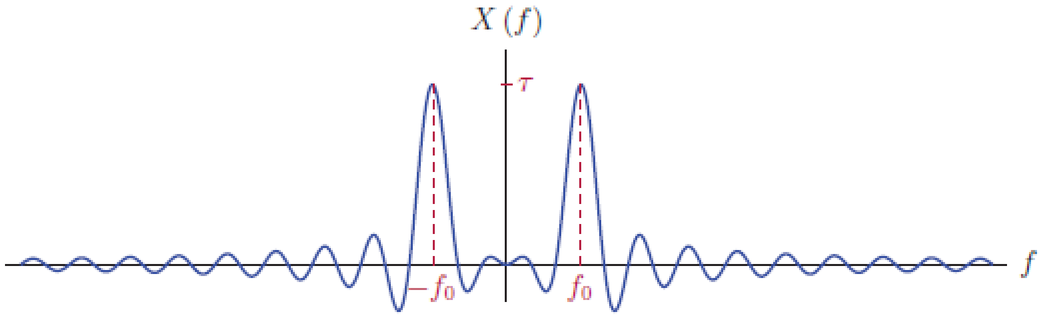
CECC507: Signals and Systems

Lecture Notes 7: Sampling and Reconstruction



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Mechatronics

Manara University

Chapter 6

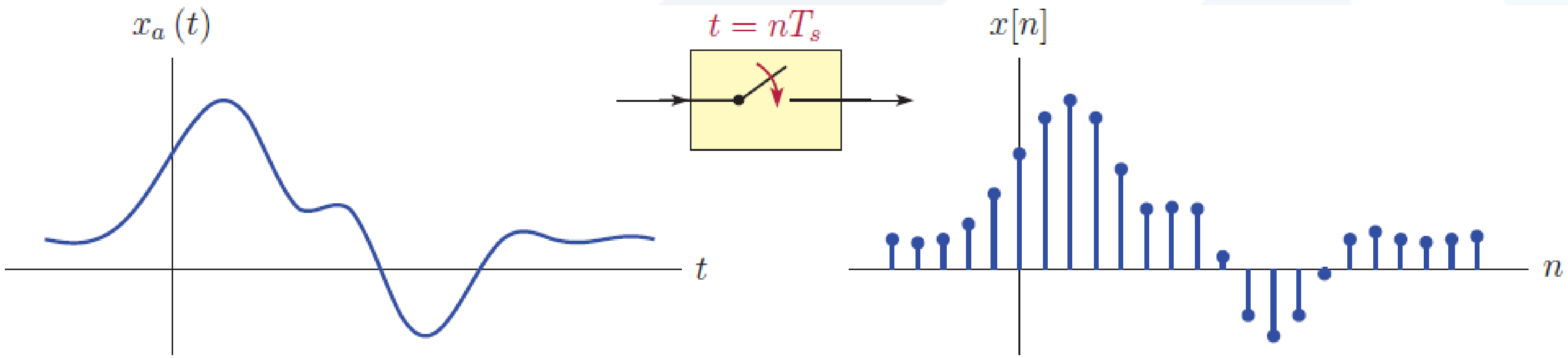
Sampling and Reconstruction

1. Sampling of a Continuous-Time Signal
2. Reconstruction of a Signal from Its Sampled Version
3. Resampling Discrete-Time Signals

Introduction

$$x[n] = x_a(t)\big|_{t=nT_s} = x_a(nT_s)$$

where T_s is the sampling interval, that is, the time interval between consecutive samples. It is also referred to as the sampling period. The reciprocal of the sampling interval is called the sampling rate or the sampling frequency: $f_s = 1/T_s$



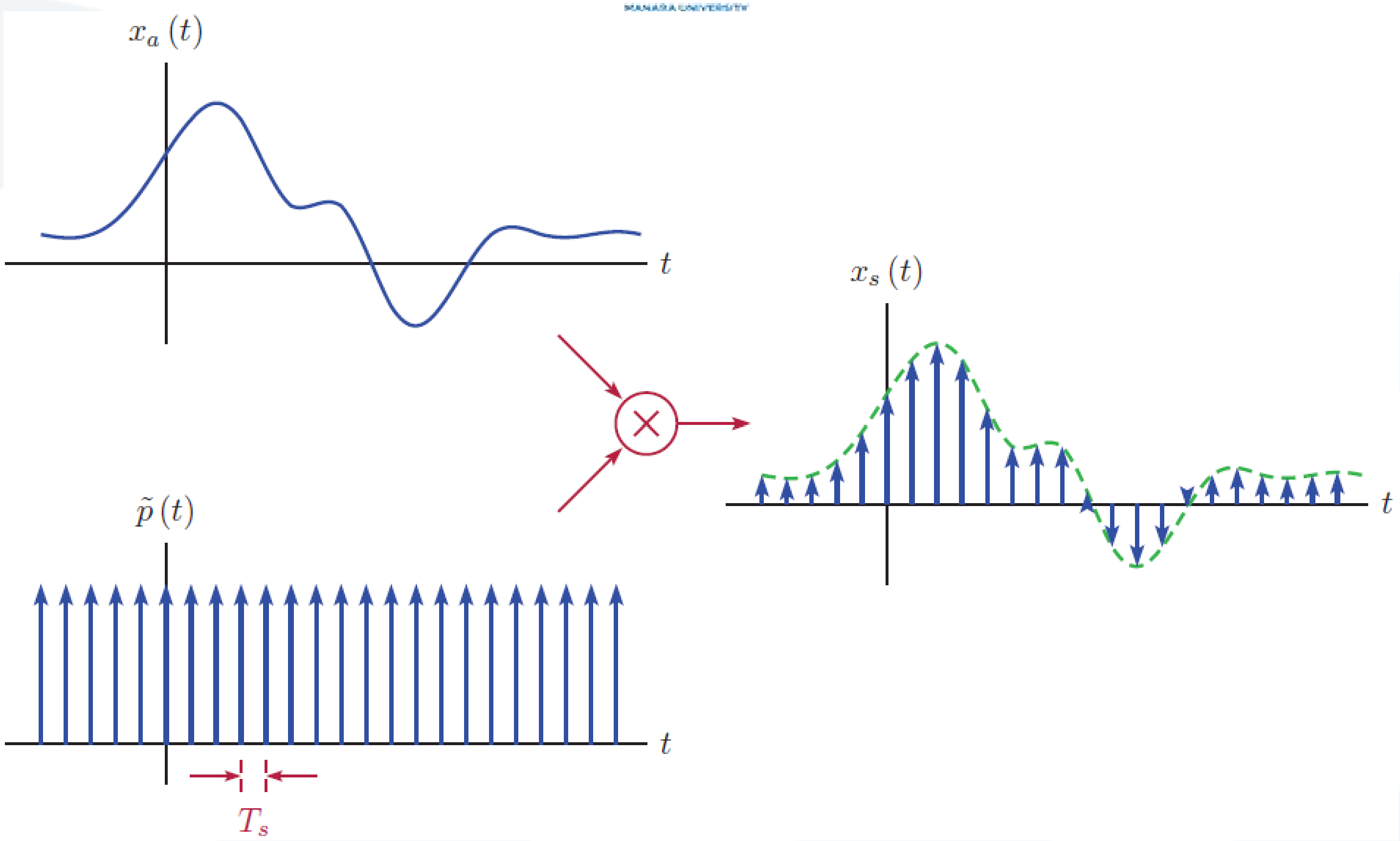
- Does the measurements (samples) taken at intervals of T_s will be sufficient to reconstruct the continuous-time signal $x_a(t)$?

1. Sampling of a Continuous-Time Signal

$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x_a(t)\tilde{p}(t) = x_a(t)\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

- We will refer to the signal $x_s(t)$ as the impulse-sampled version of $x_a(t)$.
- The impulse-sampled signal $x_s(t)$ is still a continuous time signal.
- How dense must the impulse train $\tilde{p}(t)$ be so that the impulse-sampled signal $x_s(t)$ is an accurate and complete representation of the original signal $x_a(t)$?

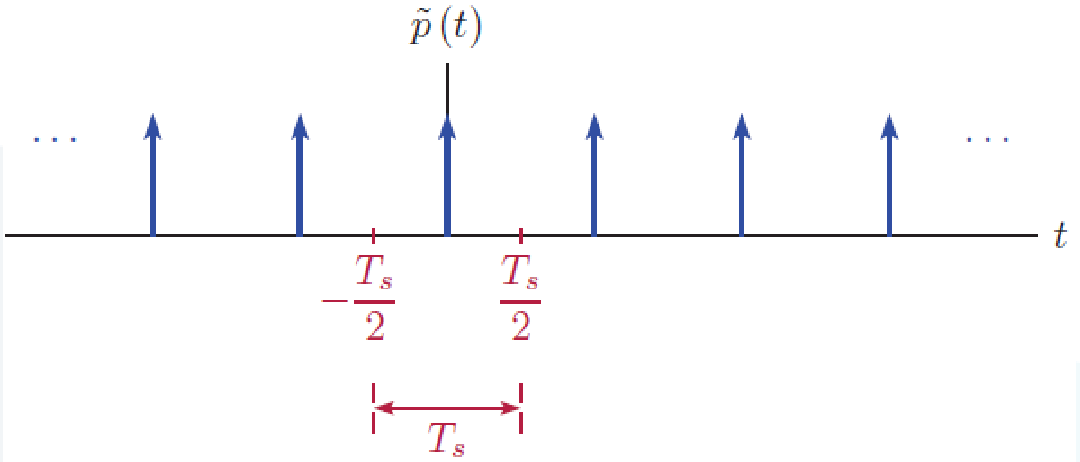


- how the frequency spectrum of the impulse-sampled signal $x_s(t)$ relates to the spectrum of the original signal $x_a(t)$?

$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$c_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \tilde{p}(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

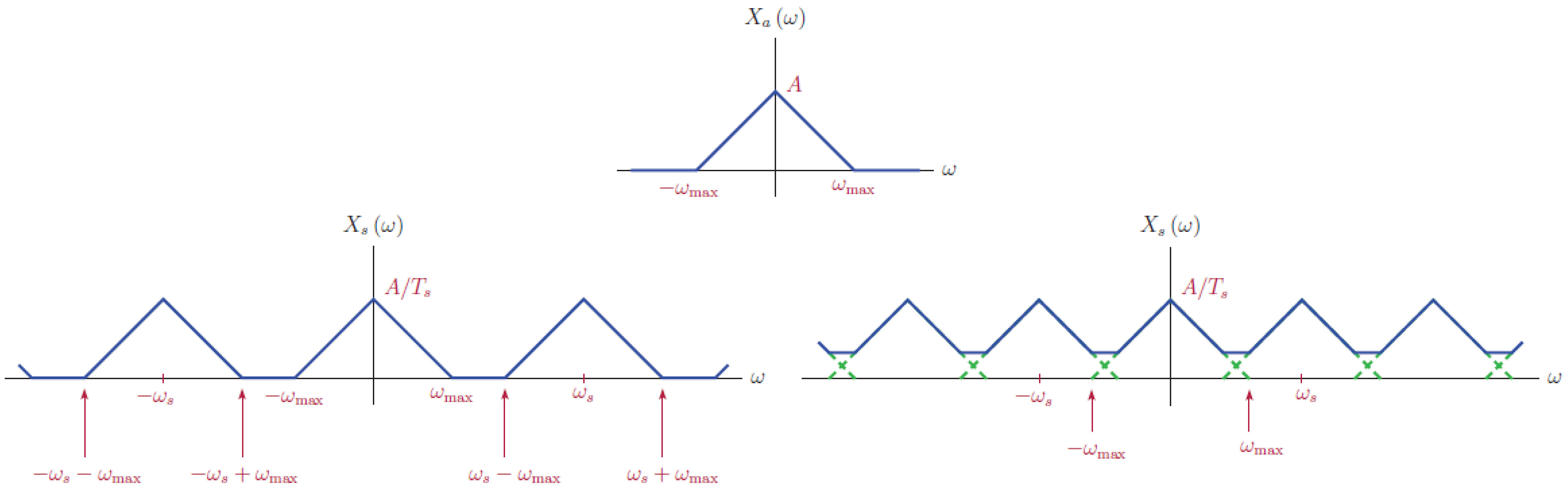


where ω_s is both the sampling rate in rad/s and the fundamental frequency of the impulse train. It is computed as $\omega_s = 2\pi f_s = 2\pi / T_s$.

$$\tilde{p}(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \Rightarrow x_s(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x_a(t) e^{jk\omega_s t}$$

