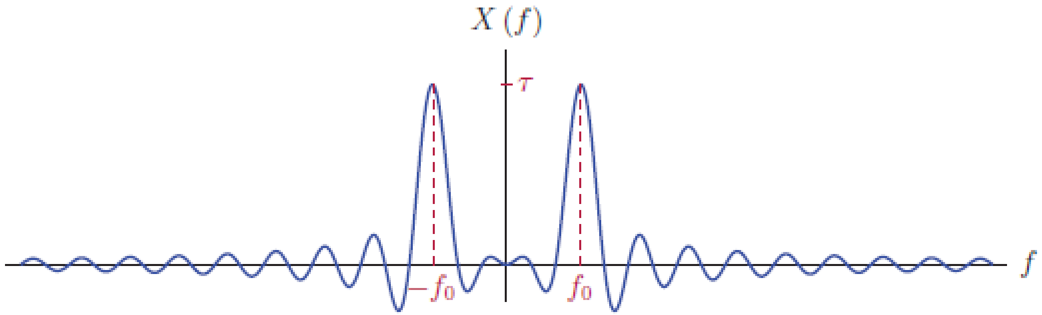
CECC507: Signals and Systems

Lecture Notes 8 & 9: Laplace Transform for Continuous-Time Signals and Systems



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Chapter 7

Laplace Transform for Continuous-Time Signals and Systems

1. Laplace Transform
2. Laplace Transform with CTLTI Systems
3. Simulation Structures for CTLTI Systems
4. Unilateral Laplace Transform

Introduction

- The Laplace transform (LT) can be viewed as a generalization of the (classical) Fourier transform.
- Certain characteristics of continuous-time (CT) systems can only be studied via the Laplace transform. Such is the case of stability, transient and steady-state responses.

1. Laplace Transform

- The Laplace transform of a continuous-time signal $x(t)$ is defined as:

$$L\{x(t)\} = X(s) = \int x(t)e^{-st}dt$$

where $s = \sigma + j\omega$, the independent variable of the transform. σ: damping factor, ω: frequency variable.

- There are two important variants:

Unilateral (or one-sided): $X(s) = \mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt;$

Bilateral (or two sided): $X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt;$

Relationship Between LT and Continuous-Time FT

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathcal{F}\{e^{-\sigma t}x(t)\}$$

$$X(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]\Bigg|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \mathcal{F}\{x(t)\}$$

- Example 1: Laplace transform of the unit impulse

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

- Example 2: Laplace transform of the unit-step signal

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt=\int_{-\infty}^{\infty}u(t)e^{-st}dt=\int_{0}^{\infty}e^{-st}dt=\frac{1}{s},\qquad \mathrm{Re}\{s\}>0$$

Regions of Convergence

- All points in the s-plane for which the Laplace transform converges is called the region of convergence (ROC). For the LT $X(s)$ of $x(t)$ to exist we need that:

$$\left|\int_{-\infty}^{\infty}x(t)e^{-\sigma t}dt\right|=\left|\int_{-\infty}^{\infty}x(t)e^{-\sigma t}e^{-jwt}dt\right|\le\int_{-\infty}^{\infty}\left|x(t)e^{-\sigma t}\right|dt<\infty$$

- Note: The frequency does not affect the ROC.

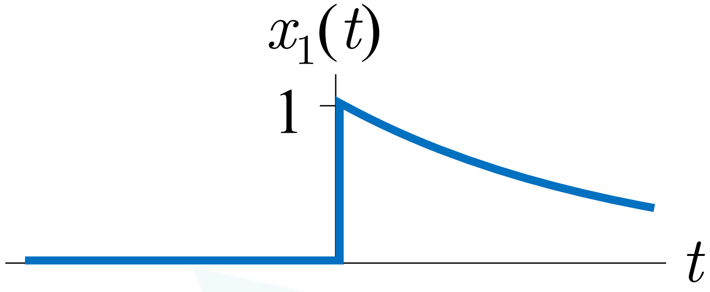
Poles and Zeros and the Region of Convergence

- Typically, $X(s)$ is rational, $X(s)=N(s)/D(s)$.

- For the Laplace The roots of $N(s)$ are called zeros, and the roots of $D(s)$ are called poles. The ROC is related to the poles of the transform.
- If $\{\sigma_i\}$ are the real parts of the poles of $X(s)$, the ROC corresponding to different types of signals is determined from its poles as follows:
- For a causal signal $x(t)$, the region of convergence of its Laplace transform $X(s)$ is a plane to the right of the poles, $R_c = \{(\sigma, \omega): \sigma > \max\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a anticausal signal $x(t)$, the ROC of its Laplace transform $X(s)$ is a plane to the left of the poles, $R_{ac} = \{(\sigma, \omega): \sigma < \min\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a noncausal signal $x(t)$, the region of convergence of its Laplace transform $X(s)$ is the intersection of the ROC corresponding to the causal component, R_c, and R_{ac} corresponding to the anticausal component, $R_c \cap R_{ac}$

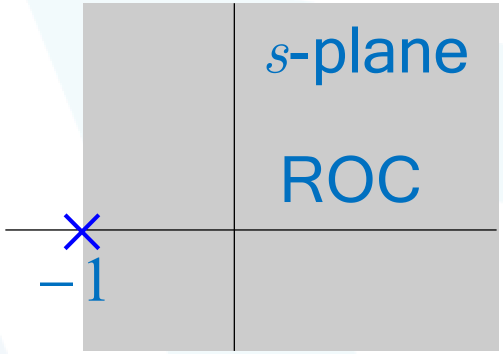
- Example 3: Find the Laplace transform of $x_1(t)$

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



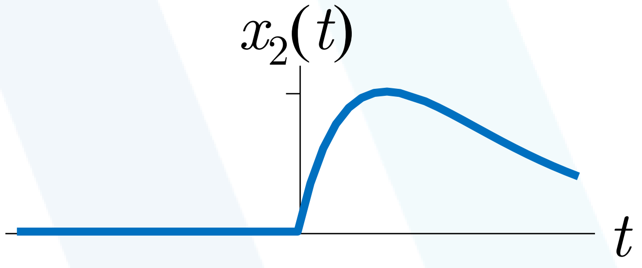
$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt = \int_0^{\infty} e^{-t} e^{-st} dt = \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^{\infty} = \frac{1}{s+1},$$

$$\text{Re}\{s\} > -1$$



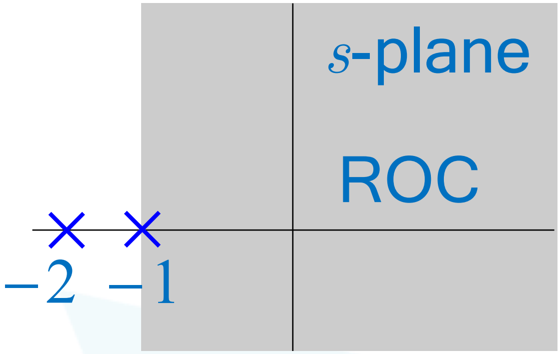
- Example 4: Find the Laplace transform of $x_2(t)$

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



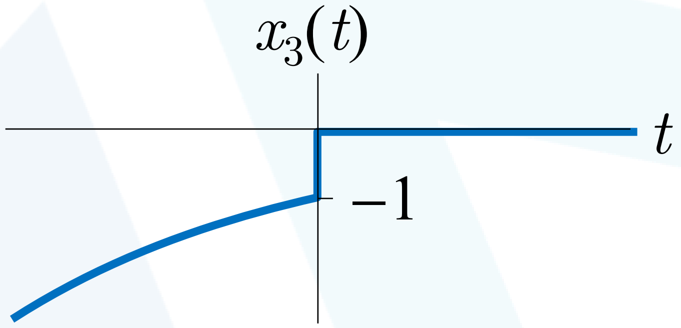
$$X_2(s) = \int_0^{\infty} (e^{-t} - e^{-2t}) e^{-st} dt$$

$$= \int_0^{\infty} e^{-t} e^{-st} dt - \int_0^{\infty} e^{-2t} e^{-st} dt = \frac{1}{(s+1)(s+2)}, \ \mathrm{Re}\{s\} > -1$$



