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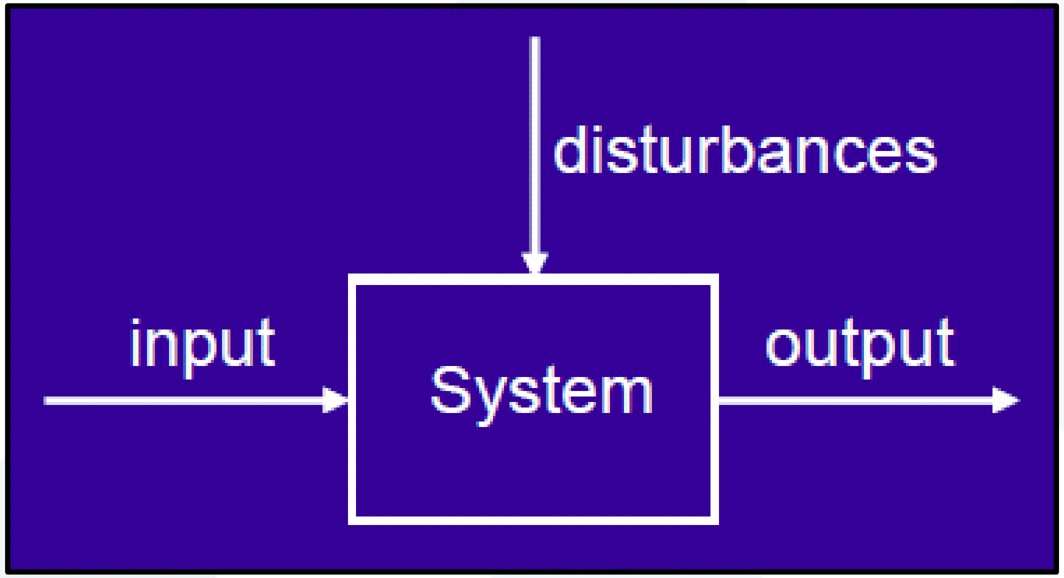
مقرر النمذجة و المطابقة

