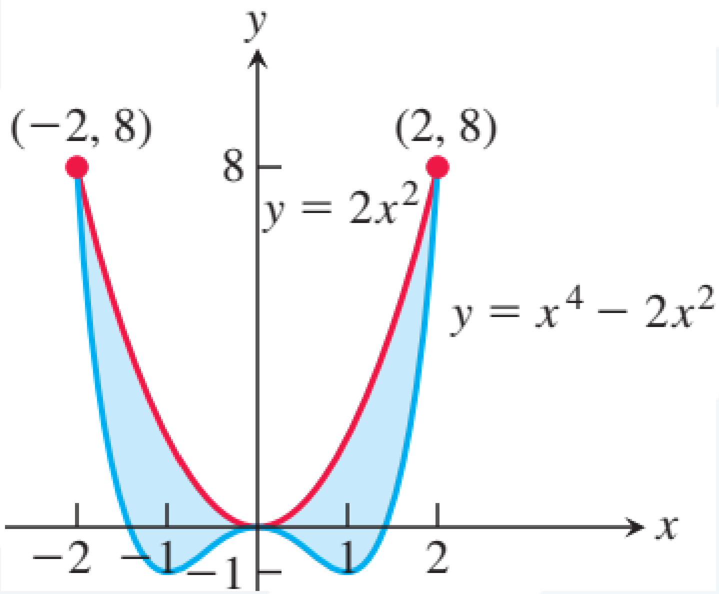
$$\int \frac{dx}{\left(x^2-1\right)^2} = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2\left(x^2-1\right)} + C$$

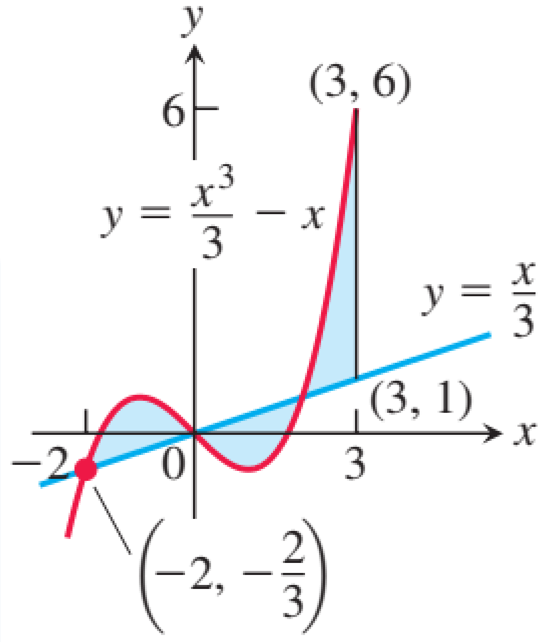
تطبيقات التكامل

تمارين

1 احسب مساحة المنطقة المظللة لكل مما يلي:

- 

- 

- 

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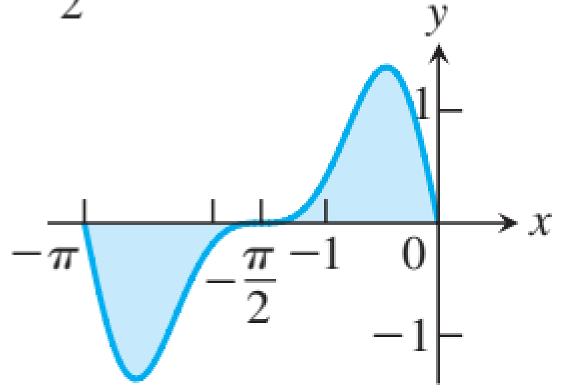
الحل

$y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))$



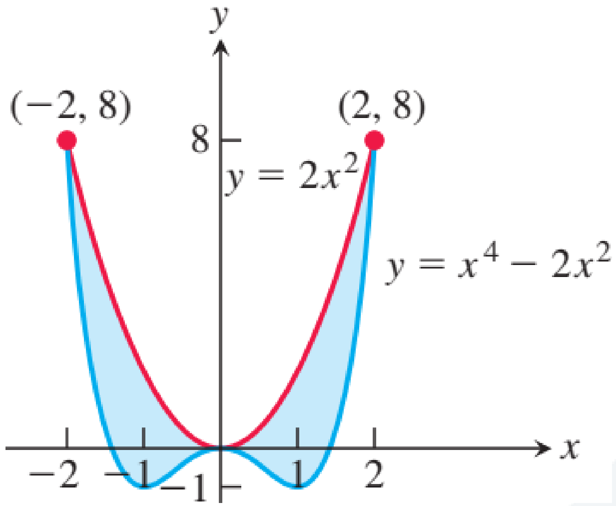
$$u = \pi + \pi \sin x \Rightarrow du = \pi \cos x\, dx \Rightarrow \frac{1}{\pi}\, du = \cos x\, dx$$

$$x = -\frac{\pi}{2} \Rightarrow u = \pi + \pi \sin\left(-\frac{\pi}{2}\right) = 0,\ x = 0 \Rightarrow u = \pi$$