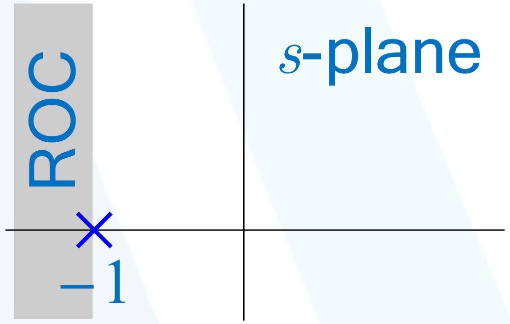
- Example 5: LT of an anti-causal exponential signal

$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$X_3(s) = \int_{-\infty}^{\infty} x_3(t) e^{-st} dt = \int_{-\infty}^{0} -e^{-t} e^{-st} dt = \left. \frac{-e^{-(s+1)t}}{-(s+1)} \right|_{-\infty}^{0} = \frac{1}{s+1},$$

$$\mathrm{Re}\{s\} < -1$$



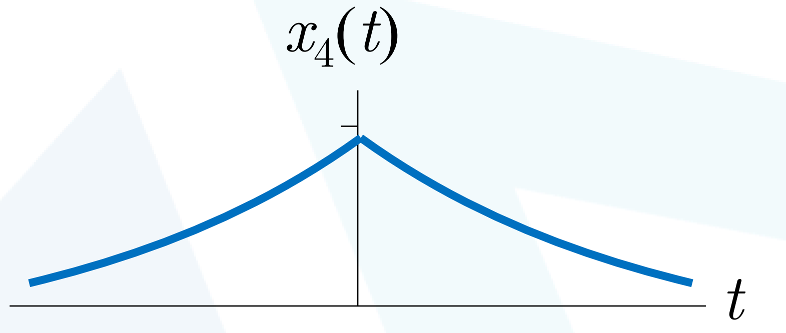
- It is possible for two different signals to have the same transform expression for $X(s)$.

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad \text{ROC: Re}\{s\} > -1$$

$$L\{-e^{-t}u(-t)\} = \frac{1}{s+1}, \quad \text{ROC: Re}\{s\} < -1$$

$\Rightarrow$ the ROC must be specified along with the transform

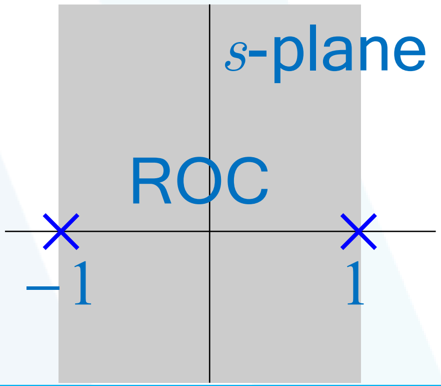
- Example 6: Find the Laplace transform of $x_4(t)$



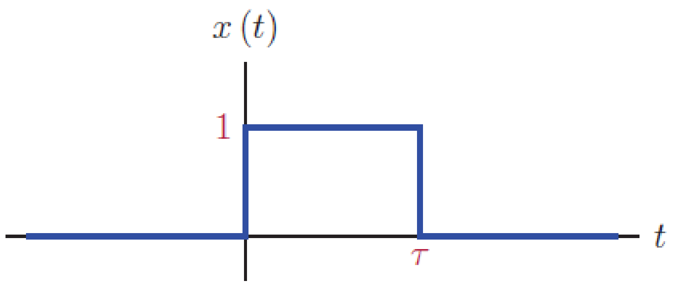
$$x_4(t) = e^{-|t|}$$

$$X_4(s) = \int_{-\infty}^{\infty} e^{-|t|}e^{-st}dt = \int_{-\infty}^{0} e^{(1-s)t}dt + \int_{0}^{\infty} e^{-(1+s)t}dt$$

$$= \left.\frac{e^{(1-s)t}}{(1-s)}\right|_{-\infty}^{0} + \left.\frac{-e^{(1+s)t}}{-(1+s)}\right|_{0}^{\infty} = \underset{\text{Re}<1}{\frac{1}{1-s}} + \underset{\text{Re}>-1}{\frac{1}{1+s}} = \frac{2}{1-s^2}, \ -1 < \text{Re}\{s\} < 1$$



- Example 7: Laplace transform of a pulse signal



$$x(t) = \Pi\left(\frac{t-\tau/2}{\tau}\right)$$

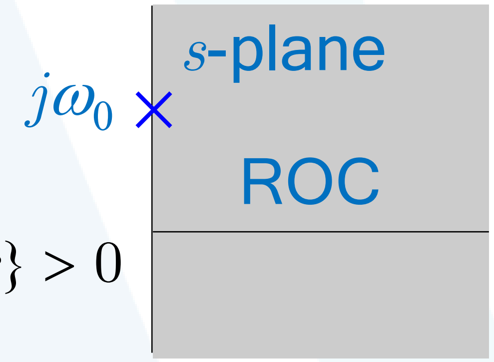
$$X(s) = \int_0^\tau (1)e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\tau = \frac{1-e^{-s\tau}}{s}$$

$$X(s)\big|_{s=0} = \left.\frac{\tau e^{-s\tau}}{1}\right|_{s=0} = \tau \Rightarrow X(s) \text{ converge at } s = 0$$

- Example 8: Laplace transform of complex exponential signal

$$x(t) = e^{j\omega_0 t}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t}u(t)e^{-st}dt = \int_0^\infty e^{(j\omega_0 - s)t}dt = \left.\frac{e^{(j\omega_0 t - st)}}{j\omega_0 - s}\right|_0^\infty = \frac{1}{s-j\omega_0},\ \mathrm{Re}\{s\} > 0$$



Properties of Laplace Transform

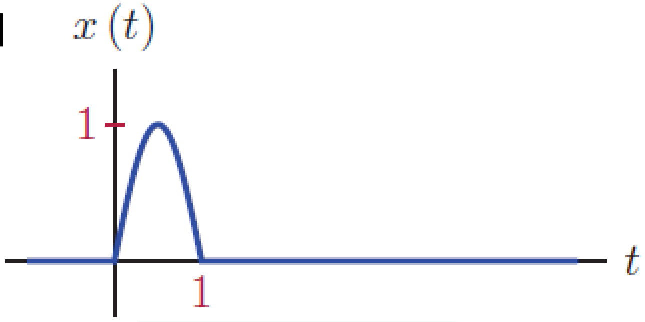
Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t - T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s + \alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at)$	$\frac{1}{\lvert a\rvert} X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Laplace Transform Pairs

1	$\delta(t)$	1	All s
2	$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
3	$-u(-t)$	$1/s$	$\text{Re}\{s\} < 0$
4	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} < 0$
6	$e^{-at} u(t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} > -a$
7	$-e^{-at} u(-t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} < -a$

8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} < -a$
10	$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
11	$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
12	$[e^{-at} \cos \omega_0 t] u(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > -a$
13	$[e^{-at} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > -a$

- Example 9: Laplace transform of a truncated sine functio



$$x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_0^1 \sin(\pi t) e^{-st} dt = \frac{1}{2j} \int_0^1 (e^{j\pi t} - e^{-j\pi t}) e^{-st} dt = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$