Minimum-Order Observer



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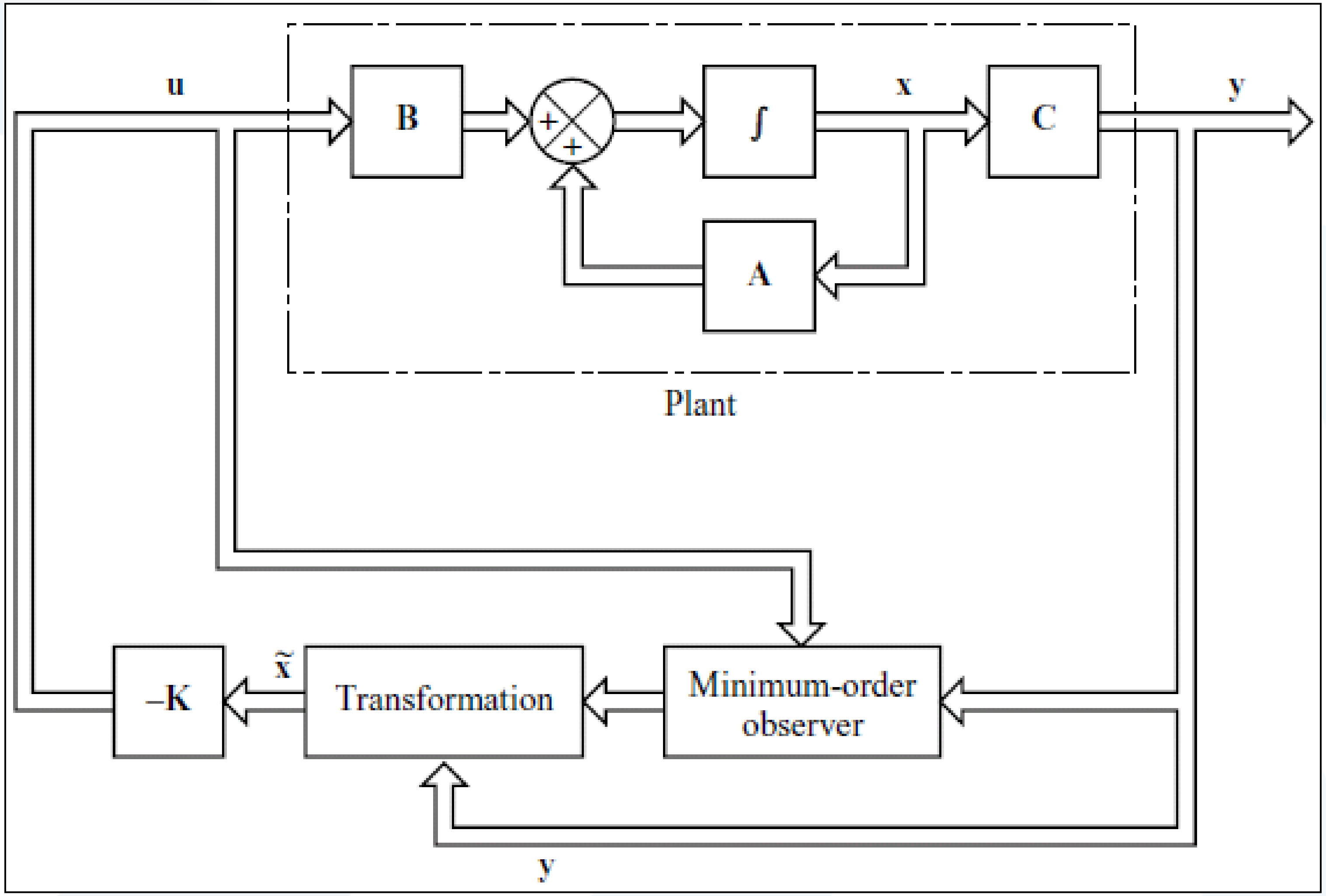
Minimum-Order Observer. The observers discussed thus far are designed to reconstruct all the state variables. In practice, some of the state variables may be accurately measured. Such accurately measurable state variables need not be estimated.

Suppose that the state vector $\mathbf{x}$ is an n-vector and the output vector $\mathbf{y}$ is an m-vector that can be measured. Since m output variables are linear combinations of the state variables, m state variables need not be estimated. We need to estimate only $n - m$ state variables. Then the reduced-order observer becomes an $(n - m)$th-order observer. Such an $(n - m)$th-order observer is the minimum-order observer. Figure shows the block diagram of a system with a minimum-order observer.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It is important to note, however, that if the measurement of output variables involves significant noises and is relatively inaccurate, then the use of the full-order observer may result in a better system performance.

To present the basic idea of the minimum-order observer, without undue mathematical complications, we shall present the case where the output is a scalar (that is, $m = 1$) and derive the state equation for the minimum-order observer. Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where the state vector $\mathbf{x}$ can be partitioned into two parts x_a (a scalar) and $\mathbf{x}_b$ [an $(n - 1)$-vector]. Here the state variable x_a is equal to the output y and thus can be directly measured, and $\mathbf{x}_b$ is the unmeasurable portion of the state vector. Then the partitioned state and output equations become

$$\begin{bmatrix} \dot{x}_a \\ \dot{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} + \begin{bmatrix} B_a \\ \mathbf{B}_b \end{bmatrix} u$$

$$y = [1 \;\vdots\; \mathbf{0}] \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix}$$

where
$$\begin{aligned} A_{aa} &= \text{scalar} \\ \mathbf{A}_{ab} &= 1 \times (n-1) \text{ matrix} \\ \mathbf{A}_{ba} &= (n-1) \times 1 \text{ matrix} \\ \mathbf{A}_{bb} &= (n-1) \times (n-1) \text{ matrix} \\ B_a &= \text{scalar} \\ \mathbf{B}_b &= (n-1) \times 1 \text{ matrix} \end{aligned}$$

$$\dot{x}_a = A_{aa} x_a + \mathbf{A}_{ab} \mathbf{x}_b + B_a u$$

$$\dot{x}_a - A_{aa} x_a - B_a u = \mathbf{A}_{ab} \mathbf{x}_b$$

In designing the minimum-order observer, we consider the left-hand side of Equation to be known quantities. Thus, Equation relates the measurable quantities and unmeasurable quantities of the state.