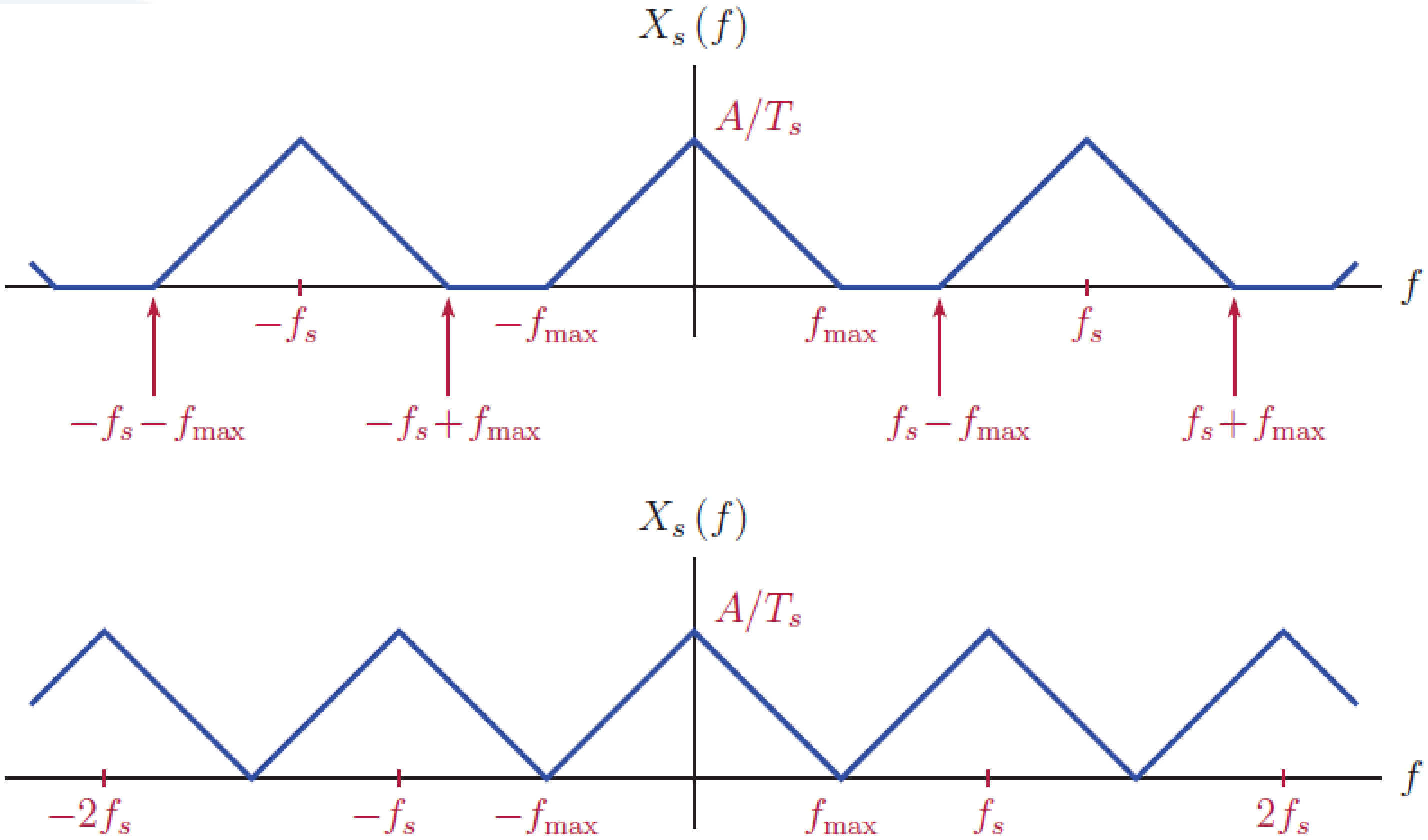
$$F\{x_s(t)\} = F\left\{\frac{1}{T_s}\sum_{k=-\infty}^{\infty} x_a(t) e^{jk\omega_s t}\right\} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} F\left\{x_a(t) e^{jk\omega_s t}\right\}$$

$$X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s), \qquad X_s(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

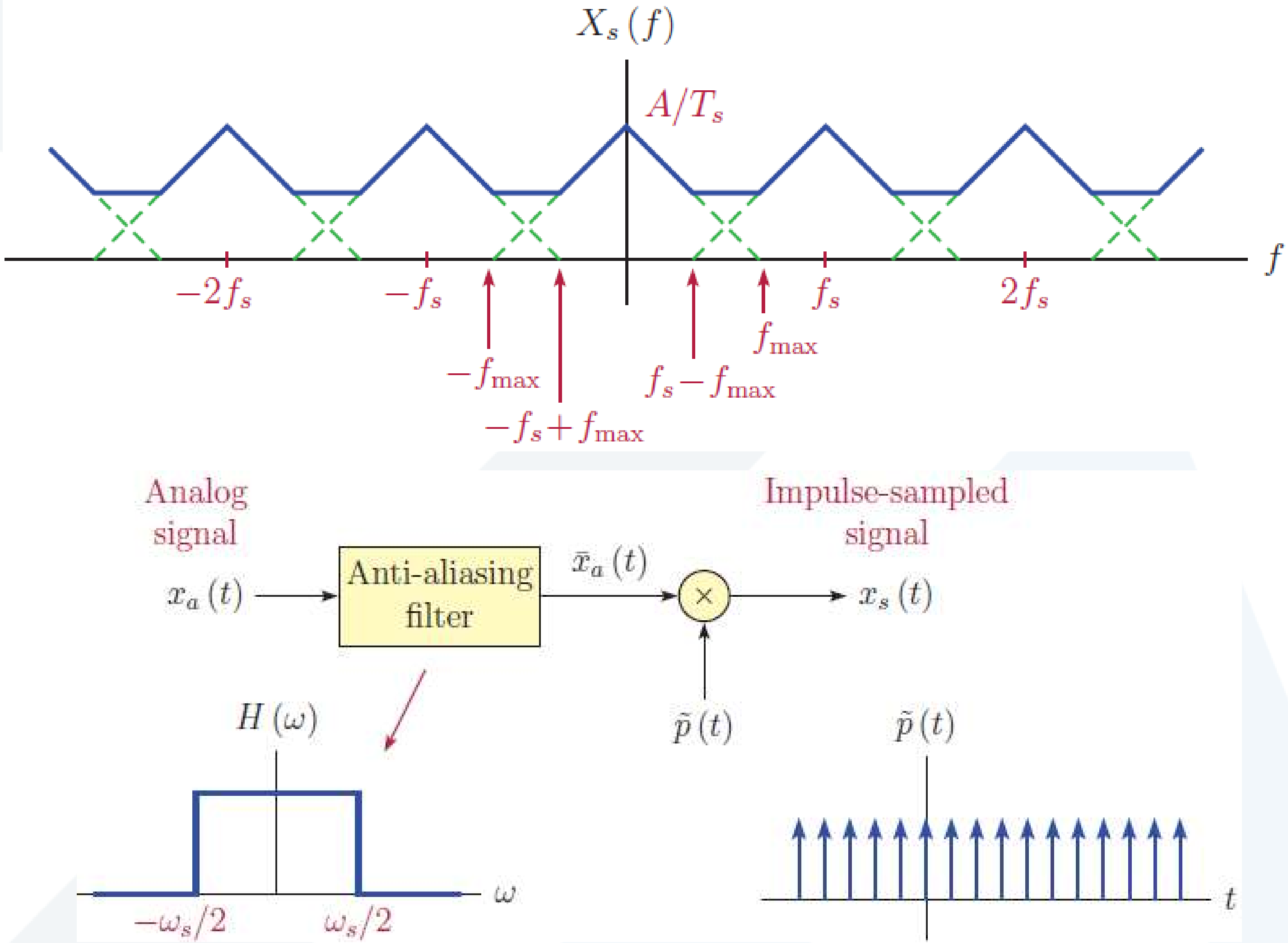


Nyquist sampling criterion

- If the range of frequencies in the signal $x_a(t)$ is not limited, then the periodic repetition of spectral components creates overlapped regions.
- This effect is known as aliasing, and it results in the shape of the spectrum $X_s(f)$ being different than the original spectrum $X_a(f)$.
- Once the spectrum is aliased, the original signal is no longer recoverable from its sampled version.
- For the impulse-sampled signal to form an accurate representation of the original signal, the sampling rate f_s must be at least twice the highest frequency in the spectrum of the original signal f_{max} ($f_s \geq 2f_{max}$). This is known as the Nyquist sampling criterion.



$X_s(f)$
A/T_s
f
$-f_s$
$-f_{\max}$
$f_{\max}$
f_s
$-f_s - f_{\max}$
$-f_s + f_{\max}$
$f_s - f_{\max}$
$f_s + f_{\max}$
$X_s(f)$
A/T_s
f
$-2f_s$
$-f_s$
$-f_{\max}$
$f_{\max}$
f_s
$2f_s$


$X_s(f)$
A/T_s
f
$-2f_s$
$-f_s$
f_s
$2f_s$
f_{max}
$-f_{max}$
$f_s - f_{max}$
$-f_s + f_{max}$
Analog signal
$x_a(t)$
Anti-aliasing filter
$\bar{x}_a(t)$
Impulse-sampled signal
$x_s(t)$
$\tilde{p}(t)$
$H(\omega)$
ω
$-\omega_s/2$
$\omega_s/2$
$\tilde{p}(t)$
t

- Use of an antialiasing filter to ensure compliance with the requirements of Nyquist sampling criterion.

DTFT of sampled signal

$$x_a(t) \Rightarrow X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt$$ CTFT

$$x[n] = x_a(nT_s) \Rightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ DTFT

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s) \right] e^{-j\omega t} dt$$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\left[\int_{-\infty}^{\infty} \delta(t - nT_s)e^{-j\omega t}dt\right] = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-j\omega nT_s}$$

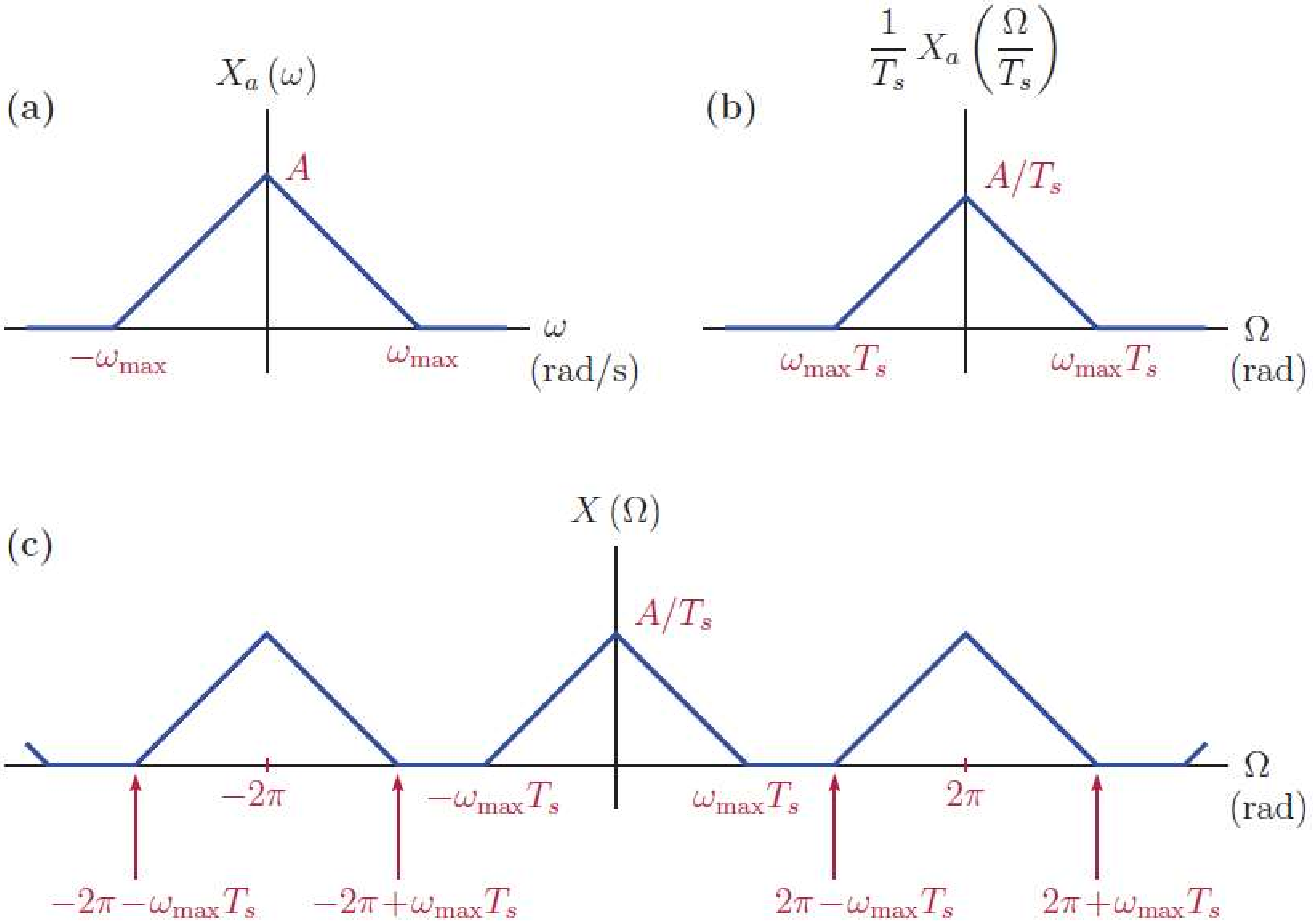
$$\Omega = \omega T_s \Rightarrow X(\Omega) = X_s\left(\frac{\Omega}{T_s}\right)$$