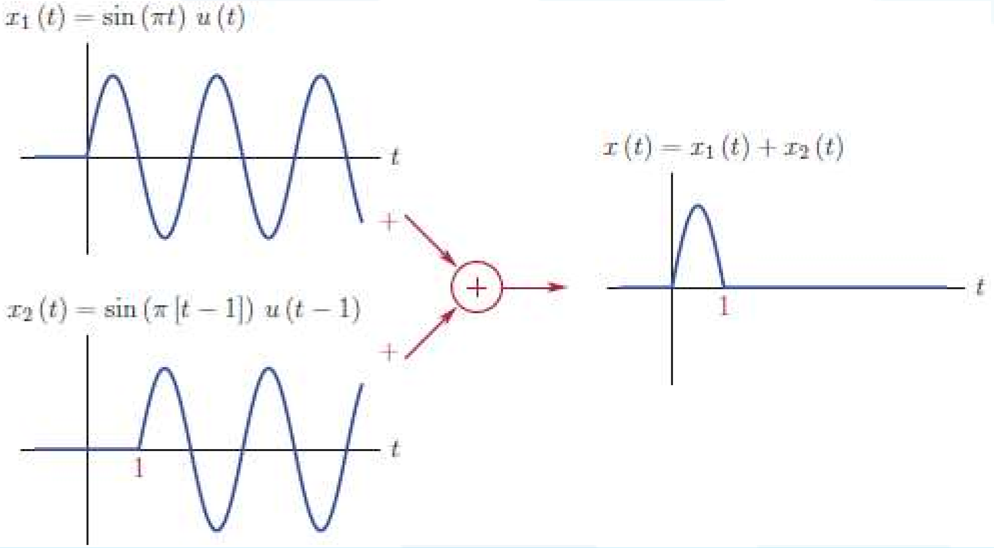
Another method

$$x(t) = \sin(\pi t) u(t) + \sin(\pi[t-1]) u(t-1)$$

$$X(s) = \frac{\pi}{s^2 + \pi^2} + \frac{\pi}{s^2 + \pi^2} e^{-s} = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$

ROC: entire s-plane except points where

$\text{Re}\{s\} \rightarrow -\infty$



- Example 10: Using the convolution property of the Laplace transform

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = \delta(t) - e^{-2t}u(t)$$

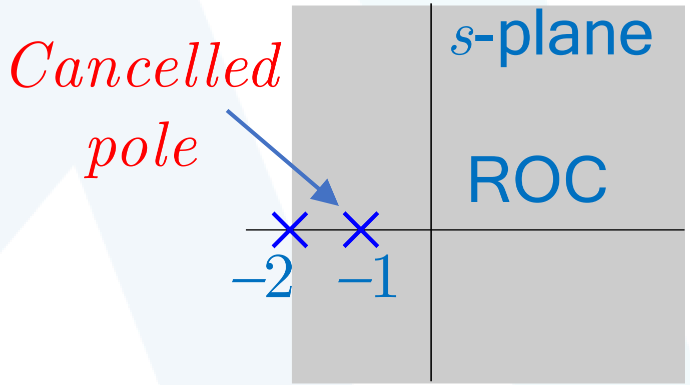
Determine $x(t) = x_1(t) * x_2(t)$ using Laplace transform techniques.

$$X_1(s) = \frac{1}{s+1}, \quad \text{ROC: Re}\{s\} > -1$$

$$X_2(s) = 1 - \frac{1}{s+2} = \frac{s+1}{s+2}, \quad \text{ROC: Re}\{s\} > -2$$

$$X(s) = X_1(s)X_2(s) = \frac{1}{s+2}, \quad \text{ROC: Re}\{s\} > -2$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$



Initial Value Theorem

For a function x with Laplace transform X, if x is causal and contains no impulses or higher order singularities at the origin, then:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

- When X is known but x is not, the initial value theorem eliminates the need to explicitly find x in order to evaluate $x(0^+)$.

- Example 11: Calculate the initial value of the function $x(t)$, whose LT is:

$$X(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification: $x(t) = 2e^{-t}\cos(5t)\,u(t)$

Final Value Theorem

For a function x with Laplace transform X, if x is causal and $x(t)$ has a finite limit as $t \to \infty$, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

- When X is known but x is not, the final value theorem eliminates the need to explicitly find x in order to evaluate limit $_{t\to\infty}$ $x(t)$.

- Example 12: Calculate the final value of the function $x(t)$, whose Laplace transform is:

$$X(s) = \frac{s+3}{s(s+1)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s+3)}{s(s+1)} = \lim_{s\to 0} \frac{s+3}{s+1} = 3$$

Verification: $x(t) = (3 - 2e^{-t})\,u(t)$

Inverse Laplace Transform

The inverse LT x of X is given by $L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$

where Re(s) $= \sigma$ is in the ROC of X.

- For rational functions, the inverse Laplace transform can be more easily computed using partial fraction expansions (PFE).
- Example 13: Calculate the inverse LT of the function $H(s) = 1/(s + a)$

  $h(t) = e^{-at}u(t)$ with ROC: Re$\{s\} > -a$

  $h(t) = -e^{-at}u(-t)$ with ROC: Re$\{s\} < -a$

- Example 14: The Laplace transform of a signal $x(t)$ is $X(s) = \frac{s+1}{s(s^2+9)}$ with the ROC specified as Re $\{s\} > 0$. Determine $x(t)$.

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s + j3} + \frac{k_3}{s - j3}$$

$$k_1 = \frac{1}{9}, \quad k_2 = -\frac{1}{18} + j\frac{1}{6}, \quad k_3 = \frac{1}{18} - j\frac{1}{6}$$

Based on the specified ROC, the signal $x(t)$ is causal

$$x(t) = \frac{1}{9}u(t) - \frac{1}{918}[e^{-j3t} + e^{j3t}]u(t) + j\frac{1}{6}[e^{-j3t} - e^{j3t}]u(t)$$

$$x(t) = \frac{1}{9}u(t) - \frac{1}{9}\cos(3t)u(t) + \frac{1}{3}\sin(3t)u(t)$$

- Example 15: Multiple-order poles

A causal signal $x(t)$ has the Laplace transform $X(s) = \dfrac{s(s+1)}{(s+1)^3(s+2)}$

$$X(s) = \frac{s(s+1)}{(s+1)^3(s+2)} = \frac{-3}{s+1} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)^3} + \frac{3}{s+2}$$

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \qquad L\{te^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{s+1}\right] = \frac{1}{(s+1)^2}$$

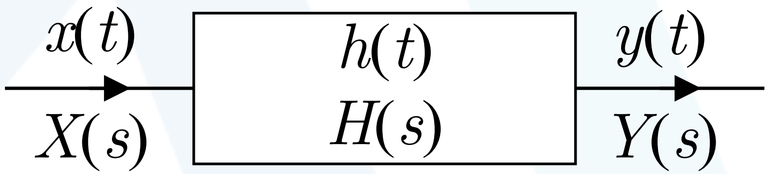
$$L\{t^2e^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2}\right] = \frac{2}{(s+1)^3}$$

$$x(t) = -3e^{-t}u(t) + 3te^{-t}u(t) - t^2e^{-3t}u(t) + 3e^{-2t}u(t)$$

2. Laplace Transform with CTLTI Systems

Transfer Function and LTI Systems

- Since $y(t) = x(t) * h(t)$, the system is characterized in the Laplace domain by $Y(s) = X(s)H(s)$.



- $H(s)$ is the transfer function (or system function) of the system.

- A LTI system is completely characterized by its transfer function $H(s)$.

Relating the transfer function to the differential equation

- Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients.