From Equation, the equation for the unmeasured portion of the state becomes

$$\dot{\mathbf{x}}_b = \mathbf{A}_{ba}x_a + \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{B}_b u$$

Noting that terms $\mathbf{A}_{ba}x_a$ and $\mathbf{B}_b u$ are known quantities, Equation describes the dynamics of the unmeasured portion of the state.

In what follows we shall present a method for designing a minimum-order observer. The design procedure can be simplified if we utilize the design technique developed for the full-order state observer.

Let us compare the state equation for the full-order observer with that for the minimum-order observer. The state equation for the full-order observer is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and the "state equation" for the minimum-order observer is

$$\dot{\mathbf{x}}_b = \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u$$

The output equation for the full-order observer is

$$y = \mathbf{C}\mathbf{x}$$

and the "output equation" for the minimum-order observer is

$$\dot{x}_a - A_{aa}x_a - B_a u = \mathbf{A}_{ab}\mathbf{x}_b$$

The design of the minimum-order observer can be carried out as follows: First, note that the observer equation for the full-order observer was given by Equation

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

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List of Necessary Substitutions for Writing the Observer Equation for the Minimum-Order State Observer

Full-Order State Observer	Minimum-Order State Observer
$\tilde{\mathbf{x}}$	$\tilde{\mathbf{x}}_b$
$\mathbf{A}$	$\mathbf{A}_{bb}$
$\mathbf{B}u$	$\mathbf{A}_{ba}x_a + \mathbf{B}_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
$\mathbf{C}$	$\mathbf{A}_{ab}$
$\mathbf{K}_e$ $(n \times 1$ matrix$)$	$\mathbf{K}_e$ $[(n-1) \times 1$ matrix$]$

Then, making the substitutions

$$\dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\tilde{\mathbf{x}}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u + \mathbf{K}_e(\dot{x}_a - A_{aa}x_a - B_a u)$$

where the state observer gain matrix $\mathbf{K}_e$ is an $(n - 1) \times 1$ matrix. In Equation notice that in order to estimate $\tilde{\mathbf{x}}_b$, we need the derivative of x_a. This presents a difficulty, because differentiation amplifies noise. If $x_a (= y)$ is noisy, the use of $\dot{x}_a$ is unacceptable.

To avoid this difficulty, we eliminate $\dot{x}_a$ in the following way. First rewrite Equation as

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_b - \mathbf{K}_e\dot{x}_a &= (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\tilde{\mathbf{x}}_b + (\mathbf{A}_{ba} - \mathbf{K}_e A_{aa})y + (\mathbf{B}_b - \mathbf{K}_e B_a)u \\ &= (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})(\tilde{\mathbf{x}}_b - \mathbf{K}_e y) \\ &\quad + [(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}]y \\ &\quad + (\mathbf{B}_b - \mathbf{K}_e B_a)u\end{aligned}$$

Define

$$\mathbf{x}_b - \mathbf{K}_e y = \mathbf{x}_b - \mathbf{K}_e x_a = \boldsymbol{\eta}$$

and

$$\tilde{\mathbf{x}}_b - \mathbf{K}_e y = \tilde{\mathbf{x}}_b - \mathbf{K}_e x_a = \tilde{\boldsymbol{\eta}}$$

$$\dot{\tilde{\boldsymbol{\eta}}} = (\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\tilde{\boldsymbol{\eta}} + [(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab})\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}]y + (\mathbf{B}_b - \mathbf{K}_e B_a)u$$

Define

$$\hat{\mathbf{A}} = \mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}$$

$$\hat{\mathbf{B}} = \hat{\mathbf{A}}\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}$$

$$\hat{\mathbf{F}} = \mathbf{B}_b - \mathbf{K}_e B_a$$

$$\dot{\tilde{\boldsymbol{\eta}}} = \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}u$$