بسبب التناظر حول $x = -\frac{\pi}{2}$

$$A = 2\int_{-\pi/2}^{0} \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))\, dx = 2\int_0^{\pi} \frac{\pi}{2}(\sin u)\left(\frac{1}{\pi}\, du\right)$$

$$= \int_0^{\pi} \sin u\, du = [-\cos u]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2$$

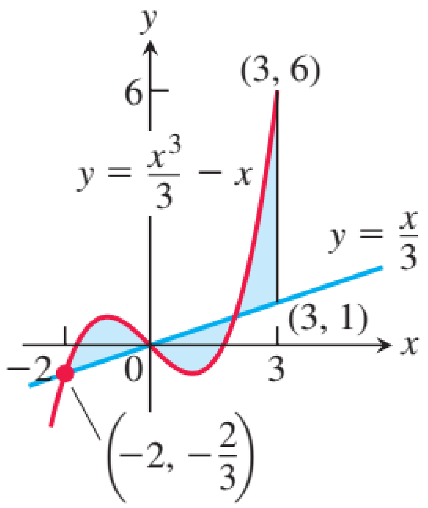


$$a = -2, b = 2;\ f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$$

$$A = \int_{-2}^{2} (4x^2 - x^4)\, dx = \left[\frac{4x^2}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right]$$

$$= \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$$

تمارين



$$AREA = A1 + A2 + A3$$

A1: $a=-2$ and $b=0$: $f(x)-g(x)=\left(\frac{x^3}{3}-x\right)-\frac{x}{3}=\frac{x^3}{3}-\frac{4}{3}x=\frac{1}{3}(x^3-4x)$

$$\Rightarrow A1=\frac{1}{3}\int_{-2}^{0}(x^3-4x)\,dx=\frac{1}{3}\left[\frac{x^4}{4}-2x^2\right]_{-2}^{0}=0-\frac{1}{3}(4-8)=\frac{4}{3};$$

A2: $a=0$

لإيجاد b علينا إيجاد نقطة تقاطع $y=\frac{x^3}{3}-x$ و $y=\frac{x}{3}$

$$\frac{x^3}{3}-x=\frac{x}{3}\Rightarrow\frac{x^3}{3}-\frac{4}{3}x=0\Rightarrow\frac{x}{3}(x-2)(x+2)=0\ \Rightarrow x=-2,\ x=0,\ x=2\ \Rightarrow\ b=2$$

$$f(x)-g(x)=\frac{x}{3}-\left(\frac{x^3}{3}-x\right)=-\frac{1}{3}(x^3-4x)\Rightarrow\ A2=-\frac{1}{3}\int_{0}^{2}(x^3-4x)\,dx=\frac{1}{3}\int_{0}^{2}(4x-x^3)=\frac{1}{3}\left[2x^2-\frac{x^4}{4}\right]_{0}^{2}=\frac{1}{3}(8-4)=\frac{4}{3};$$

A3: $a=2$ and $b=3$: $f(x)-g(x)=\left(\frac{x^3}{3}-x\right)-\frac{x}{3}=\frac{1}{3}(x^3-4x)$

$$\Rightarrow A3=\frac{1}{3}\int_{2}^{3}(x^3-4x)\,dx=\frac{1}{3}\left[\frac{x^4}{4}-2x^2\right]_{2}^{3}=\frac{1}{3}\left[\left(\frac{81}{4}-2\cdot 9\right)-\left(\frac{16}{4}-8\right)\right]=\frac{1}{3}\left(\frac{81}{4}-14\right)=\frac{25}{12};$$

$$AREA = A1 + A2 + A3=\frac{4}{3}+\frac{4}{3}+\frac{25}{12}=\frac{32+25}{12}=\frac{19}{4}$$

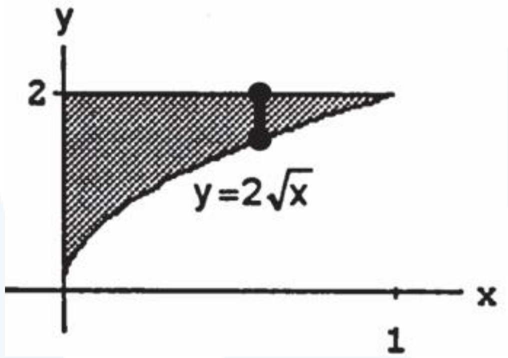
تمارين

2 أوجد حجم المجسم الناتج عن تدوير المنطقة المحدودة بالمستقيمات والمنحنيات الآتية حول المحور-x