- Consider a system with input x and output y that is characterized by an equation of the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where the a_k and b_k are complex constants and

$$\mathcal{L}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{L}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \Rightarrow \sum_{k=0}^{N} \mathcal{L}\left\{a_k \frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} \mathcal{L}\left\{b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

- The transfer function is always rational.
- The impulse response of the system $h(t) = \mathcal{L}^{-1}\{H(s)\}$.
- The convolution operation is only applicable to problems involving LTI systems.
- Therefore it follows that the transfer function concept is meaningful only for systems that are both linear and time invariant.
- In determining the transfer function from the differential equation, all initial conditions must be assumed to be zero.

- Example 16: Finding the transfer function from the DE

  A CTLTI system is defined by means of the differential equation:

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 17\frac{dy(t)}{dt} + 13y(t) = \frac{d^2x(t)}{dt^2} + x(t)$$

$$s^3Y(s) + 5s^2Y(s) + 17sY(s) + 13Y(s) = s^2X(s) + X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 1}{s^3 + 5s^2 + 17s + 13}$$

Transfer function and causality

- Theorem: For a LTI system with a rational transfer function H, causality of the system is equivalent to the ROC of H being the right sided to the right of the rightmost pole or, if H has no poles, the entire complex plane.

- For a CTLTI system to be causal, its impulse response $h(t)$ needs to be equal to zero for $t < 0$.

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} h(t)e^{-st}dt$$

- Consider a transfer function in the form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

For the system described by $H(s)$ to be causal we need:

$$\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_M}{a_N} s^{M-N} < \infty \Leftrightarrow M - N \leq 0 \Rightarrow M \leq N$$

Causality condition:

- In the transfer function of a causal CTLTI system the order of the numerator must not be greater than the order of the denominator.

Transfer function and stability:

- For a CTLTI system to be stable its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty$$

Stability condition:

- For a CTLTI system to be stable, the ROC of its s-domain transfer function must include the imaginary axis.
- For a causal system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s-plane.
- For a anticausal system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s-plane.

- For a noncausal system the ROC for the TF, if it exists, is the region in the form $\sigma_1 < \text{Re}\{s\} < \sigma_2$. For stability we need $\sigma_1 < 0$ and $\sigma_2 > 0$. The poles of the TF may be either:

    a. On or to the left of the vertical line $\sigma = \sigma_1$

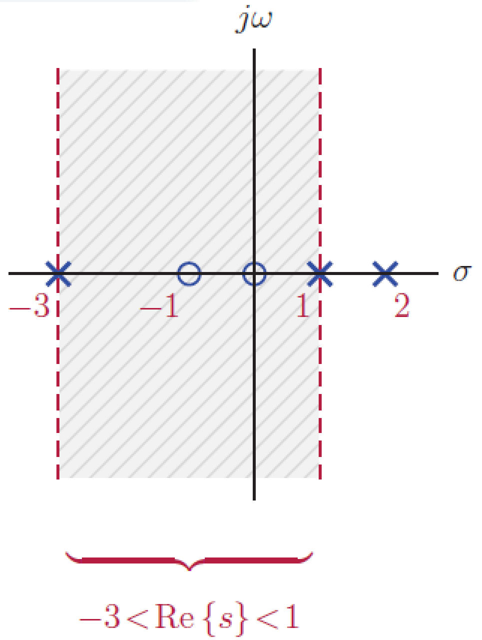
    b. On or to the right of the vertical line $\sigma = \sigma_2$

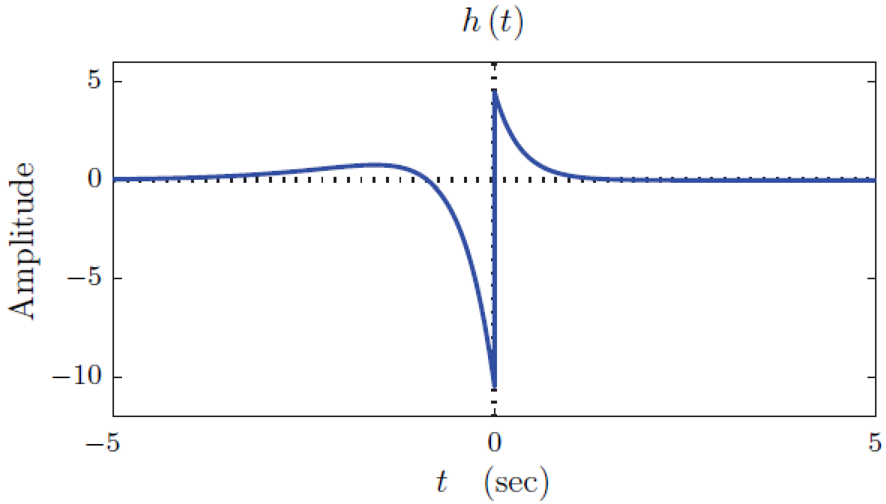
- Example 17: Impulse response of a stable system $H(s) = \dfrac{15s(s+1)}{(s+3)(s-1)(s-2)}$

  Determine the ROC of the TF. Afterwards find the impulse response.

  The 3 poles are at $s = -3, 1, 2$. Since the system is known to be stable, its ROC must include the $j\text{–}\omega$ axis. The only possible choice is $-3 < \text{Re}\{s\} < 1$.

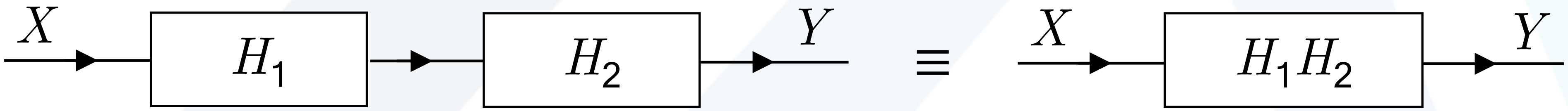
$$H(s) = \frac{4.5}{s+3} - \frac{7.5}{s-1} + \frac{18}{s-2} \qquad \Rightarrow h(t) = 4.5e^{-3t}u(t) + 7.5e^{t}u(-t) - 18e^{2t}u(-t)$$



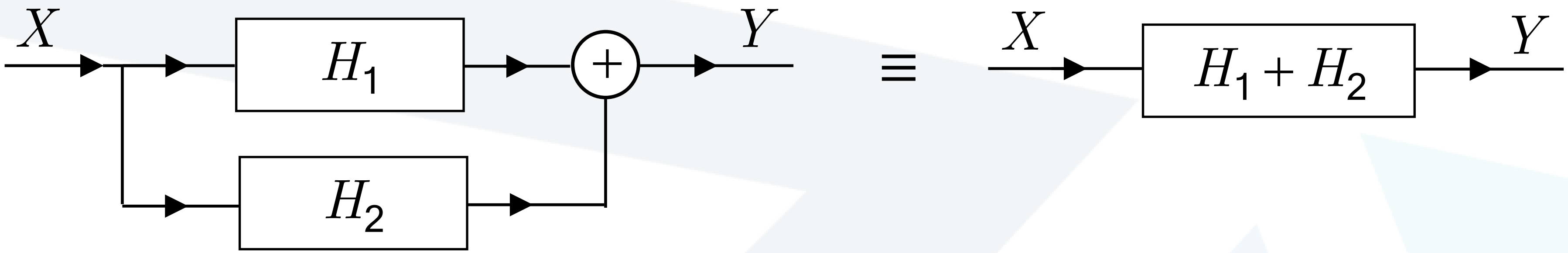


Interconnection of LTI Systems

- The series interconnection of the LTI systems with TFs H_1 and H_2.



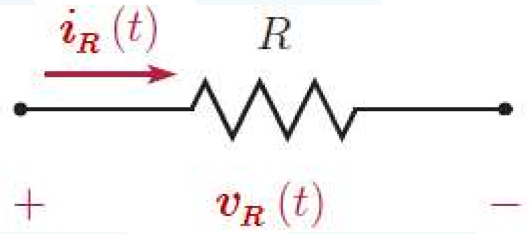
- The parallel interconnection of the LTI systems with TFs H_1 and H_2.