Since

$$y = [1 \;\vdots\; \mathbf{0}]\begin{bmatrix} x_a \\ \cdots \\ \mathbf{x}_b \end{bmatrix}$$

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_a \\ \hline \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} y \\ \hline \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{I}_{n-1} \end{bmatrix}[\tilde{\mathbf{x}}_b - \mathbf{K}_e y] + \begin{bmatrix} 1 \\ \hline \mathbf{K}_e \end{bmatrix} y$$

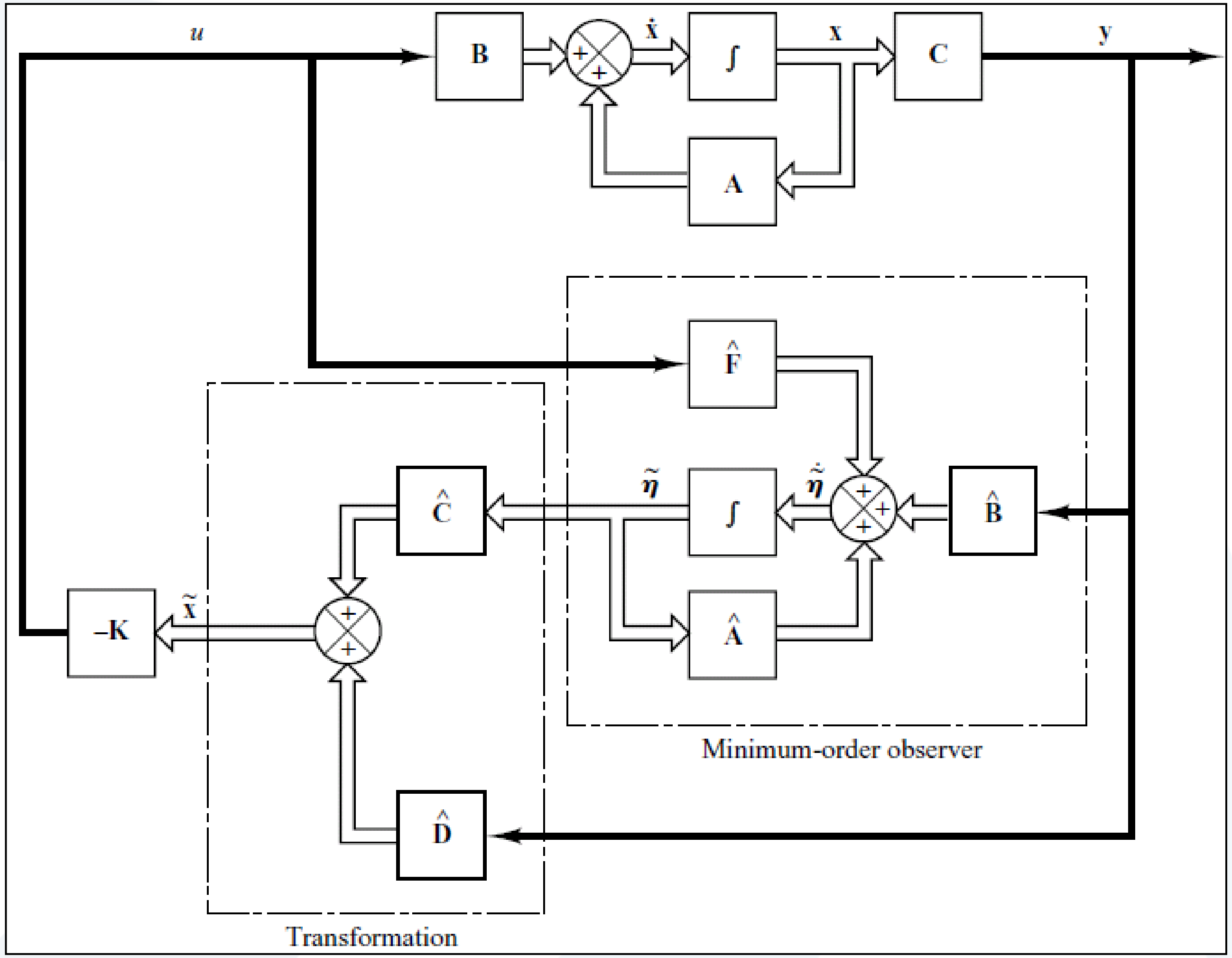
where $\mathbf{0}$ is a row vector consisting of $(n - 1)$ zeros, if we define