- The relationship between the spectrum of the original continuous-time signal and the DTFT of the discrete-time signal obtained by sampling it:

$$X(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{T_s}\right)$$

- In order to avoid overlaps between repetitions of the segments of the spectrum, we need:

$$\omega_{\max} T_s \le \pi \Rightarrow f_{\max} \le \frac{f_s}{2}$$

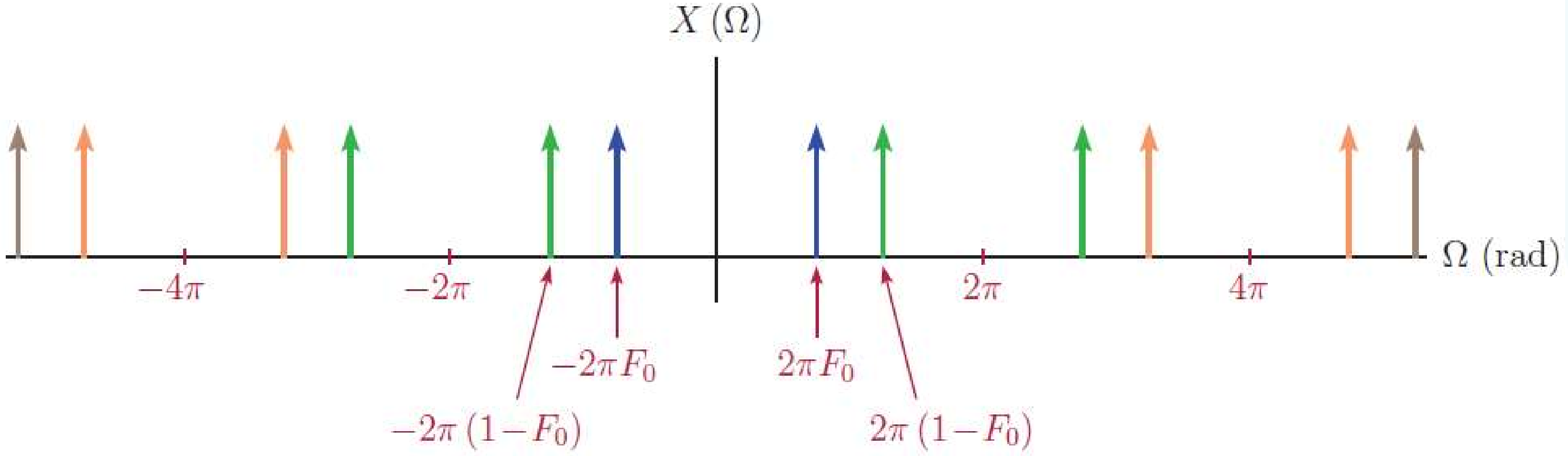

(a)
$X_a(\omega)$
A
ω
(rad/s)
$-\omega_{\max}$
$\omega_{\max}$
(b)
$\frac{1}{T_s} X_a\left(\frac{\Omega}{T_s}\right)$
A/T_s
Ω
(rad)
$\omega_{\max} T_s$
$\omega_{\max} T_s$
(c)
$X(\Omega)$
A/T_s
Ω
(rad)
-2π
$-\omega_{\max} T_s$
$\omega_{\max} T_s$
2π
$-2\pi - \omega_{\max} T_s$
$-2\pi + \omega_{\max} T_s$
$2\pi - \omega_{\max} T_s$
$2\pi + \omega_{\max} T_s$

Sampling of sinusoidal signals

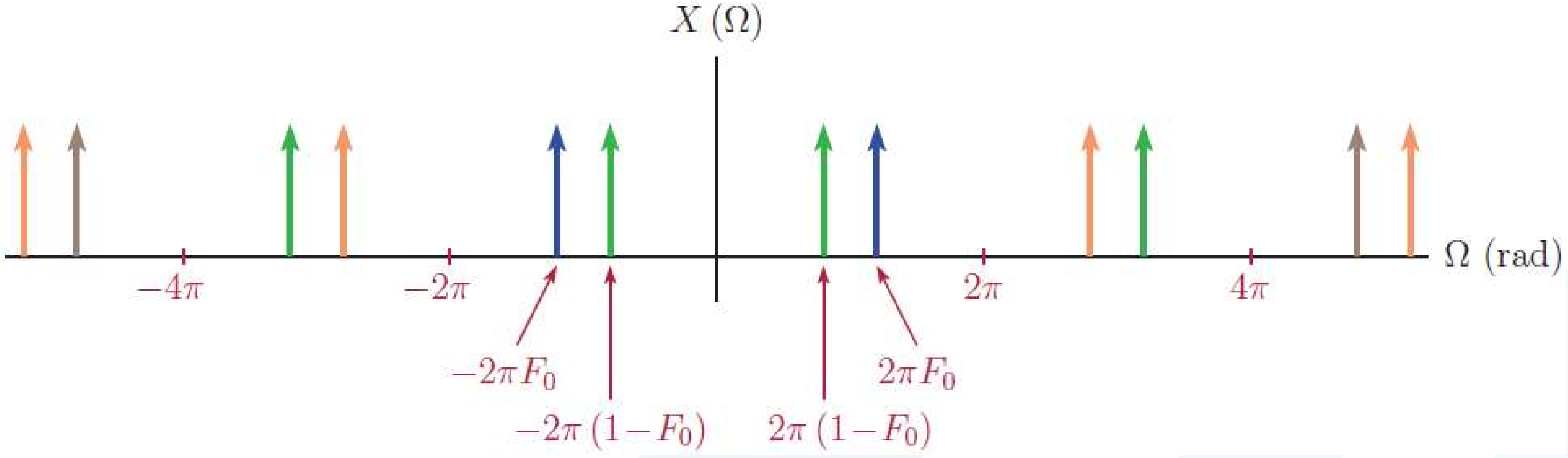
$$x_a(t) = \cos(2\pi f_0 t) \Rightarrow x[n] = x_a(nT_s) = \cos(2\pi f_0 n T_s)$$

Using the normalized frequency $F_0 = f_0 T_s = f_0 / f_s$, $x[n]$ becomes $x[n] = x_a(nT_s) = \cos(2\pi F_0 n)$.

$$X(\Omega) = \sum_{k=-\infty}^{\infty} [\pi\delta(\Omega + 2\pi F_0 - 2\pi k) + \pi\delta(\Omega - 2\pi F_0 - 2\pi k)]$$



$$proper\ sampling\ rate,\ |F_0| < 0.5$$



$improper\ sampling\ rate, |F_0| > 0.5$

- Example 1: Sampling a sinusoidal signal

  The three sinusoidal signals are sampled using the sampling rate $f_s = 16$ Hz and $T_s = 1/f_s = 0.0625$.

$$x_{1a}(t) = \cos(12\pi t),\ x_{2a}(t) = \cos(20\pi t),\ x_{3a}(t) = \cos(44\pi t)$$

$$x_1[n] = x_{1a}(0.0625n),\ x_2[n] = x_{2a}(0.0625n),\ x_3[n] = x_{3a}(0.0625n)$$

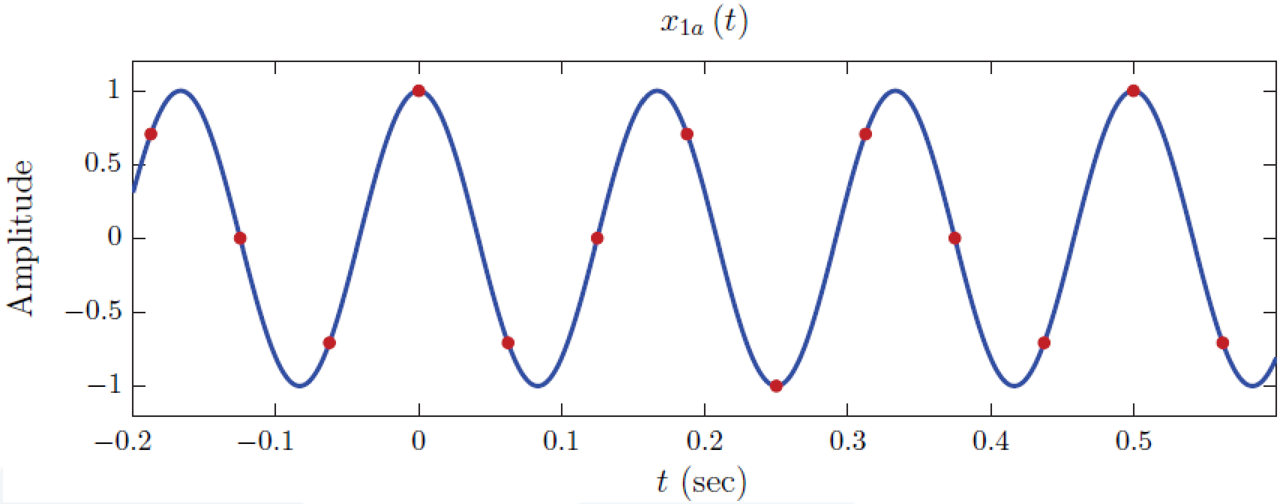
Show that the three discrete-time signals are identical, that is, $x_1[n] = x_2[n] = x_3[n]$, for all n.

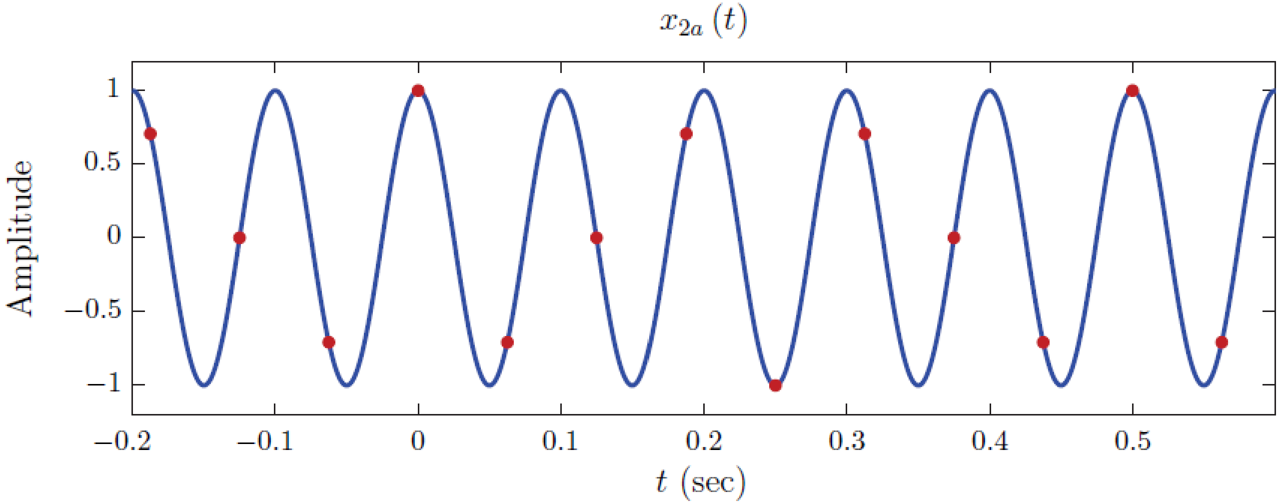
$$x_{1a}(t) = \cos(12\pi t),\ x_{2a}(t) = \cos(20\pi t),\ x_{3a}(t) = \cos(44\pi t)$$

$$x_1[n] = \cos(0.75\pi n),\ x_2[n] = \cos(1.25\pi n),\ x_3[n] = \cos(2.75\pi n)$$