Application: Circuit Analysis: Electronic Circuits

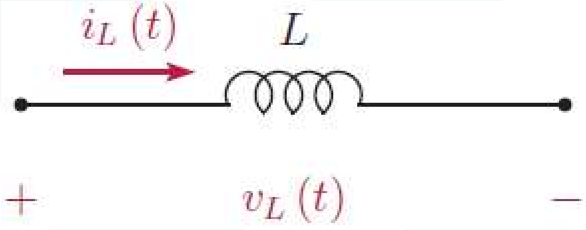
- A resistor $v_R(t) = Ri_R(t)$ or $i_R(t) = \frac{1}{R} v_R(t)$

$$V_R(s) = RI_R(s) \quad \text{or} \quad I_R(s) = \frac{1}{R} V_R(s)$$

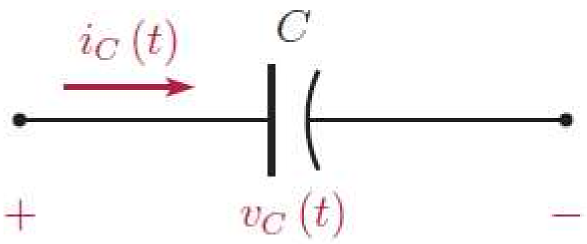


- An inductor $v_L(t) = L \frac{d}{dt} i_L(t)$ or $i_L(t) = \frac{1}{L} \int_{-\infty}^{t} v_L(\tau) d\tau$

$$V_L(s) = sLI_L(s) \quad \text{or} \quad I_L(s) = \frac{1}{sL} V_L(s)$$



- A capacitor $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$ or $i_C(t) = C\frac{d}{dt}v_C(t)$



$$V_C(s) = \frac{1}{sC}I_C(s) \quad \text{or} \quad I_C(s) = sCV_C(s)$$

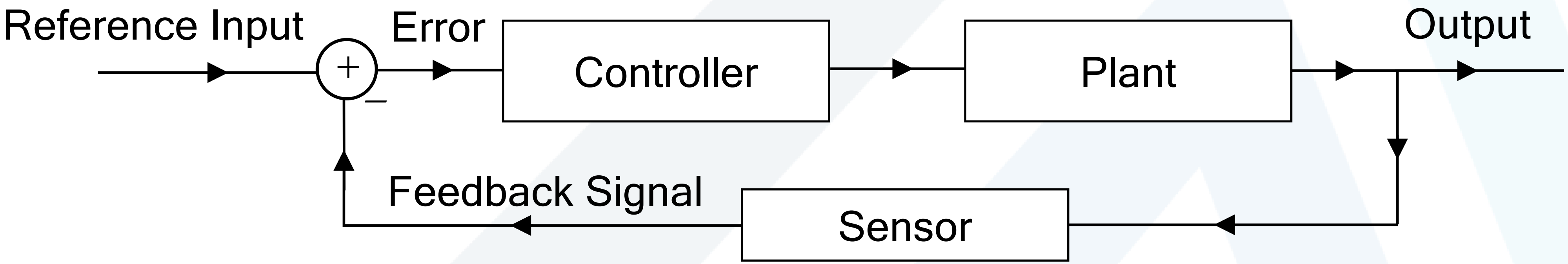
Application: Design and Analysis of Control Systems

Control Systems

- The desired values of the quantities being controlled are collectively viewed as the input of the control system.
- The actual values of the quantities being controlled are collectively viewed as the output of the control system.
- A control system whose behavior is not influenced by the actual values of the quantities being controlled is called an open loop (or non-feedback) system.

- A control system whose behavior is influenced by the actual values of the quantities being controlled is called a closed loop (or feedback) system.
- An example of a simple control system would be a thermostat system, which controls the temperature in a room or building.

Feedback Control Systems



- input: desired value of the quantity to be controlled.
- output: actual value of the quantity to be controlled.

- error: difference between the desired and actual values.
- plant: system to be controlled.
- controller: device that monitors the error and changes the input of the plant. with the goal of forcing the error to zero.
- sensor: device used to measure the actual output.

A control system includes two very important components:

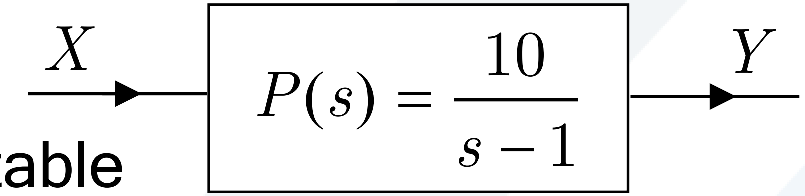
- Transducer: Since it is possible that the output signal $y(t)$ and the reference signal $x(t)$ might not be of the same type, a transducer is used to change $y(t)$ so it is compatible with the reference input $x(t)$.
- Actuator: A device that makes possible the execution of the control action on the plant, so that the output of the plant follows the reference input.

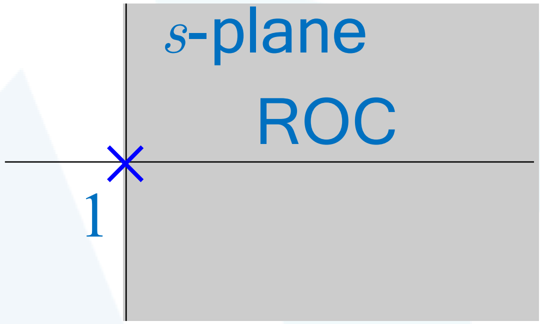
Stability Analysis of Feedback Systems

- Often, we want to ensure that a system is BIBO stable. BIBO stability property is more easily characterized in the Laplace domain than in the time domain.

Example 18: Stabilization Example: Unstable Plant

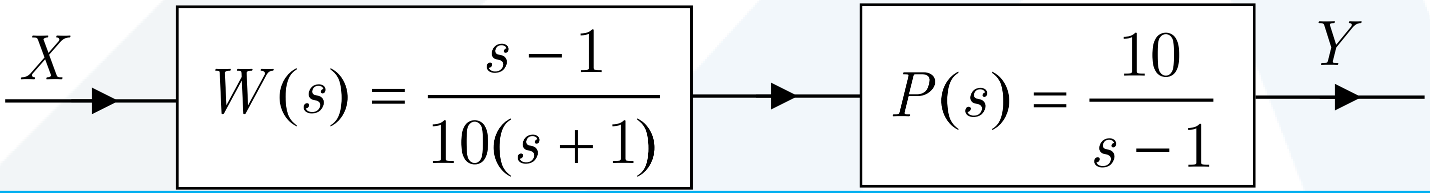
- Causal LTI plant
- System is not BIBO stable





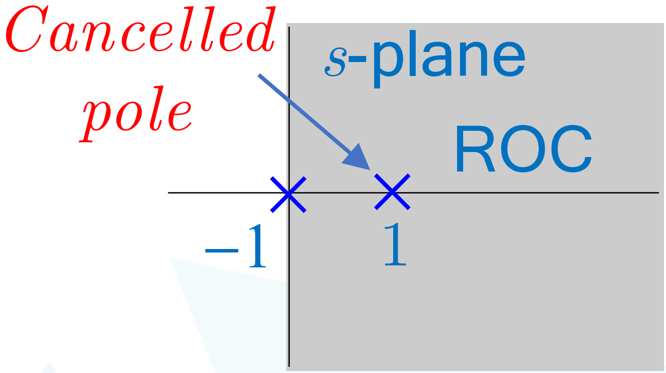
Example 19: Stabilization Example: Using Pole-Zero Cancellation

- System formed by series interconnection of plant and causal LTI compensator:



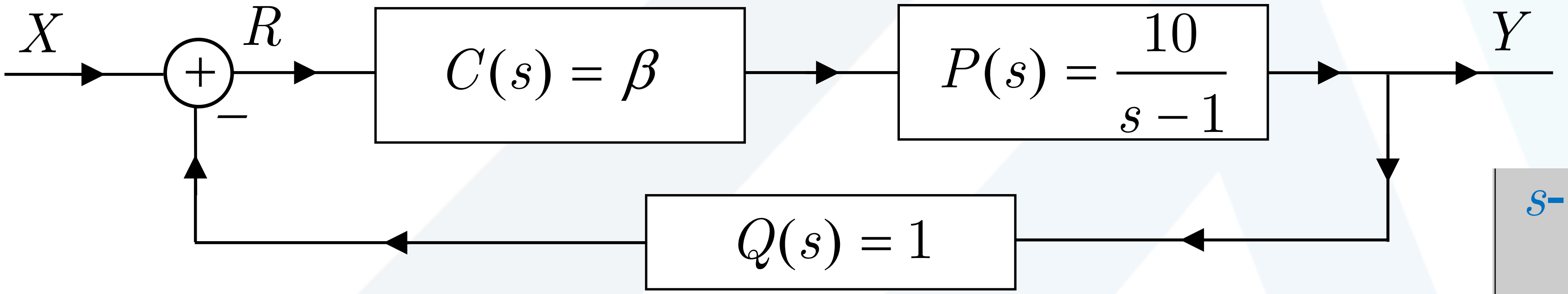
- Transfer function H of overall system (BIBO stable):

$$H(s) = W(s)P(s) = \frac{s-1}{10(s+1)}\frac{10}{s-1} = \frac{1}{(s+1)}$$



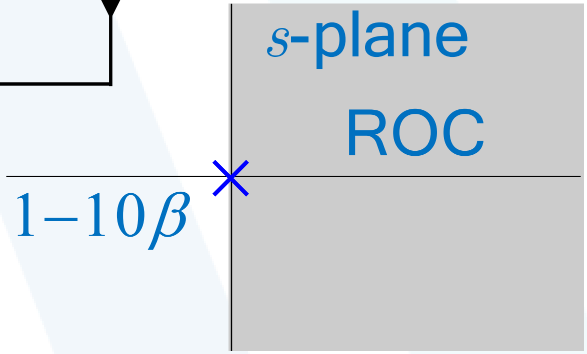
Example 20: Stabilization Example: Using Feedback

- Feedback system (with causal LTI compensator and sensor):



$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)} = \frac{10\beta}{s-(1-10\beta)}$$

BIBO stable iff $1-10\beta < 0$



3. Simulation Structures for CTLTI Systems

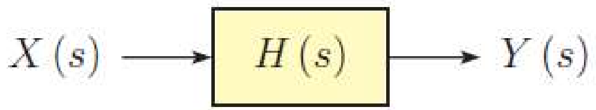
Direct-form implementation

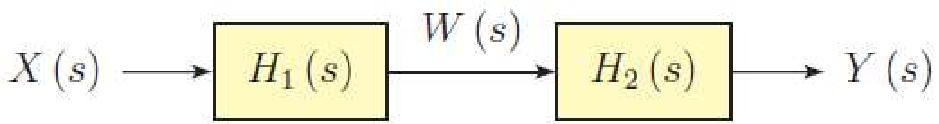
- Consider a third-order CTLTI system described by a TF $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Let us use an intermediate function $W(s)$

$$H(s) = \frac{Y(s)}{W(s)}\frac{W(s)}{X(s)} = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}$$

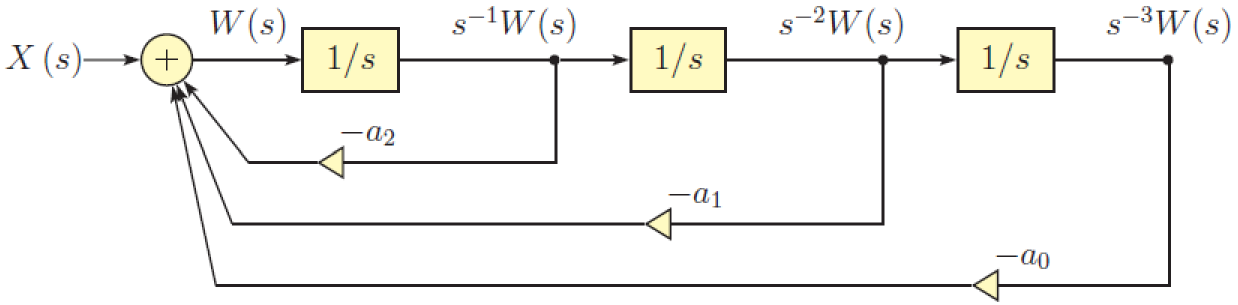


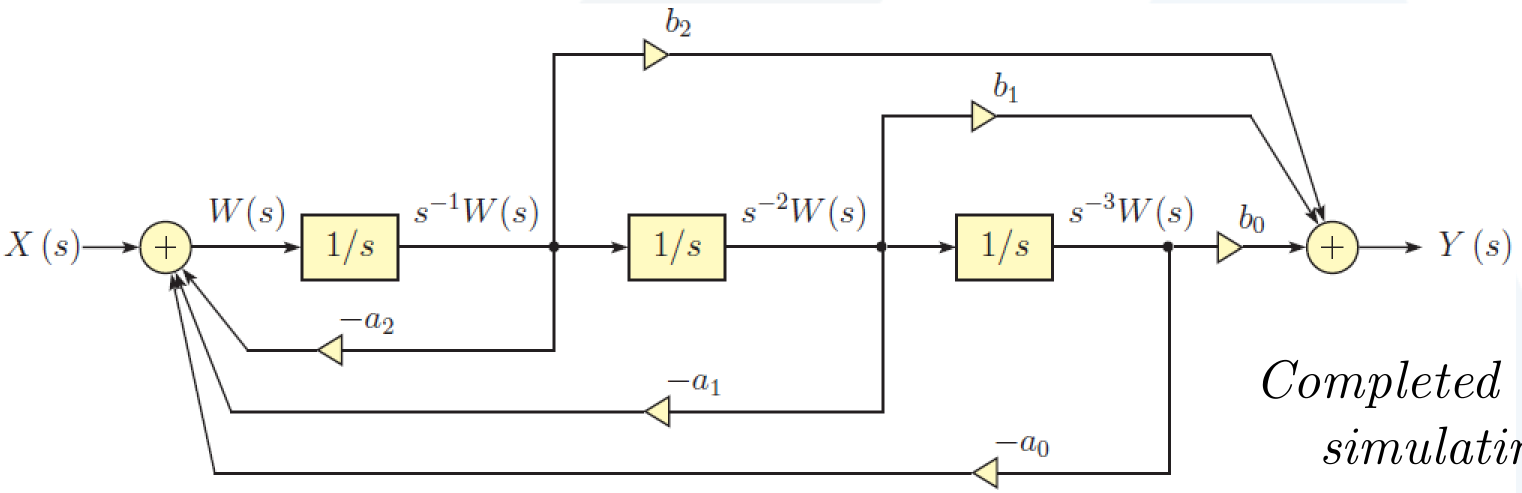


$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}, \qquad H_2(s) = \frac{Y(s)}{W(s)} = b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}$$

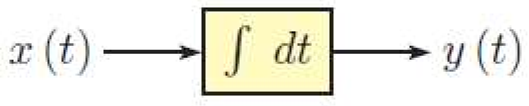
$$W(s) = X(s) - a_2 s^{-1} W(s) - a_1 s^{-2} W(s) - a_0 s^{-3} W(s)$$