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{I}_{n-1} \end{bmatrix}, \qquad \hat{\mathbf{D}} = \begin{bmatrix} 1 \\ \hline \mathbf{K}_e \end{bmatrix}$$

then we can write $\tilde{\mathbf{x}}$ in terms of $\tilde{\boldsymbol{\eta}}$ and y as follows:

$$\tilde{\mathbf{x}} = \hat{\mathbf{C}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{D}}y$$

This equation gives the transformation from $\tilde{\boldsymbol{\eta}}$ to $\tilde{\mathbf{x}}$.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Minimum-order observer
Transformation

Next we shall derive the observer error equation

$$\dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\tilde{\mathbf{x}}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u + \mathbf{K}_e\mathbf{A}_{ab}\mathbf{x}_b$$

$$\dot{\mathbf{x}}_b - \dot{\tilde{\mathbf{x}}}_b = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})(\mathbf{x}_b - \tilde{\mathbf{x}}_b)$$

Define

$$\mathbf{e} = \mathbf{x}_b - \tilde{\mathbf{x}}_b = \boldsymbol{\eta} - \tilde{\boldsymbol{\eta}}$$

$$\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{e}$$

This is the error equation for the minimum-order observer. Note that $\mathbf{e}$ is an $(n - 1)$-vector.

The error dynamics can be chosen as desired by following the technique developed for the full-order observer.

The characteristic equation for the minimum-order observer is

$$|s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e \mathbf{A}_{ab}| = (s - \mu_1)(s - \mu_2) \cdots (s - \mu_{n-1})$$

where $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are desired eigenvalues for the minimum-order observer.

where $\mathbf{K}_e$ is an $(n - 1) \times 1$ matrix

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

Let us assume that we want to place the closed-loop poles at

$$s_1 = -2 + j2\sqrt{3}, \qquad s_2 = -2 - j2\sqrt{3}, \qquad s_3 = -6$$

Then the necessary state-feedback gain matrix **K** can be obtained as follows:

$$\mathbf{K} = [90 \quad 29 \quad 4]$$

Next, let us assume that the output y can be measured accurately so that state variable x_1 (which is equal to y) need not be estimated. Let us design a minimum-order observer. (The minimum-order observer is of second order.) Assume that we choose the desired observer poles to be at

$$s = -10, \qquad s = -10$$

the characteristic equation for the minimum-order observer is

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{K}_e \mathbf{A}_{ab}| &= (s - \mu_1)(s - \mu_2) \\ &= (s + 10)(s + 10) \\ &= s^2 + 20s + 100 = 0 \end{aligned}$$

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_a \\ \hdashline \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} x_1 \\ \hline \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}, \quad \mathbf{A} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ -6 & -11 & -6 \end{array}\right], \quad \mathbf{B} = \begin{bmatrix} 0 \\ \hline 0 \\ 1 \end{bmatrix}$$

$$A_{aa} = 0, \quad \mathbf{A}_{ab} = [1 \quad 0], \quad \mathbf{A}_{ba} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$$

$$\mathbf{A}_{bb} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}, \quad B_a = 0, \quad \mathbf{B}_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{K}_e = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$$

```
A = [0 1 0;0 0 1;-6 -11 -6];
B = [0;0;1];
J = [-2+j*2*sqrt(3) -2-j*2*sqrt(3) -6];
K = acker(A,B,J)

K =
  90.0000   29.0000    4.0000
```

```
Abb = [0 1;-11 -6];
Aab = [1 0];
L = [-10 -10];
Ke = acker(Abb',Aab',L)'

Ke =
   14
    5
```

$$\dot{\tilde{\boldsymbol{\eta}}} = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\tilde{\boldsymbol{\eta}} + [(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{K}_e + \mathbf{A}_{ba} - \mathbf{K}_e A_{aa}]y + (\mathbf{B}_b - \mathbf{K}_e B_a)u$$

where

$$\tilde{\boldsymbol{\eta}} = \tilde{\mathbf{x}}_b - \mathbf{K}_e y = \tilde{\mathbf{x}}_b - \mathbf{K}_e x_1$$