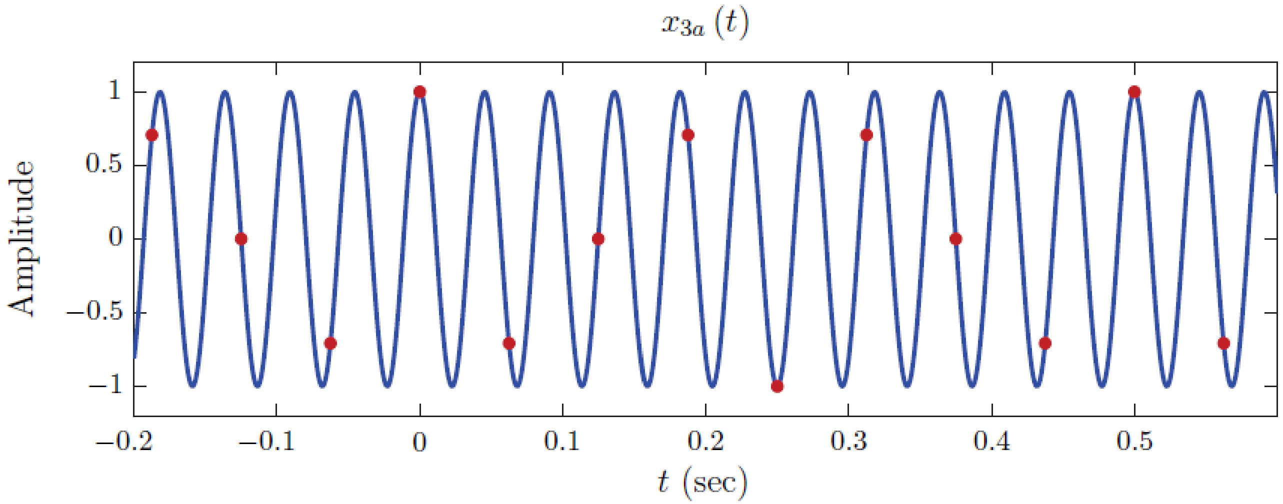
$$x_2[n] = \cos(1.25\pi n - 2\pi n) = \cos(-0.75\pi n) = \cos(0.75\pi n) = x_1[n]$$

$$x_3[n] = \cos(2.75\pi n - 2\pi n) = \cos(0.75\pi n) = x_1[n]$$



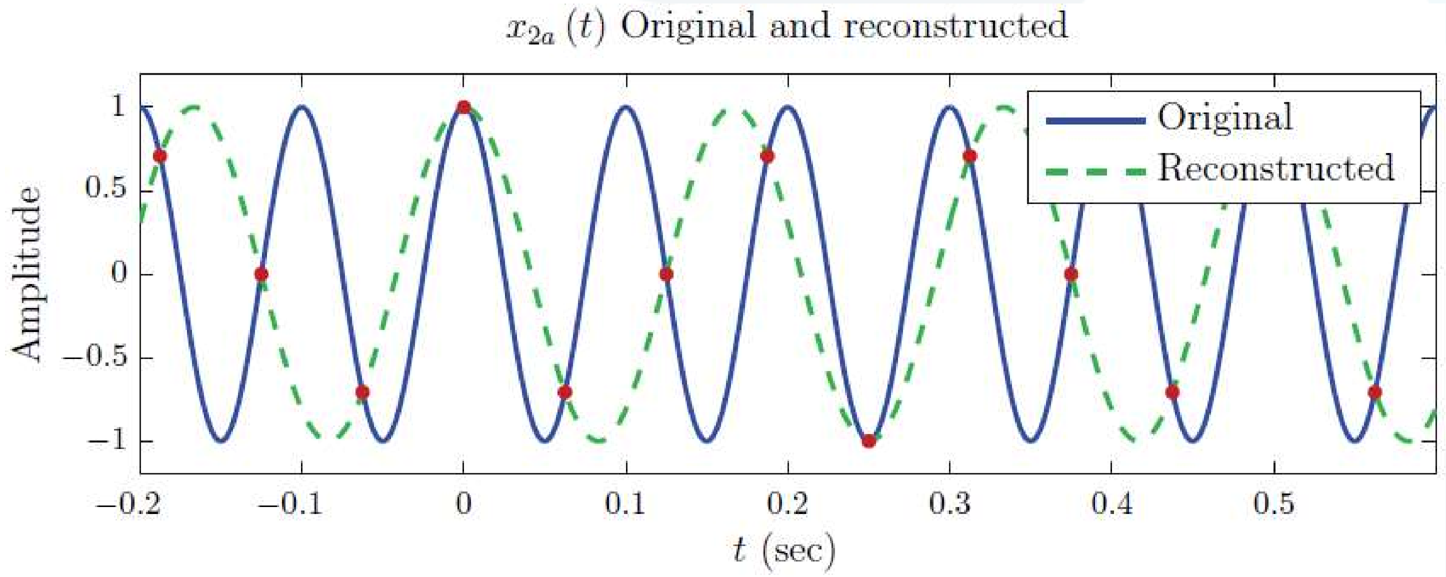


$x_{2a}(t)$
Amplitude
1
0.5
0
−0.5
−1
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
t (sec)



$x_{3a}(t)$
Amplitude
1
0.5
0
−0.5
−1
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
t (sec)

- Is the reconstruction of the original signal from its sampled version?
- An infinite number of different sinusoidal signals can be passed through the points shown with red dots in the figure below.



- Let us determine the actual and the normalized frequencies for the three signals:

$$f_1 = 6 \text{ Hz}, \qquad F_1 = 6/16 = 0.375$$
$$f_2 = 10 \text{ Hz}, \qquad F_2 = 10/16 = 0.625$$
$$f_3 = 22 \text{ Hz}, \qquad F_3 = 22/16 = 1.375$$

- Of the three normalized frequencies only F_1 satisfies the condition $|F| \leq 0.5$, and the other two violate it.
- In terms of the Nyquist sampling theorem, the signal $x_{1a}(t)$ is sampled using a proper sampling rate, that is, $f_s > 2f_1$. The other two signals are sampled improperly since $f_s < 2f_2$ and $f_s < 2f_3$.
- Example 2: Spectral relationships in sampling a sinusoidal signal

  Find the frequency spectrum for each of the 3 signals of example 1, and then use it in obtaining the DTFT spectrum of the sampled signal.

The Fourier transform of $x_{1a}(t)$ is:

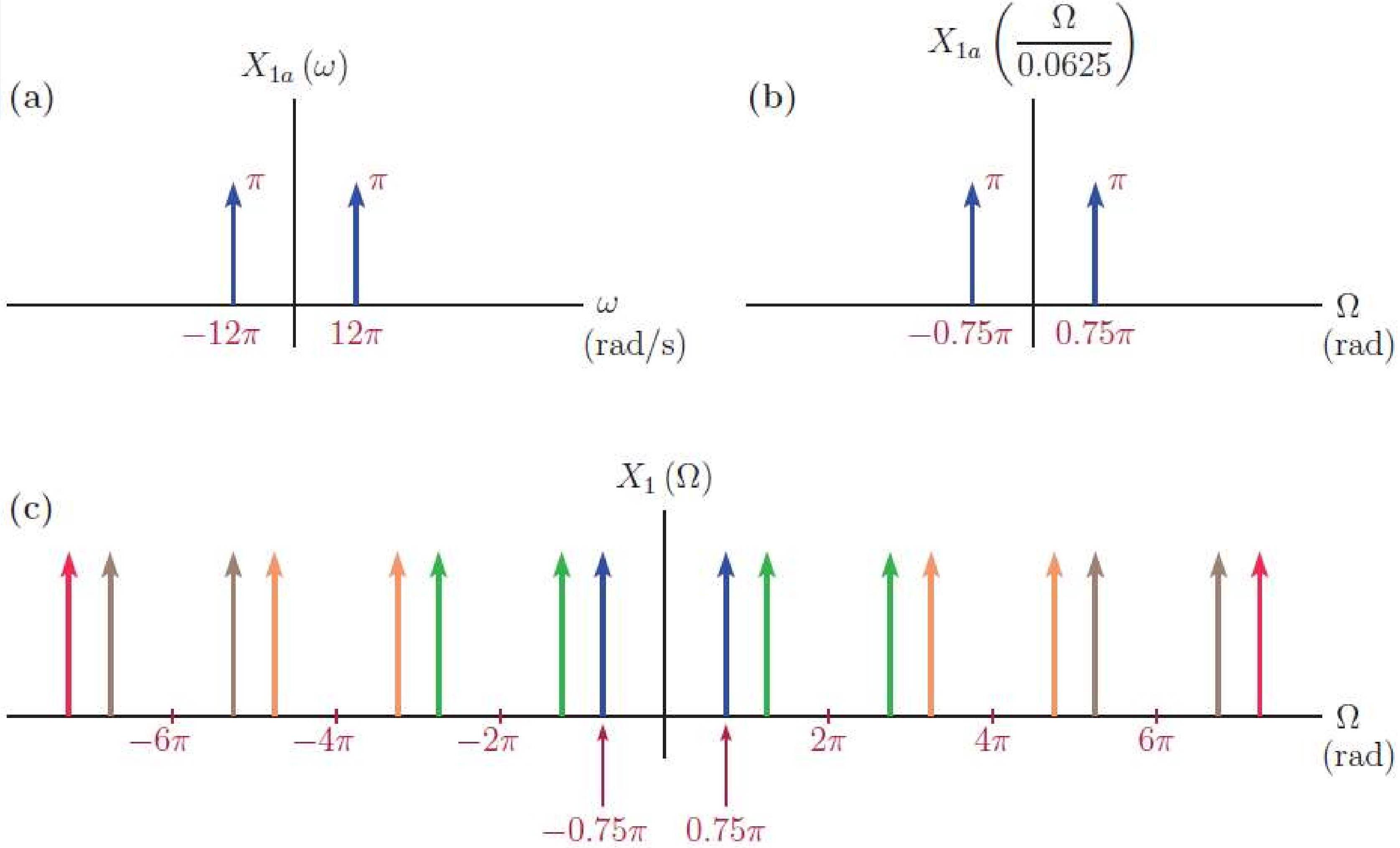
$$X_{1a}(\omega) = \pi\delta(\omega + 12\pi) + \pi\delta(\omega - 12\pi)$$

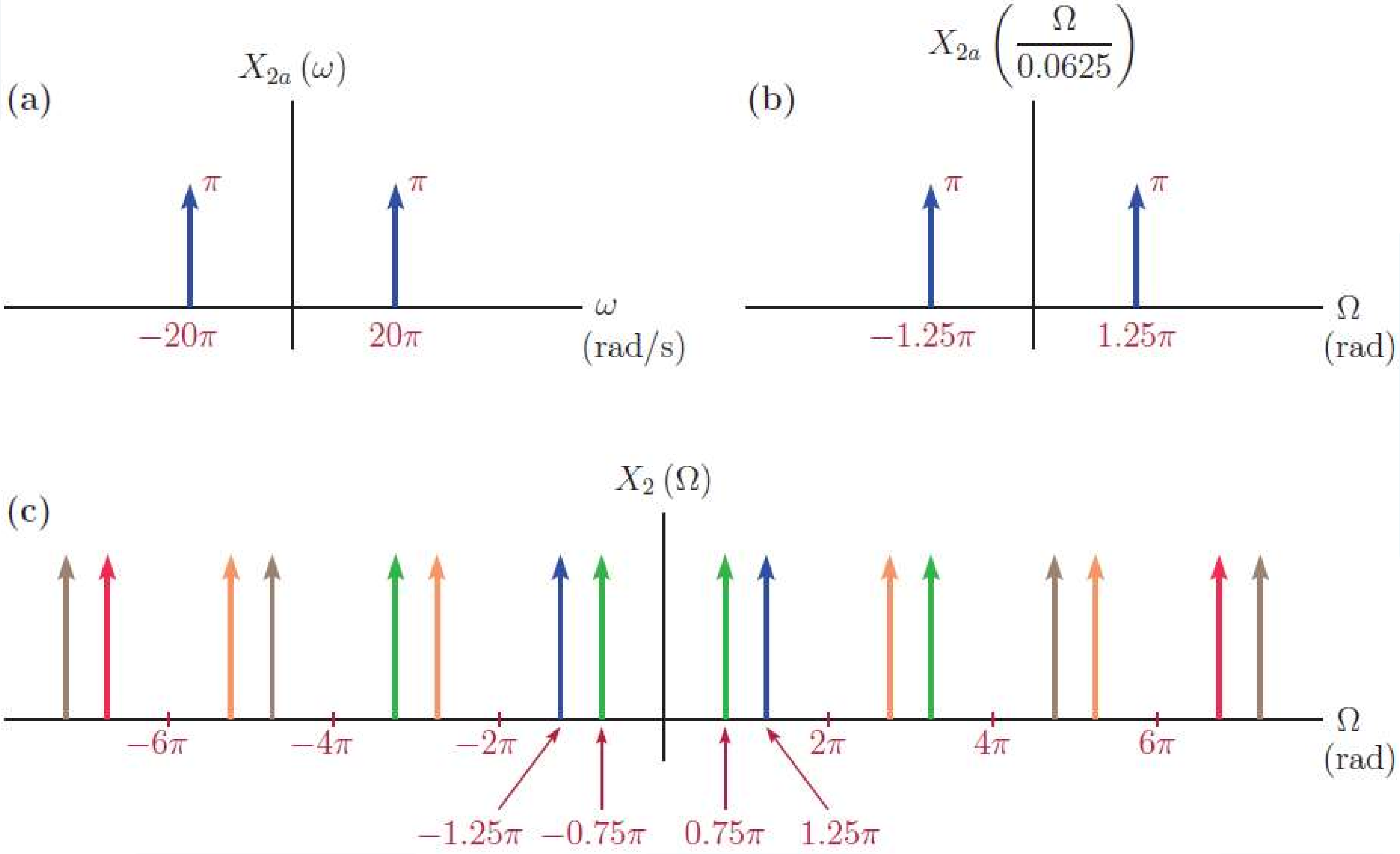
$$X_{1a}\left(\frac{\Omega}{T_s}\right) = \pi\delta\left(\frac{\Omega + 0.75\pi}{0.0625}\right) + \pi\delta\left(\frac{\Omega - 0.75\pi}{0.0625}\right)$$

$$X_1(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_{1a}\left(\frac{\Omega - 2\pi k}{T_s}\right)$$