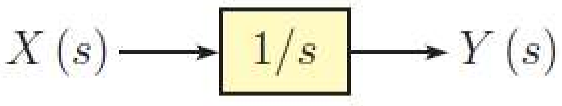
$$Y(s) = b_2 s^{-1} W(s) + b_1 s^{-2} W(s) + b_0 s^{-3} W(s)$$





Completed block diagram for simulating the TF H(s)

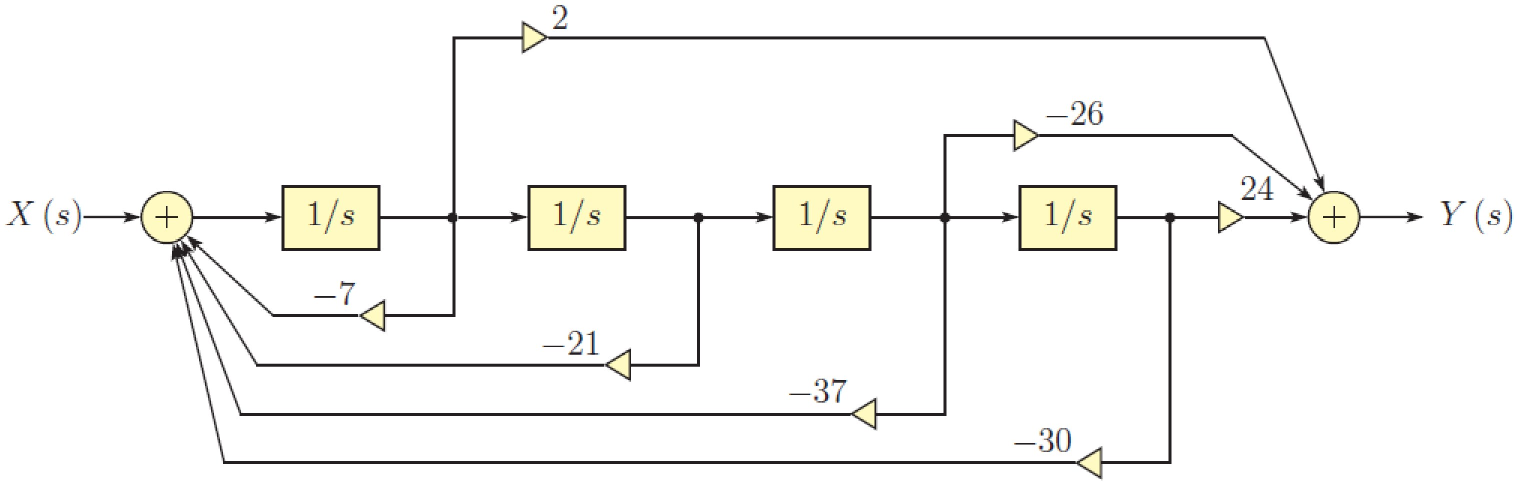




- Example 21: Obtaining a block diagram (BD) from transfer function

A CTLTI system is described through the transfer function:

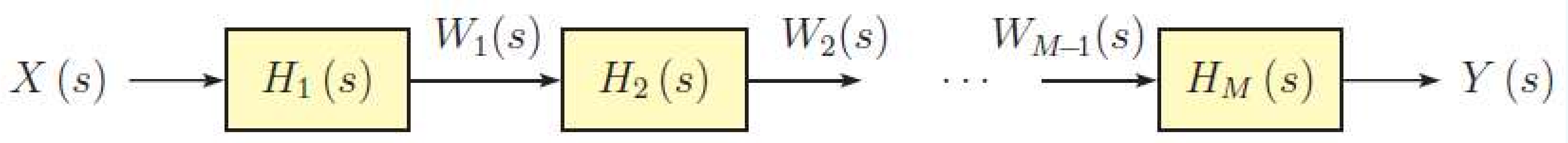
$$H(s)=\frac{Y(s)}{X(s)}=\frac{2s^{3}-26s+24}{s^{4}+7s^{3}+21s^{2}+37s+30}$$



Cascade and parallel forms

Cascade form

$$H(s) = H_1(s) H_2(s) \cdots H_M(s) = \frac{W_1(s)}{X(s)} \frac{W_2(s)}{W_1(s)} \cdots \frac{Y(s)}{W_{M-1}(s)}$$



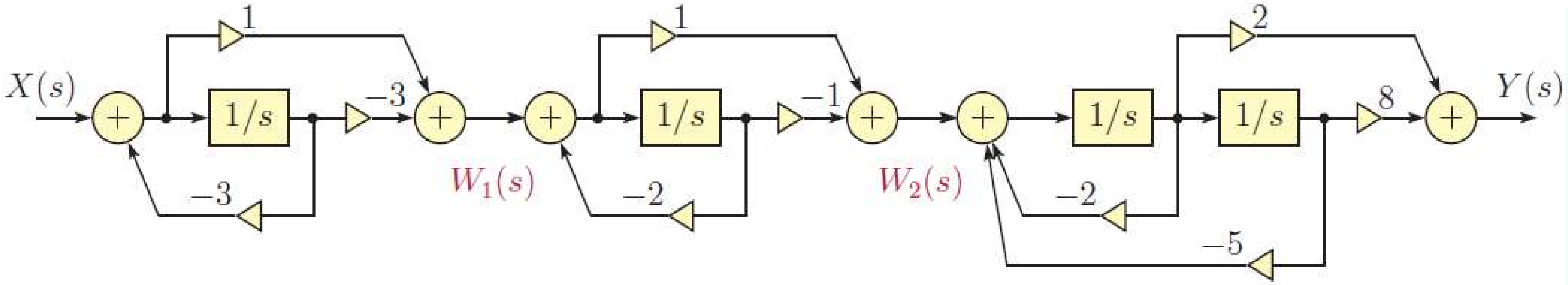
- Example 22: Obtaining a block diagram from transfer function

  Develop a cascade form block diagram for simulating the system used in example 21.

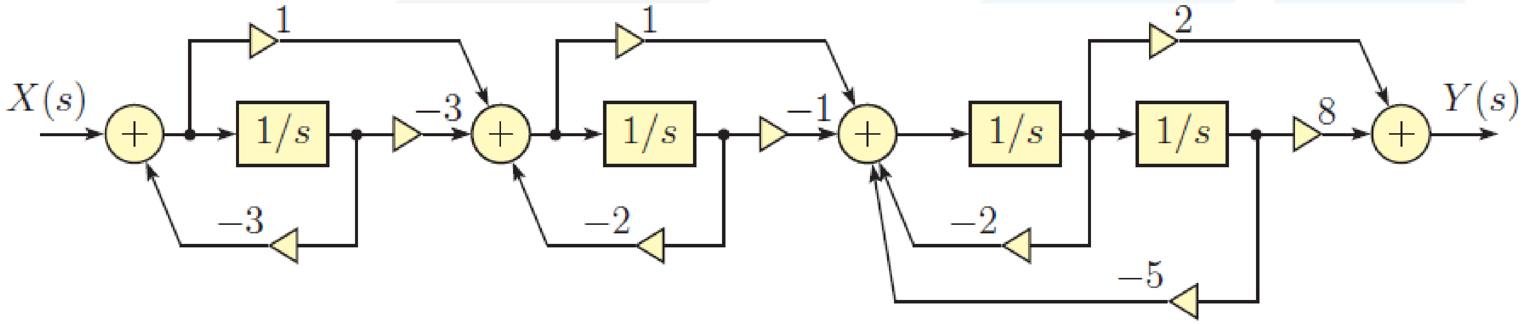
$$H(s) = \frac{Y(s)}{X(s)} = \frac{2(s+4)(s-3)(s-1)}{(s+1-j2)(s+1+j2)(s+3)(s+2)}$$