Noting that

$$\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix}[1 \quad 0] = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \left\{ \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} 14 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} 0 \right\} y + \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} 0 \right\} u$$

or

$$\begin{bmatrix} \dot{\tilde{\eta}}_2 \\ \dot{\tilde{\eta}}_3 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \begin{bmatrix} -191 \\ -260 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where

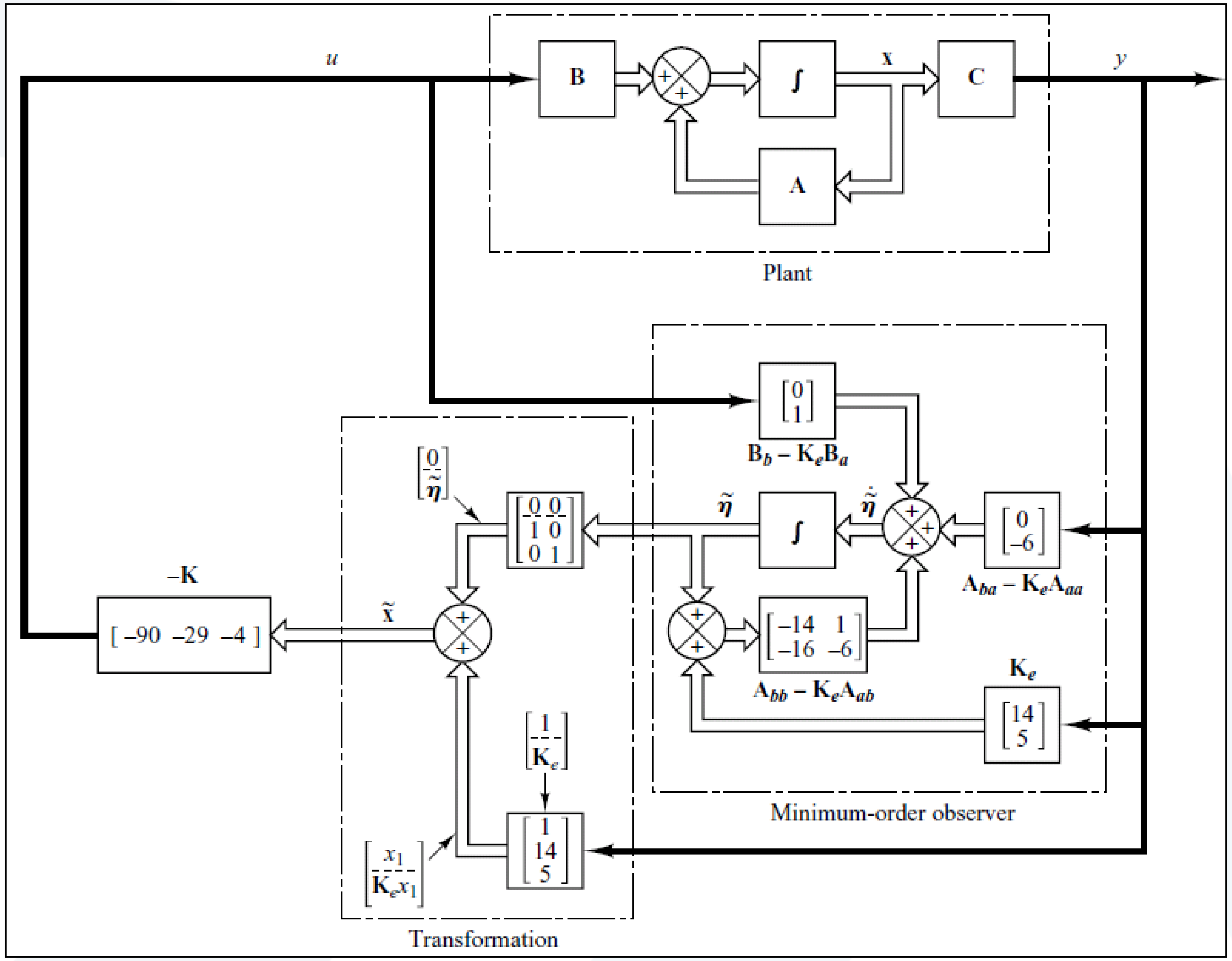
$$\begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} - \mathbf{K}_e y$$