- $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$
- $y = x^2 + 1, \quad y = x + 3$
- $y = 4 - x^2, \quad y = 2 - x$

الحل

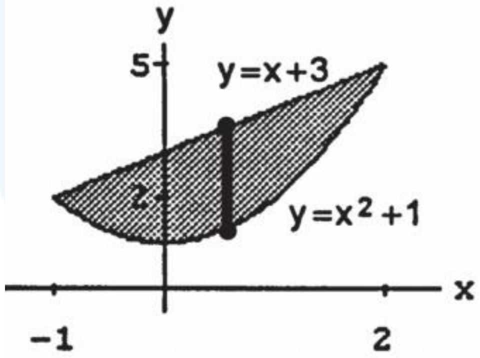
$y = 2\sqrt{x}, \quad y = 2, \quad x = 0$



$$r(x) = 2\sqrt{x} \quad R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \pi\int_0^1 (4 - 4x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$$

$y = x^2 + 1, \quad y = x + 3$



$$r(x) = x^2 + 1 \quad R(x) = x + 3 \quad \Rightarrow V = \int_{-1}^2 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \pi\int_{-1}^2 \left[(x+3)^2 - \left(x^2+1\right)^2\right] dx = \pi\int_{-1}^2 \left[\left(x^2 + 6x + 9\right) - \left(x^4 + 2x^2 + 1\right)\right] dx$$

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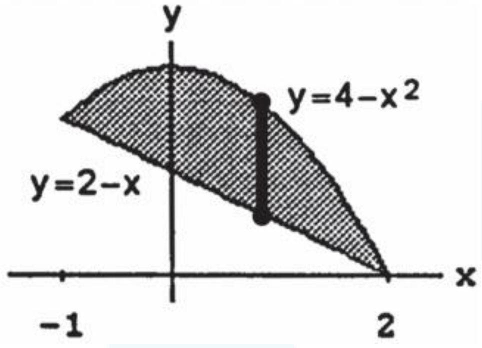
$$= \pi\int_{-1}^{2}\left(-x^4 - x^2 + 6x + 8\right)dx = \pi\left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^{2} = \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right]$$

$$= \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5\cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$$

$y = 4 - x^2, \quad y = 2 - x$ $\qquad r(x) = 2 - x \qquad R(x) = 4 - x^2$



$$\Rightarrow V = \int_{-1}^{2}\pi\left([R(x)]^2 - [r(x)]^2\right)dx = \pi\int_{-1}^{2}\left[\left(4 - x^2\right)^2 - (2 - x)^2\right]dx$$

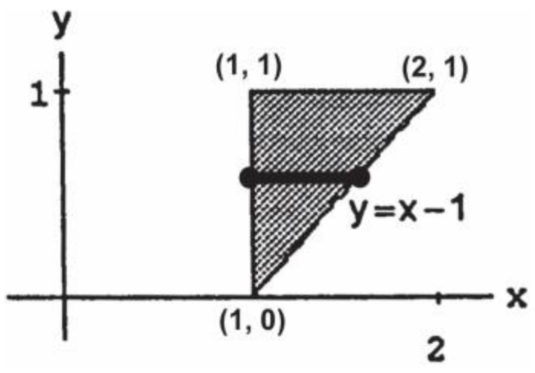
$$= \pi\int_{-1}^{2}\left[\left(16 - 8x^2 + x^4\right) - \left(4 - 4x + x^2\right)\right]dx$$

$$= \pi\int_{-1}^{2}\left(12 + 4x - 9x^2 + x^4\right)dx = \pi\left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$$

تمارين

2 أوجد حجم المجسم الناتج عن تدوير المنطقة المحصورة ضمن المثلث ذي الرؤوس $(1,\ 0),\ (2,\ 1),\ (1,\ 1)$ حول المحور y-



الحل

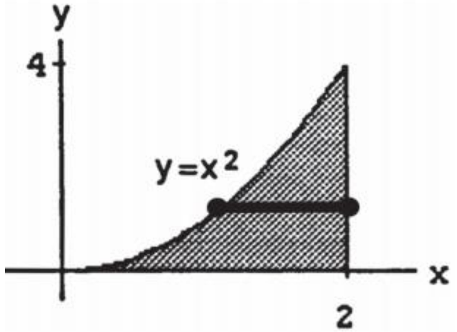
$$r(y)=1 \quad R(y)=1+y \Rightarrow V=\int_0^1 \pi\left([R(y)]^2-[r(y)]^2\right)dy$$

$$=\pi\int_0^1\left[(1+y)^2-1\right]dy=\pi\int_0^1\left(1+2y+y^2-1\right)dy$$

$$=\pi\int_0^1\left(2y+y^2\right)dy=\pi\left[y^2+\frac{y^3}{3}\right]_0^1=\pi\left(1+\frac{1}{3}\right)=\frac{4\pi}{3}$$