$$H(s) = H_1(s)H_2(s)H_3(s)$$

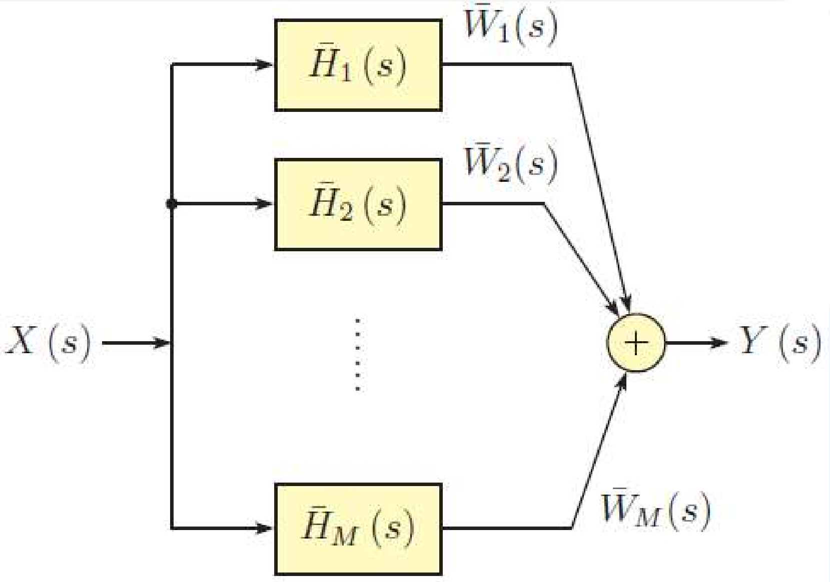
$$H_1(s) = \frac{2(s+4)}{(s+1-j2)(s+1+j2)} = \frac{2s+8}{s^2+2s+5}, \quad H_2(s) = \frac{s-3}{s+3}, \quad H_3(s) = \frac{s-1}{s+2}$$



Further simplified cascade form block diagram



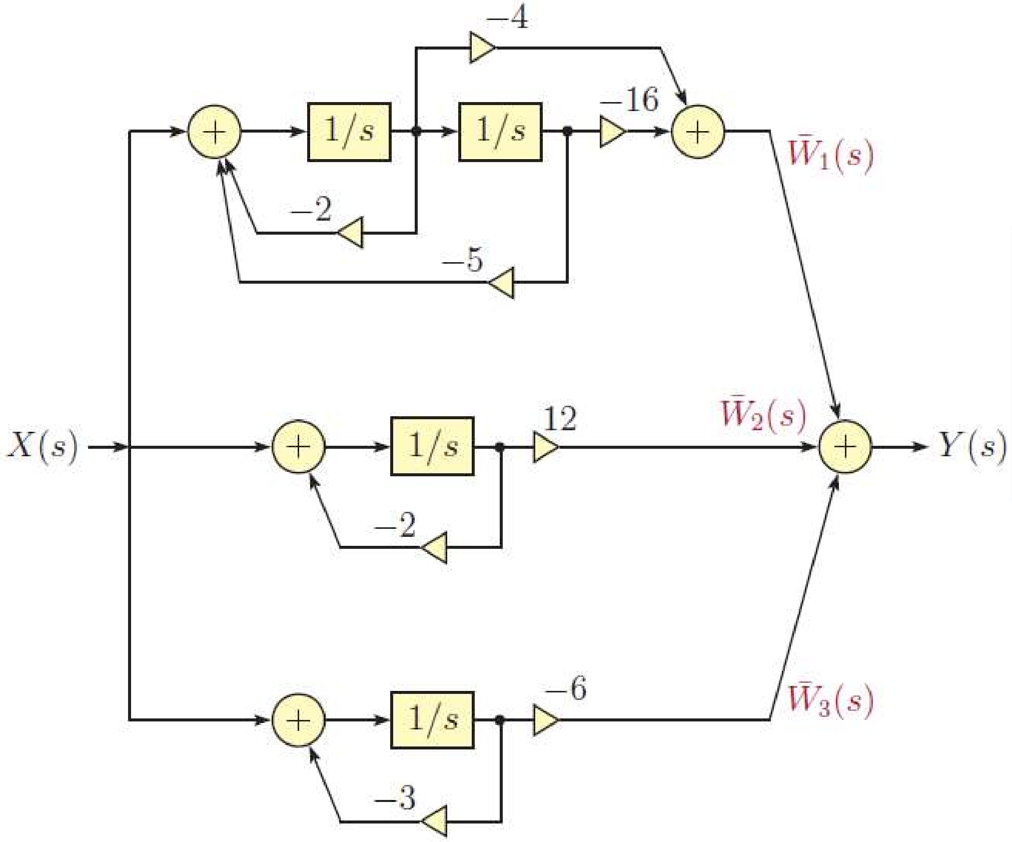
Parallel form $H(s)=\bar{H}_1(s)+\bar{H}_2(s)+\cdots+\bar{H}_M(s)=\dfrac{\bar{W}_1(s)}{X(s)}+\dfrac{\bar{W}_2(s)}{X(s)}+\cdots+\dfrac{\bar{W}_M(s)}{X(s)}$



- Example 23: Obtaining a block diagram from TF

  Develop a parallel form BD for simulating the system used in example 21.

$$H(s) = \frac{2s+8}{s^2+2s+5} + \frac{12}{s+2} + \frac{-6}{s+3}$$



4. Unilateral Laplace Transform

The unilateral Laplace transform of the function x is defined as:

$$\mathcal{L}_u\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- The unilateral LT is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \mathcal{L}\{x(t)u(t)\}$$

- With the unilateral LT, the same inverse transform equation is used as in the bilateral case.
- The unilateral LT is only invertible for causal functions.
- For a noncausal function x, we can only recover $x(t)$ for $t \geq 0$.

Unilateral Versus Bilateral Laplace Transform

In the unilateral case:

- The time-domain convolution property has the additional requirement that the functions being convolved must be causal.
- The time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive.
- The time-domain differentiation property has an extra term in the expression of $\mathcal{L}_u(dx(t)/dt)$, namely $-x(0^-)$.
- The time-domain integration property has a different lower limit in the time-domain integral (0^- instead of $-\infty$);
- The time-domain shifting property does not hold (except in special cases).

Properties of the Unilateral Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Multiply by t	$t\,x(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at),\ a > 0$	$\frac{1}{a}X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s) - x(0^-)$	$\supset R$
Integrate in t	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Unilateral Laplace Transform Pairs

Pair	$x(t);\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

Pair	$x(t);\ t \geq 0$	$X(s)$
6	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

- **Example 24:** Response of a second-order system (RLC circuit)

A voltage $x(t) = 10e^{-3t}u(t)$ is applied at the input of the RLC circuit. Find the output voltage $v_C(t) = y(t)$ for $t \geq 0$ if the initial inductor current is $i_L(0^-) = 0$, and the initial capacitor voltage $v_C(0^-) = 5$ V. Use $R = 3\ \Omega$, $L = 1$ H and $C = 1/2$ F.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t) \qquad y(0^-) = 5, \quad \frac{dy}{dt}(0^-) = \frac{i(0^-)}{C} = 0$$

$$s^2Y(s) - 5s - 0 + 3(sY(s) - 5) + 2Y(s) = 2X(s) = \frac{20}{s+3}$$

$$(s^2 + 3s + 2)Y(s) = 5s + 15 + \frac{20}{s+3}$$

$$Y(s) = \frac{5s^2 + 30s + 65}{(s+3)(s^2 + 3s + 2)}$$

$$y(t) = \underbrace{20e^{-t} - 25e^{-2t} + 10e^{-3t}}_{y_t(t)},\ t \geq 0$$

$$y(t) = \underbrace{20e^{-t} - 25e^{-2t}}_{y_n(t)} + \underbrace{10e^{-3t}}_{y_\phi(t)},\ t \geq 0$$