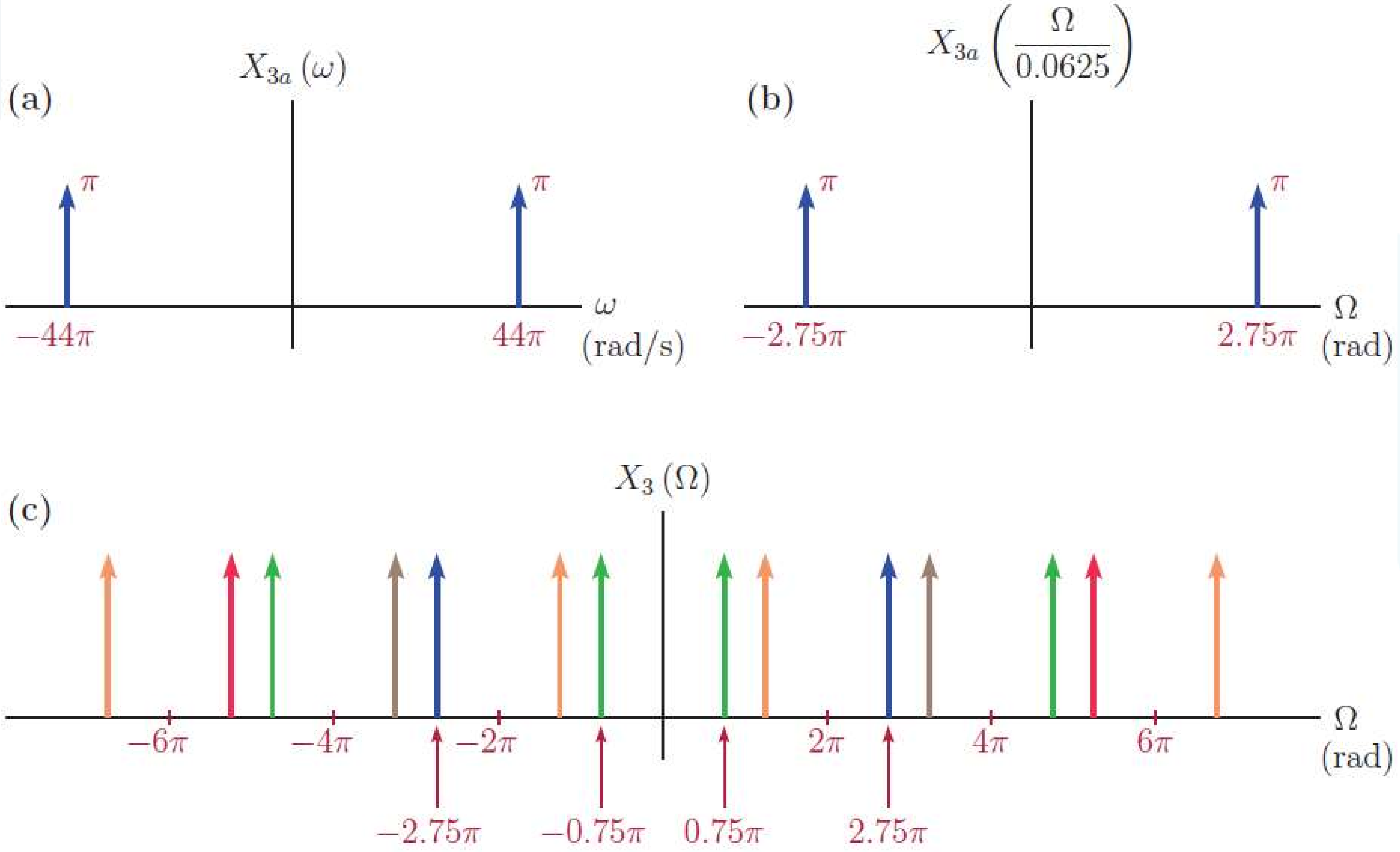
$$= 16\sum_{k=-\infty}^{\infty}\left[\pi\delta\left(\frac{\Omega + 0.75 - 2\pi k}{0.0625}\right) + \pi\delta\left(\frac{\Omega - 0.75 - 2\pi k}{0.0625}\right)\right]$$

The term for $k = 0$ is shown in blue whereas the terms for $k = \mp 1$, $k = \mp 2$, $k = \mp 3$ are shown in green, orange and brown respectively.


(a)
$X_{1a}(\omega)$
π
π
-12π
12π
ω (rad/s)
(b)
$X_{1a}\left(\frac{\Omega}{0.0625}\right)$
π
π
-0.75π
0.75π
Ω (rad)
(c)
$X_1(\Omega)$
-6π
-4π
-2π
2π
4π
6π
Ω (rad)
-0.75π
0.75π



(a)
$X_{2a}(\omega)$
π
π
-20π
20π
ω
(rad/s)
(b)
$X_{2a}\left(\frac{\Omega}{0.0625}\right)$
π
π
-1.25π
1.25π
Ω
(rad)
(c)
$X_2(\Omega)$
-6π
-4π
-2π
2π
4π
6π
Ω
(rad)
-1.25π
-0.75π
0.75π
1.25π



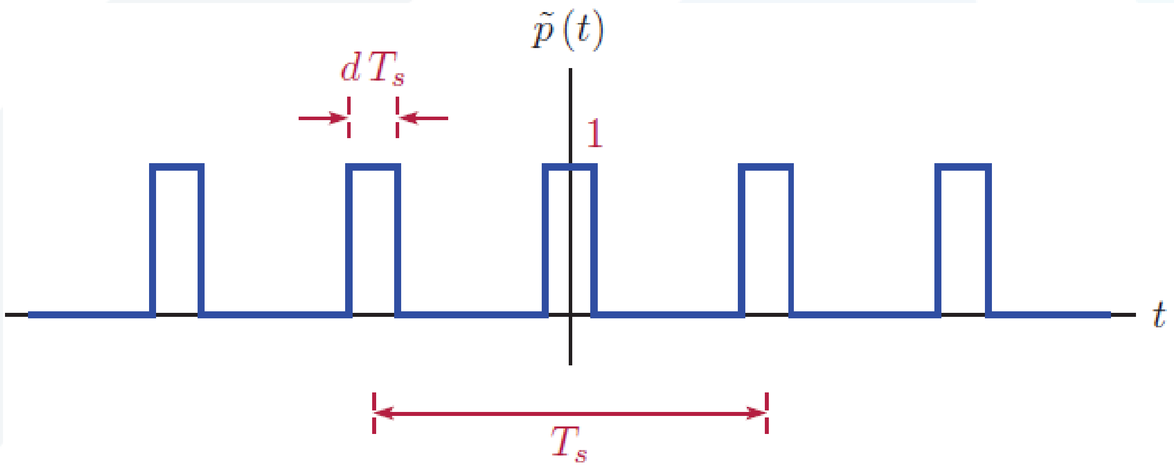
(a)
$X_{3a}(\omega)$
π
π
-44π
44π
ω
(rad/s)
(b)
$X_{3a}\left(\frac{\Omega}{0.0625}\right)$
π
π
-2.75π
2.75π
Ω
(rad)
(c)
$X_3(\Omega)$
-6π
-4π
-2π
2π
4π
6π
Ω
(rad)
-2.75π
-0.75π
0.75π
2.75π

Practical issues in sampling

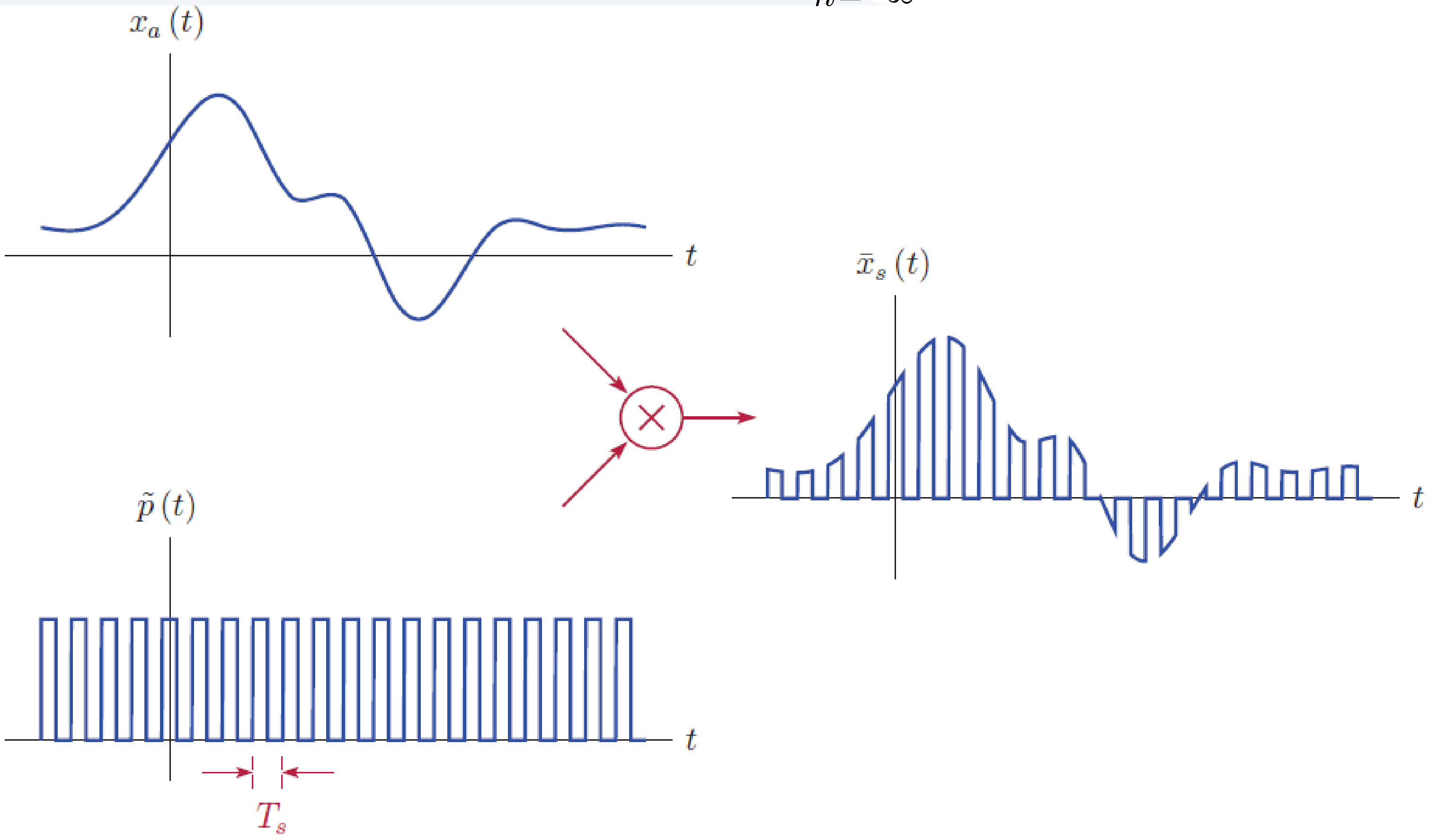
- Pulses are used instead of impulses, there are two variations of the sampling operation that can be used, namely natural sampling and zero-order hold sampling.