or

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \mathbf{K}_e x_1$$

If the observed-state feedback is used, then the control signal u becomes

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -\mathbf{K}\begin{bmatrix} x_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$



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$B_b - K_e B_a$
$\tilde{\eta}$
$\dot{\tilde{\eta}}$
[0 -6]
$A_{ba} - K_e A_{aa}$
[0 $\tilde{\eta}$]
[0 0; 1 0; 0 1]
–K
[–90 –29 –4]
$\tilde{x}$
[–14 1; –16 –6]
$A_{bb} - K_e A_{ab}$
K_e
[14 5]
[1 K_e]
[1 14 5]
[x_1 $K_e x_1$]
Minimum-order observer
Transformation

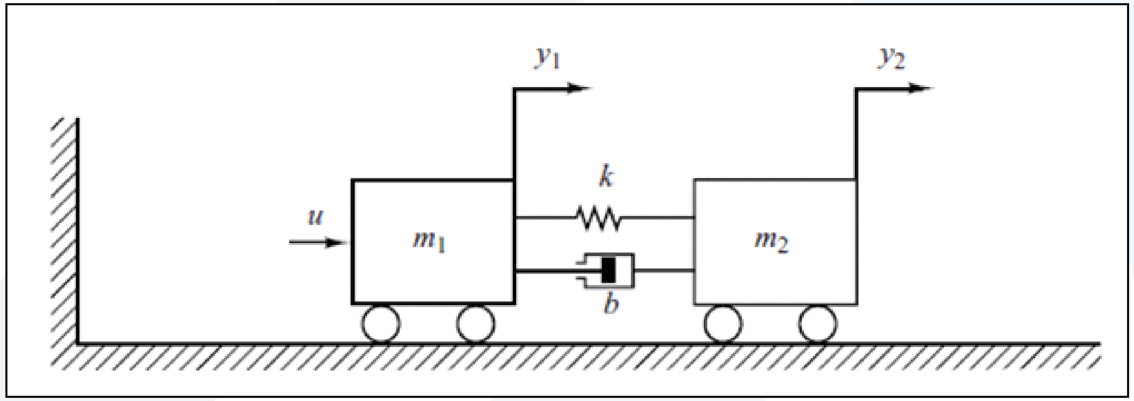
Example

Consider the system shown in Figure . Using the pole-placement-with-observer approach, design a regulator system such that the system will maintain the zero position $(y_1 = 0 \text{ and } y_2 = 0)$ in the presence of disturbances. Choose the desired closed-loop poles for the pole-placement part to be

$$s = -2 + j2\sqrt{3}, \qquad s = -2 - j2\sqrt{3}, \qquad s = -10, \qquad s = -10$$

and the desired poles for the minimum-order observer to be

$$s = -15, \qquad s = -16$$



First, determine the state feedback gain matrix **K** and observer gain matrix $\mathbf{K}_e$. Then, obtain the response of the system to an arbitrary initial condition—for example,

$$y_1(0) = 0.1, \quad y_2(0) = 0, \quad \dot{y}_1(0) = 0, \quad \dot{y}_2(0) = 0$$

$$e_1(0) = 0, \quad e_2(0) = 0$$

where e_1 and e_2 are defined by

$$e_1 = y_1 - \tilde{y}_1$$

$$e_2 = y_2 - \tilde{y}_2$$

Assume that $m_1 = 1$ kg, $m_2 = 2$ kg, $k = 36$ N/m, and $b = 0.6$ N-s/m.

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