3 أوجد حجم المجسم الناتج عن تدوير المنطقة الموجودة في الربع الأول والمحدودة من الأعلى بالقطع المكافئ $y=x^2$ ومن الأسفل بالمحور x- ومن اليمين بالمستقيم $x=2$ حول المحور y-

الحل



$$R(y)=2 \quad r(y)=\sqrt{y} \Rightarrow V=\int_0^4 \pi\left([R(y)]^2-[r(y)]^2\right)dy$$

$$=\pi\int_0^4 (4-y)\,dy=\pi\left[4y-\frac{y^2}{2}\right]_0^4=\pi(16-8)=8\pi$$

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3 أوجد طول كل من الأقواس الآتية

- $y = (1/3)(x^2 + 2)^{3/2}$ $0 \le x \le 3$
- $y = x^{3/2}$ $0 \le x \le 4$
- $y = \dfrac{x^3}{3} + \dfrac{1}{4x},\quad 1 \le x \le 3$

الحل

$y = (1/3)(x^2 + 2)^{3/2}$ $\qquad \dfrac{dy}{dx} = \frac{1}{3}\cdot\frac{3}{2}\left(x^2+2\right)^{1/2}\cdot 2x = \sqrt{\left(x^2+2\right)}\cdot x$

$$\Rightarrow L = \int_0^3 \sqrt{1+\left(x^2+2\right)x^2}\,dx = \int_0^3 \sqrt{1+2x^2+x^4}\,dx = \int_0^3 \sqrt{\left(1+x^2\right)^2}\,dx = \int_0^3 \left(1+x^2\right)dx = \left[x+\frac{x^3}{3}\right]_0^3 = 3+\frac{27}{3} = 12$$

$y = x^{3/2}$ $\qquad \dfrac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx;$

$$\left[u = 1+\frac{9}{4}x \Rightarrow du = \frac{9}{4}dx \Rightarrow \frac{4}{9}du = dx;\ x=0 \Rightarrow u=1;\ x=4 \Rightarrow u=10\right]$$

$$L = \int_1^{10} u^{1/2}\left(\frac{4}{9}du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}\left(10\sqrt{10}-1\right)$$

تمارين

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 3$$

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(x^2 - \frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}$$

$$\Rightarrow L = \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx =$$

$$\int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_1^3 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx =$$

$$\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \left(9 + \frac{1}{12}\right) - \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{53}{6}$$

تمارين

4 أوجد مساحة سطح المجسم الناتج عن تدوير المنحني الآتي حول المحور المشار إليه:

- $y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$
- $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$
- $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

الحل

$y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$

$$\frac{dy}{dx} = \frac{x^2}{3} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^4}{9} \Rightarrow S = \int_0^2 \frac{2\pi x^3}{9}\sqrt{1+\frac{x^4}{9}}\,dx;$$

$$\left[u = 1+\frac{x^4}{9} \Rightarrow du = \frac{4}{9}x^3 dx \Rightarrow \frac{1}{4}du = \frac{x^3}{9}dx;\ x=0 \Rightarrow u=1,\ x=2 \Rightarrow u = \frac{25}{9}\right]$$

$$S = 2\pi\int_1^{25/9} u^{1/2}\cdot\frac{1}{4}du = \frac{\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{25/9} = \frac{\pi}{3}\left(\frac{125}{27}-1\right) = \frac{\pi}{3}\left(\frac{125-27}{27}\right) = \frac{98\pi}{81}$$

$y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(2-2x)}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(1-x)^2}{2x-x^2}$$

$$\Rightarrow S = \int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\frac{(1-x)^2}{2x-x^2}}\,dx = 2\pi\int_{0.5}^{1.5}\sqrt{2x-x^2}\,\frac{\sqrt{2x-x^2+1-2x+x^2}}{\sqrt{2x-x^2}}\,dx = 2\pi\int_{0.5}^{1.5}dx = 2\pi[x]_{0.5}^{1.5} = 2\pi$$

تمارين

$x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

$$\frac{dx}{dy} = y^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 \Rightarrow S = \int_0^1 \frac{2\pi y^3}{3}\sqrt{1+y^4}\ dy;$$

$$\left[u = 1 + y^4 \Rightarrow du = 4y^3 dy \Rightarrow \tfrac{1}{4}\ du = y^3\ dy;\ y = 0 \Rightarrow u = 1,\ y = 1 \Rightarrow u = 2\right]$$

$$S = \int_1^2 2\pi\left(\tfrac{1}{3}\right)u^{1/2}\left(\tfrac{1}{4}du\right) = \tfrac{\pi}{6}\int_1^2 u^{1/2}du = \tfrac{\pi}{6}\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = \tfrac{\pi}{9}(\sqrt{8}-1)$$