Natural sampling

$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_s}{dT_s}\right)$$



$$\bar{x}_s(t) = x_a(t)\tilde{p}(t) = x_a(t)\sum_{n=-\infty}^{\infty}\Pi\left(\frac{t-nT_s}{dT_s}\right)$$



- The EFS coefficients for a pulse train with duty cycle d is: $c_k = d\,\text{sinc}\,(kd)$.

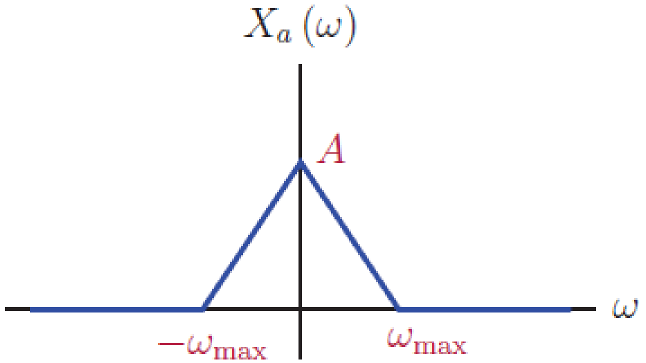
$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} = \sum_{k=-\infty}^{\infty} d\,\text{sinc}(kd)\,e^{jk\omega_s t}$$

Fundamental frequency is the same as the sampling rate $\omega_s = 2\pi/T_s$.

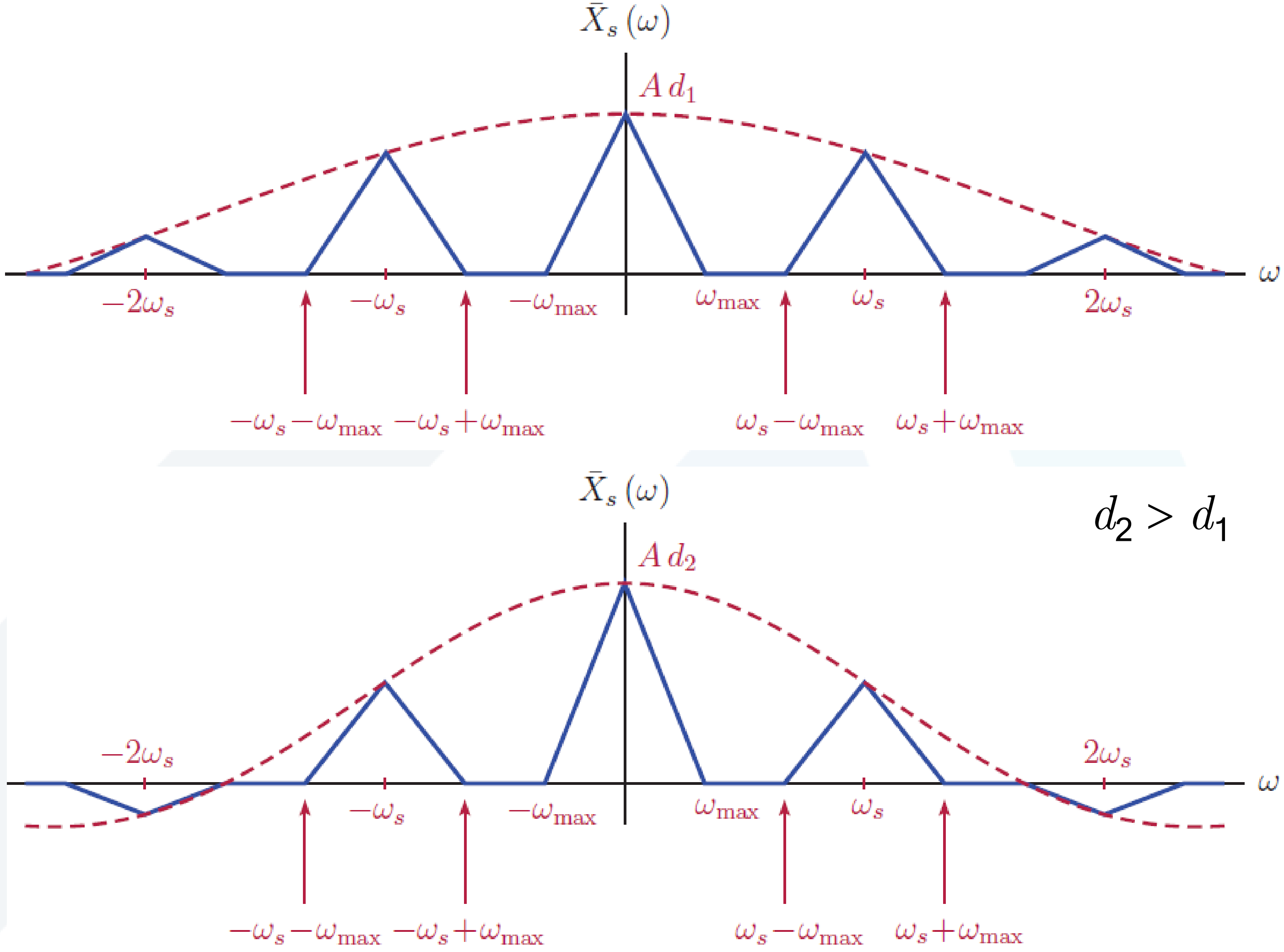
$$\bar{x}_s(t) = x_a(t)\sum_{k=-\infty}^{\infty} d\,\text{sinc}(kd)\,e^{jk\omega_s t} \Rightarrow \bar{X}_s(\omega) = F\{\bar{x}_s(t)\} = \int_{-\infty}^{\infty} \bar{x}_s(t)\,e^{-j\omega t}dt$$

$$\bar{X}_s(\omega) = F\{\bar{x}_s(t)\} = \int_{-\infty}^{\infty} x_a(t)\sum_{k=-\infty}^{\infty} d\,\text{sinc}(kd)\,e^{jk\omega_s t}e^{-j\omega t}dt$$

$$\bar{X}_s(\omega) = d\sum_{k=-\infty}^{\infty}\ \text{sinc}(kd)\left[\int_{-\infty}^{\infty} x_a(t)\,e^{-j(\omega-k\omega_s)t}dt\right] = d\sum_{k=-\infty}^{\infty}\ \text{sinc}(kd)X_a(\omega-k\omega_s)$$

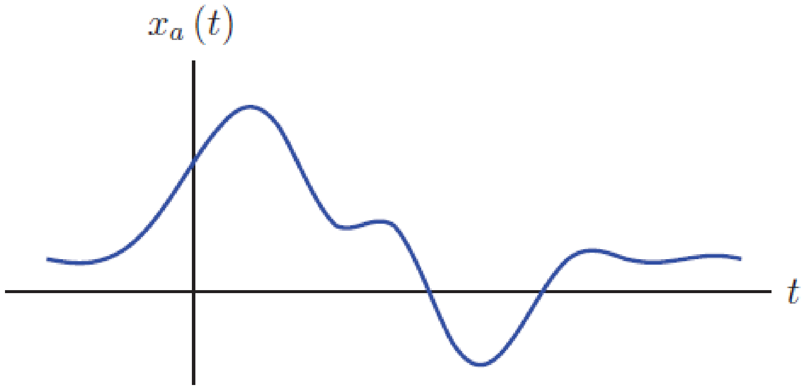


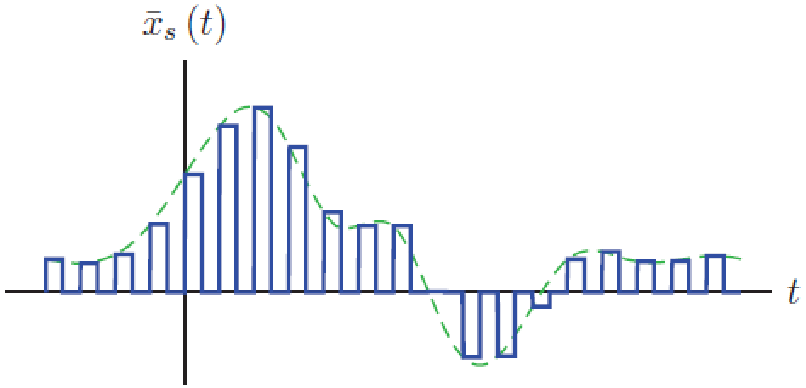
$X_a(\omega)$
A
ω
$-\omega_{\max}$
$\omega_{\max}$



$\bar{X}_s(\omega)$
$A\,d_1$
ω
$-2\omega_s$
$-\omega_s$
$-\omega_{\max}$
$\omega_{\max}$
ω_s
$2\omega_s$
$-\omega_s-\omega_{\max}$
$-\omega_s+\omega_{\max}$
$\omega_s-\omega_{\max}$
$\omega_s+\omega_{\max}$
$\bar{X}_s(\omega)$
$A\,d_2$
$d_2 > d_1$
ω
$-2\omega_s$
$2\omega_s$
$-\omega_s$
$-\omega_{\max}$
$\omega_{\max}$
ω_s
$-\omega_s-\omega_{\max}$
$-\omega_s+\omega_{\max}$
$\omega_s-\omega_{\max}$
$\omega_s+\omega_{\max}$

Zero-order hold sampling

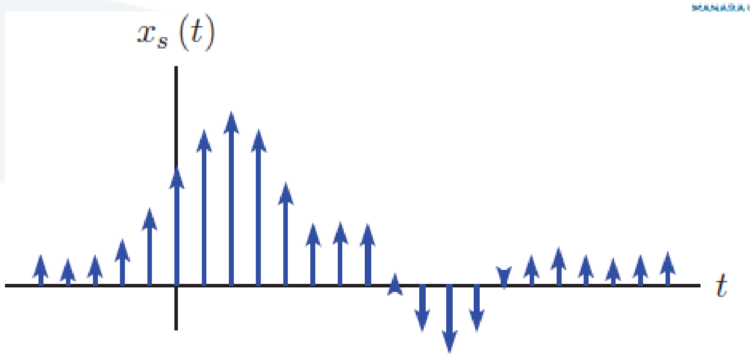


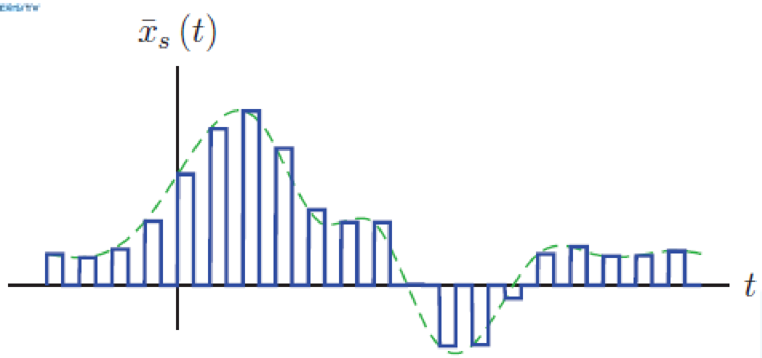


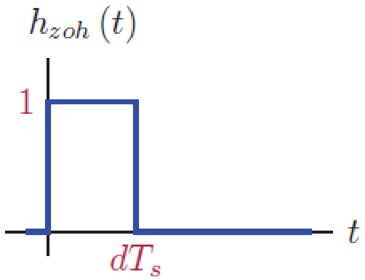
- The signal $\bar{x}_s(t)$ can be modeled as the convolution of the impulse sampled signal $x_s(t)$ and a rectangular pulse with unit amplitude and a duration of dT_s.

$$h_{zoh}(t) = \Pi\left(\frac{t - 0.5dT_s}{dT_s}\right) = u(t) - u(t - dT_s)$$

$$\bar{x}_s(t) = h_{\text{zoh}}(t) * x_s(t) \Rightarrow \bar{X}_s(\omega) = H_{\text{zoh}}(\omega) X_s(\omega)$$







$$H_{zoh}(\omega) = dT_s \,\text{sinc}\left(\omega dT_s/2\pi\right) e^{-j\omega dT_s/2}$$

$$\bar{X}_s(\omega) = d \,\text{sinc}\left(\omega dT_s/2\pi\right) e^{-j\omega dT_s/2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

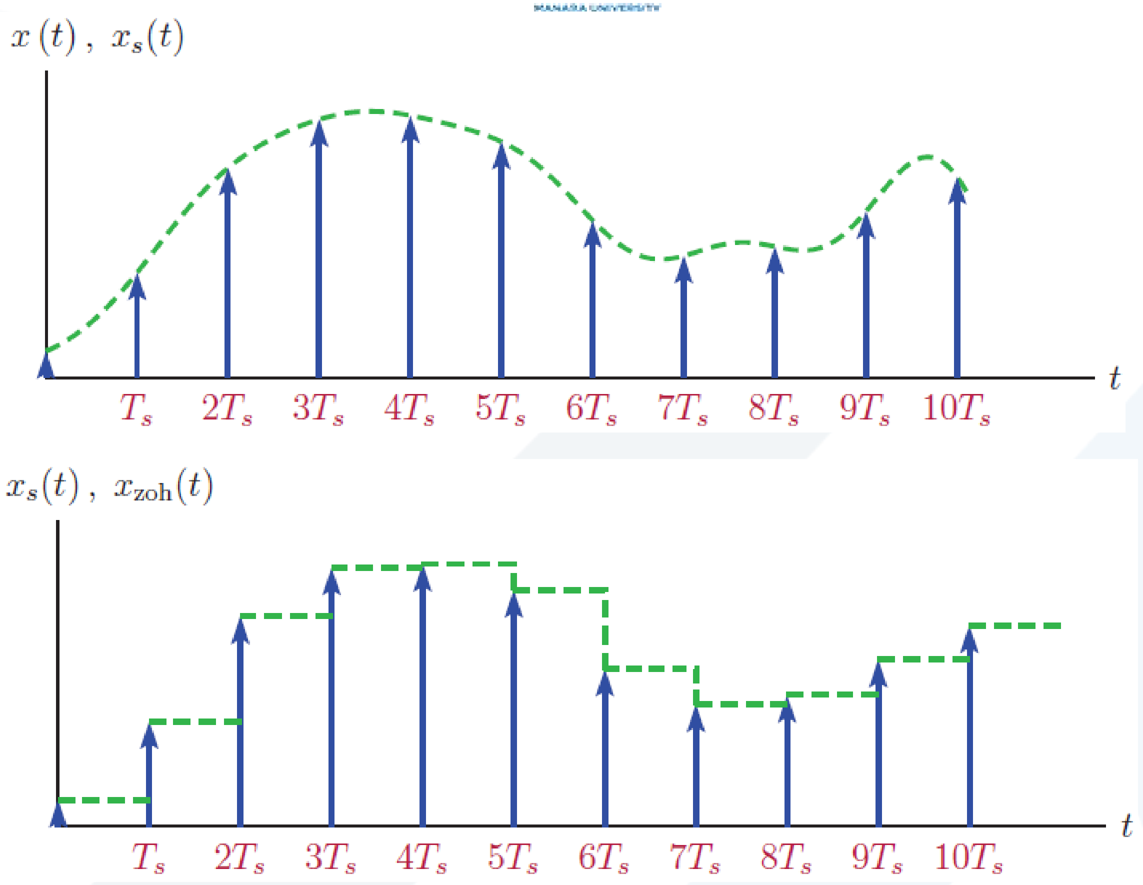
2. Reconstruction of a Signal from Its Sampled Version

- Often the purpose of sampling an analog signal is to store, process and/or transmit it digitally, and to later convert it back to analog format.

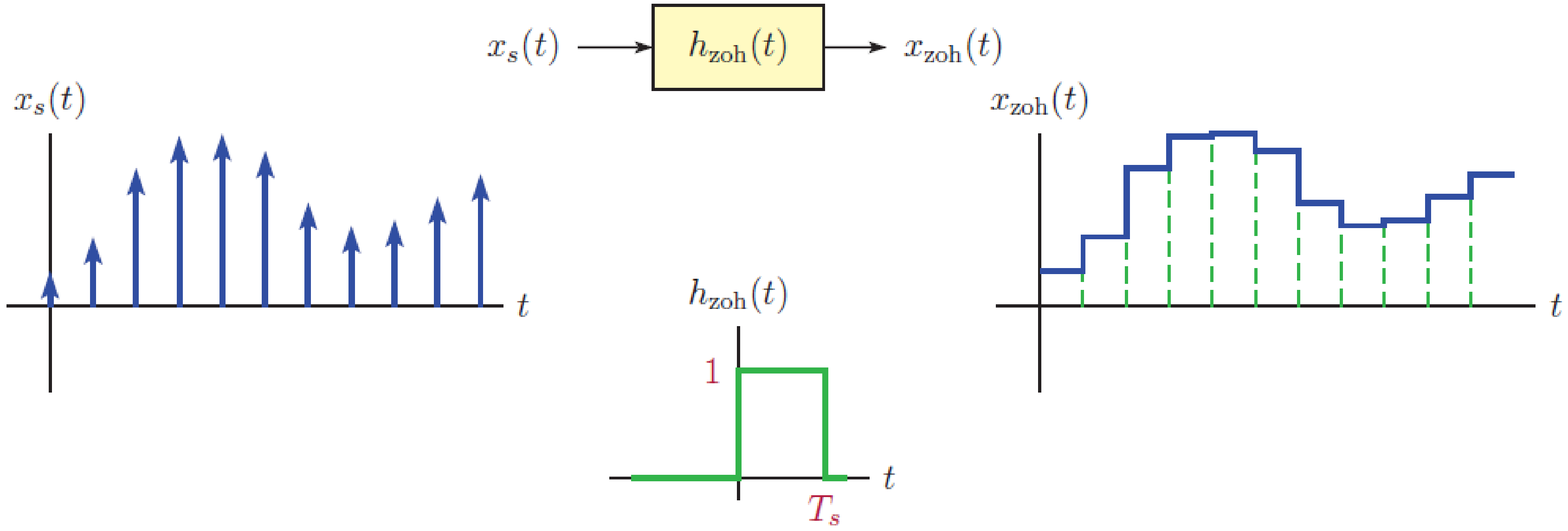
- How can the original analog signal be reconstructed from its sampled version?

- Let us first consider the possibility of obtaining a signal similar to $x_a(t)$ using simple methods. One such method would be to start with the impulse sampled signal $x_s(t)$.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

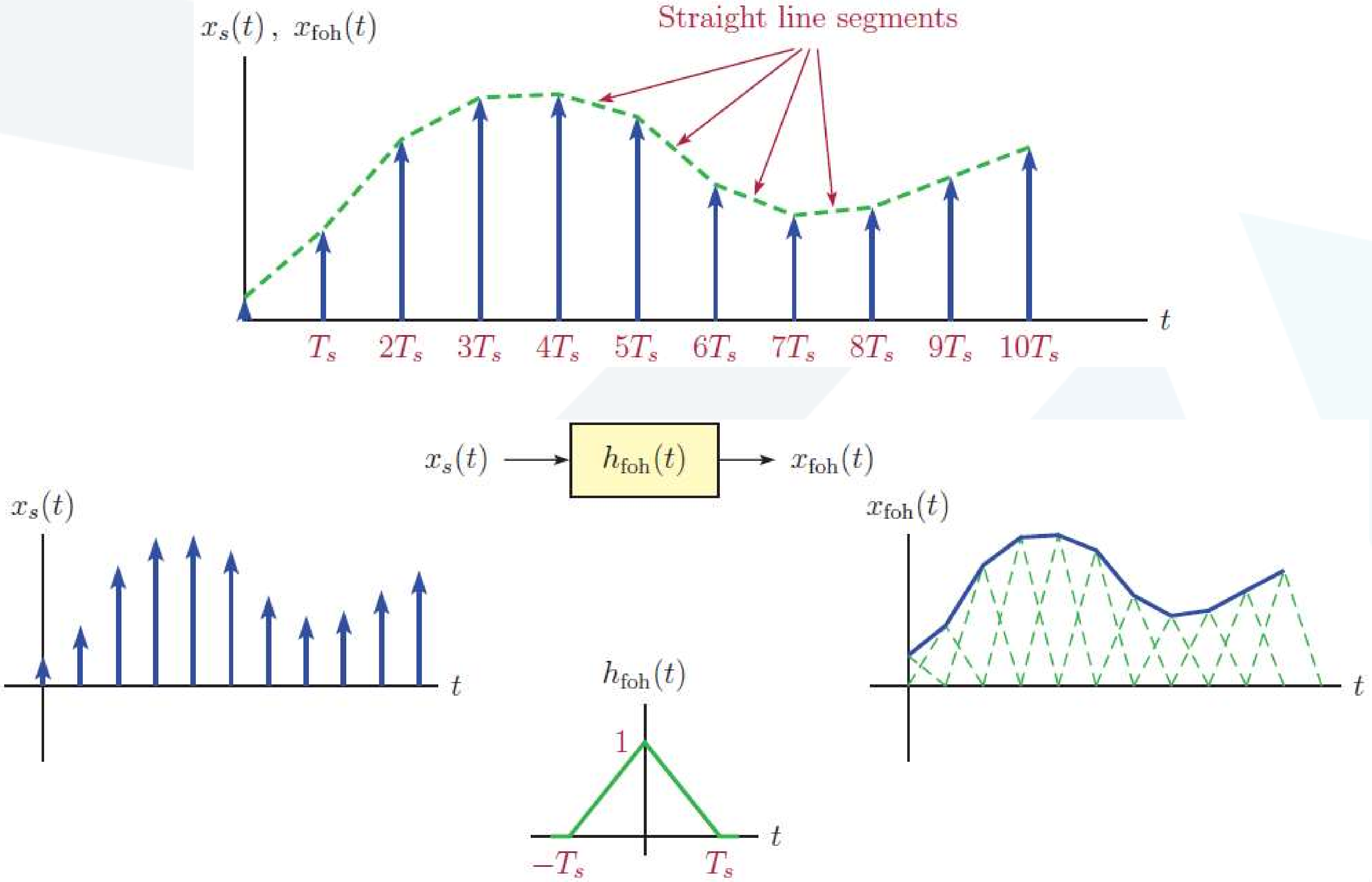


Zero-order hold reconstruction filter: $h_{zoh}(t) = \Pi\left(\frac{t - T_s/2}{T_s}\right)$



$$X_{zoh}(\omega) = \mathrm{sinc}(\omega T_s/2\pi)\, e^{-j\omega T_s/2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

First-order hold reconstruction filter: $h_{foh}(t) = \begin{cases} 1 + t/T_s, & -T_s < t < 0 \\ 1 - t/T_s, & 0 < t < T_s \\ 0, & |t| \geq T_s \end{cases}$


$x_s(t)$, $x_{foh}(t)$
Straight line segments
t
T_s $2T_s$ $3T_s$ $4T_s$ $5T_s$ $6T_s$ $7T_s$ $8T_s$ $9T_s$ $10T_s$
$x_s(t)$
$h_{foh}(t)$
$x_{foh}(t)$
$x_s(t)$
t
$x_{foh}(t)$
t
$h_{foh}(t)$
1
t
$-T_s$ T_s

- The impulse response $h_{foh}(t)$ of the first-order hold interpolation filter is non-causal since it starts at $t=-T_s$.

$$\bar{h}_{foh}(t) = h_{foh}(t - T_s)$$ (to achieve causality)

$$H_{zoh}(f) = T_s \operatorname{sinc}(fT_s) e^{-j\pi T_s}$$

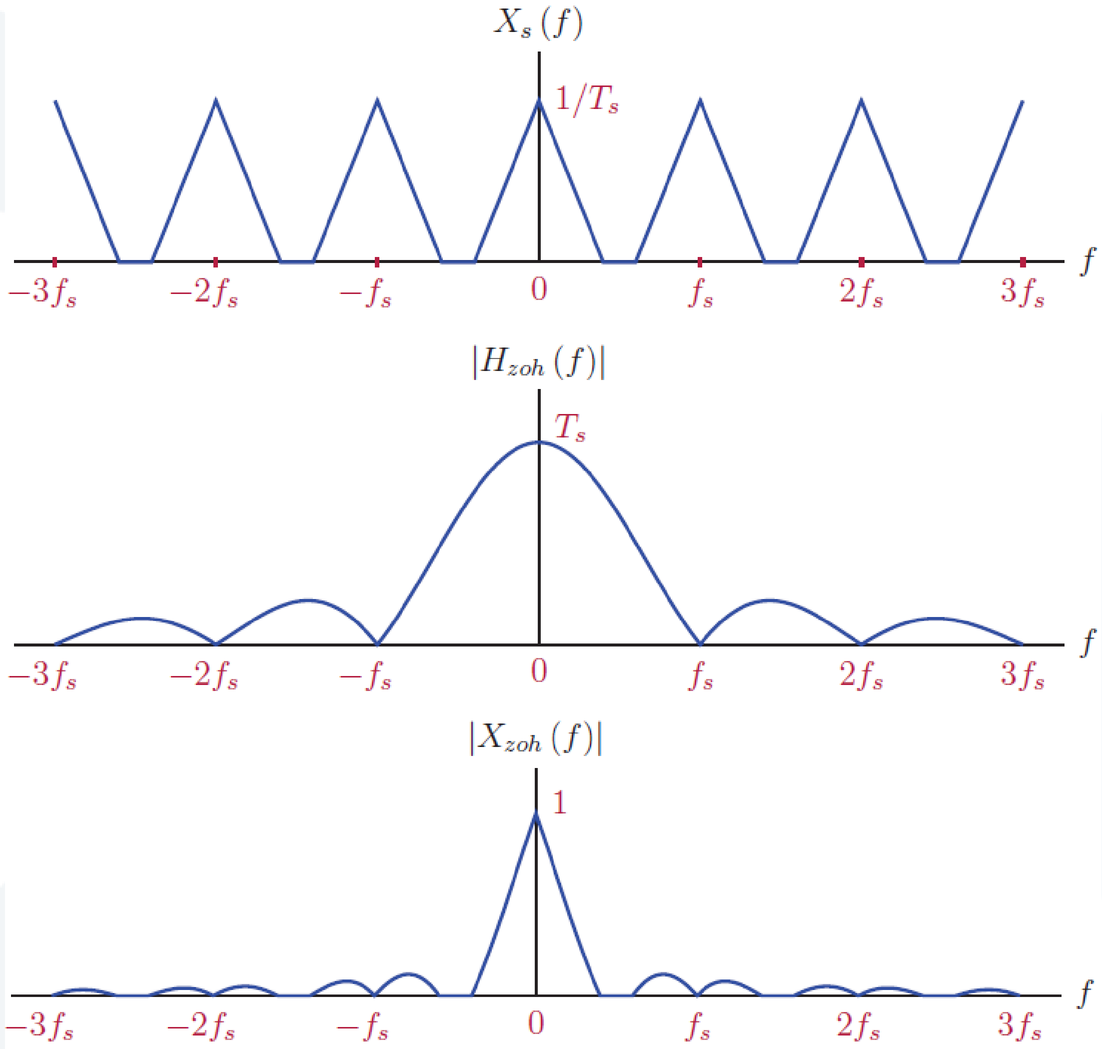
- The spectrum of the analog signal constructed using the zero-order hold filter is:

$$X_{zoh}(f) = H_{zoh}(f) X_s(f)$$

$$H_{foh}(f) = T_s \operatorname{sinc}^2(fT_s)$$

- The spectrum of the analog signal constructed using the zero-order hold filter is:

$$X_{foh}(f) = H_{foh}(f) X_s(f)$$


$X_s(f)$
$1/T_s$
f
$-3f_s$ $-2f_s$ $-f_s$ 0 f_s $2f_s$ $3f_s$
$|H_{zoh}(f)|$
T_s
f
$-3f_s$ $-2f_s$ $-f_s$ 0 f_s $2f_s$ $3f_s$
$|X_{zoh}(f)|$
1
f
$-3f_s$ $-2f_s$ $-f_s$ 0 f_s $2f_s$ $3f_s$

- What kind of interpolation is needed for perfect reconstruction of the analog signal from its impulse-sampled version?
- The answer must be found through the frequency spectrum of the sampled signal.

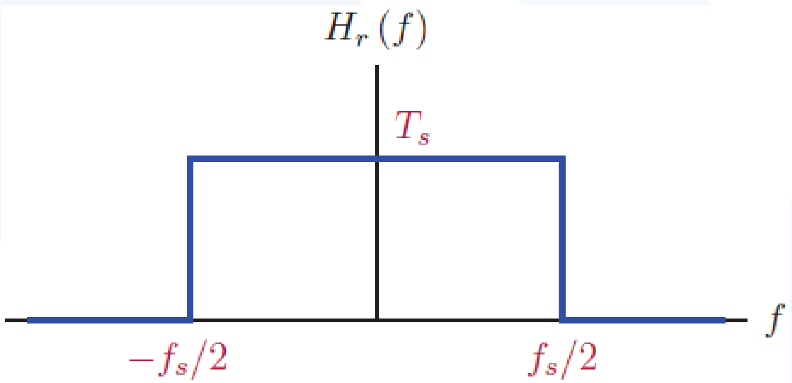
$$X_s(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

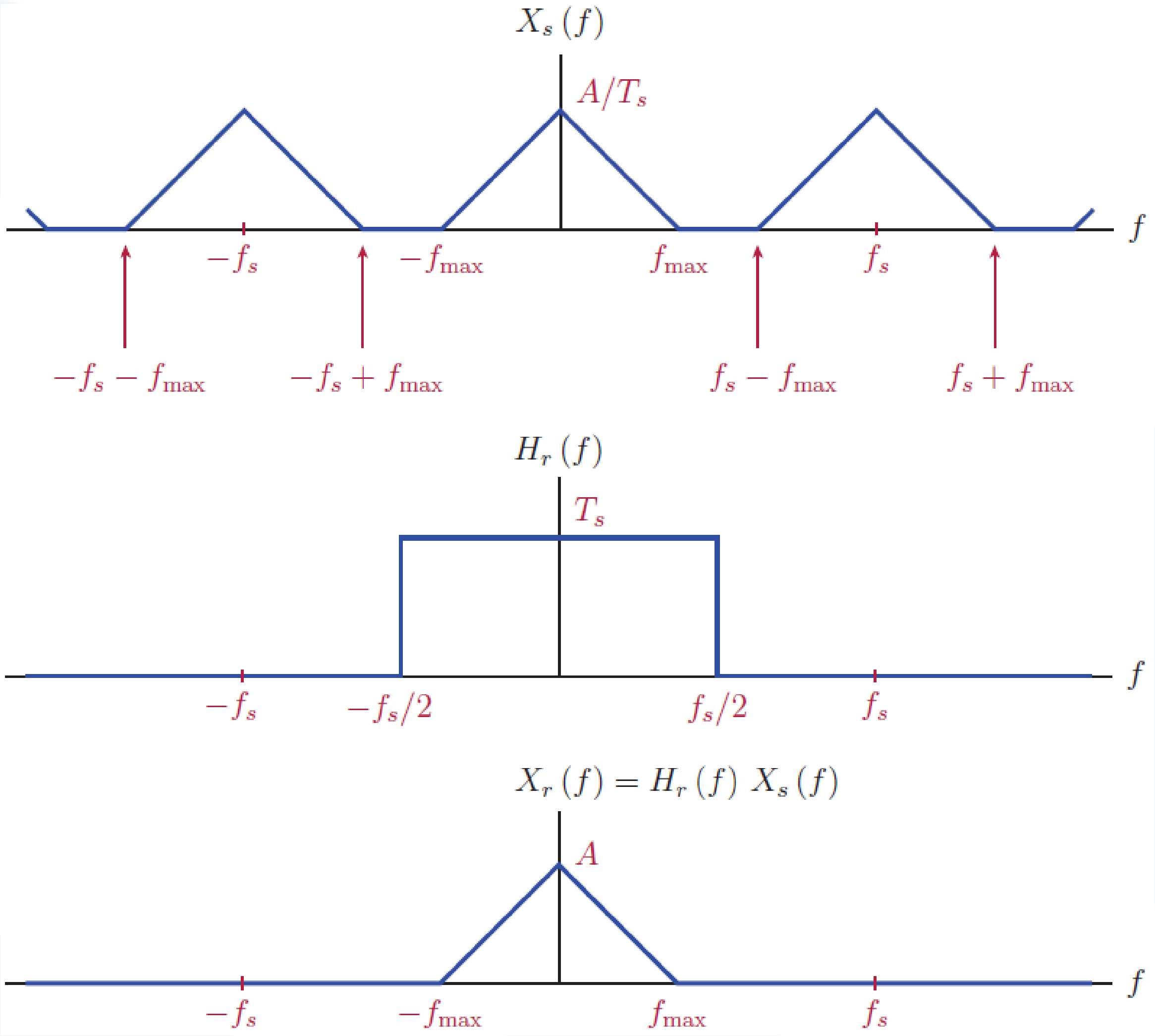
- An ideal lowpass reconstruction filter with cutoff frequency set equal to $f_s/2$ is needed.

$$H_r(f) = T_s \Pi\left(f/f_s\right)$$

$$X_r(f) = H_r(f)X_s(f)$$

$$= T_s \Pi\left(f/f_s\right)\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - kf_s) = X_a(f)$$




$X_s(f)$
A/T_s
f
$-f_s$
$-f_{\max}$
$f_{\max}$
f_s
$-f_s - f_{\max}$
$-f_s + f_{\max}$
$f_s - f_{\max}$
$f_s + f_{\max}$
$H_r(f)$
T_s
f
$-f_s$
$-f_s/2$
$f_s/2$
f_s
$X_r(f) = H_r(f)\, X_s(f)$
A
f
$-f_s$
$-f_{\max}$
$f_{\max}$
f_s

- Since $X_r(f) = X_a(f)$ we have $x_r(t) = x_a(t)$

$$h_r(t) = \text{sinc}\left(tf_s\right) = \text{sinc}\left(t/T_s\right)$$

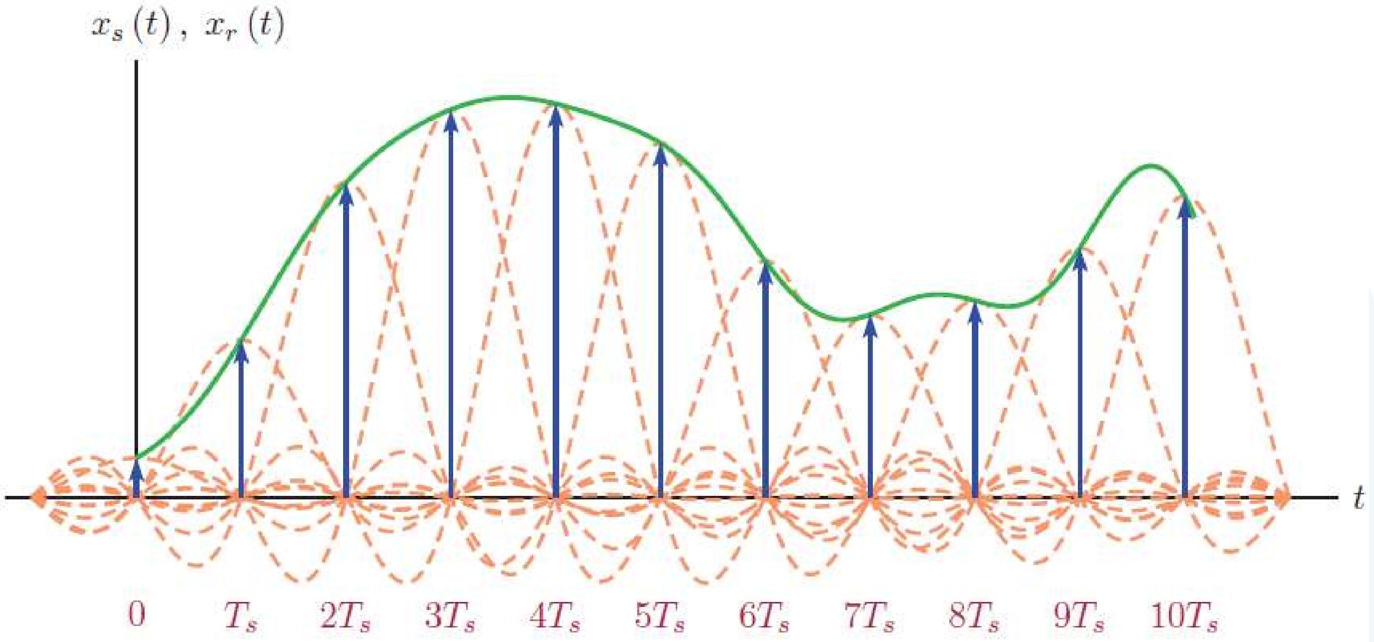
$$x_r(t) = h_r(t) * x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\text{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

- Let us consider the output of the filter at one of the sampling instants, say $t = kT_s$.

$$x_r(kT_s) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\text{sinc}\left(\frac{kT_s-nT_s}{T_s}\right) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\text{sinc}\,(k-n)$$

$$\text{sinc}\,(k-n) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \Rightarrow x_r(kT_s) = x_a(kT_s)$$

1. The output $x_r(t)$ of the ideal lowpass reconstruction filter is equal to the sampled signal at each sampling instant.
2. Between sampling instants, $x_r(t)$ is obtained by interpolation through the use of sinc functions. This is referred to as bandlimited interpolation.



- All 3 methods (zero-order hold, first-order hold, and bandlimited interpolation) result in reconstructed signals that have the correct amplitude values at the sampling instants, and interpolated amplitude values between them.

- What makes the signal obtained by bandlimited interpolation more accurate than the other two?

- The signal obtained by bandlimited interpolation is the only signal among the three that is limited to a bandwidth of $f_s/2$.

- The bandwidth of each of the other two signals is greater than $f_s/2$, therefore, neither of them could have been the signal that produced a properly sampled $x_s(t)$.